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Theory of Function Spaces III

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Preface

This book may be considered as the continuation of the monographs [Triβ] and [Triγ] with the same title. It deals with the theory of function spaces of type B^s_{pq} and F^s_{pq} as it stands at the beginning of this century. These two scales of spaces cover many well-known spaces of functions and distributions such as Hölder-Zygmund spaces, (fractional and classical) Sobolev spaces, Besov spaces and Hardy spaces.

On the one hand this book is essentially self-contained. On the other hand we concentrate principally on those developments in recent times which are related to the nowadays numerous applications of function spaces to some neighboring areas such as numerics, signal processing and fractal analysis, to mention only a few of them.

Chapter 1 in [Triγ] is a self-contained historically-oriented survey of the function spaces considered and their roots up to the beginning of the 1990s entitled

How to measure smoothness.

Chapter 1 of the present book has the same heading indicating continuity. As far as the history is concerned we will now be very brief, restricting ourselves to the essentials needed to make this book self-contained and readable. We complement [Triγ], Chapter 1, by a few points omitted there. But otherwise we jump to the 1990s, describing more recent developments. Some of them will be treated later on in detail. In other words, [Triγ], Chapter 1, and Chapter 1 of the present book complement each other, providing a sufficiently comprehensive picture of the theory of the spaces B^s_{pq} and F^s_{pq} and their roots from the beginning up to our time. But quite obviously as far as very recent topics are concerned we are somewhat selective, emphasizing those developments which are near to our own interests.

This book has 9 chapters. Chapter 1 is the indicated self-contained survey.

Chapters 2 and 3 deal with building blocks in (isotropic) spaces of type B^s_{pq} and F^s_{pq} in $\mathbb{R}^n$, especially with (non-smooth) atoms (Chapter 2) and with wavelet bases and wavelet frames (Chapter 3). We discuss some consequences: pointwise multiplier assertions, positivity properties and local smoothness problems.

In recent times there is a growing interest in function spaces in (bounded) Lipschitz domains in $\mathbb{R}^n$. Here we split our presentation, collecting some old and a few new results in the introductory Section 1.11 and returning to this subject in greater detail in Chapter 4.

Wavelet representations of anisotropic function spaces and of weighted function spaces on $\mathbb{R}^n$ will be treated in Chapters 5 and 6, respectively.

Chapter 7 might be considered as the direct continuation of our studies in [Triδ] and [Triϵ] about fractal quantities of measures and spectral assertions of fractal elliptic operators.

Finally in Chapters 8 and 9 we develop a new theory for function spaces on quasi-metric spaces and on sets.

Formulas are numbered within the nine chapters. Furthermore, within each of these chapters all definitions, theorems, propositions, corollaries, remarks and examples are jointly and consecutively numbered. Chapter n is divided in subsections $n.k$, which occasionally are subdivided in subsubsections $n.k.l$. But when quoted we refer simply to Section $n.k$ or Section $n.k.l$ instead of Subsection $n.k$ or Subsubsection $n.k.l$, respectively.

It is a pleasure to acknowledge the great help I have received from my colleagues and friends round the world who made valuable suggestions which have been incorporated in the text. This applies in particular to Chapter 1 of this book. I am especially indebted to Dorothee D. Haroske for her remarks and for producing all the figures. Last, but not least, I wish to thank my friend David Edmunds in Brighton who looked through the whole manuscript and offered many comments.

Jena, Spring 2006 Hans Triebel

Chapter 1

How to Measure Smoothness

1.1 Introduction

This chapter has the same title as the historically-oriented survey in [Triγ], Chapter 1. But our aim now is somewhat different. As far as the background is concerned we will be very brief, restricting ourselves to the bare minimum and referring to [Triγ], Chapter 1, for more details. We are now mainly interested in a description of the theory of function spaces from the 1990s up to our time. Quite obviously we are somewhat selective, emphasizing topics of our own interest. Furthermore we prepare to some extent what follows in the subsequent chapters.

The function spaces B^s_{pq} and F^s_{pq} on $\mathbb{R}^n$ and on domains with respect to the full range of the parameters

$$s \in \mathbb{R}, \quad 0 < p \leq \infty, \quad 0 < q \leq \infty, \tag{1.1}$$

were introduced between 1959 and 1975. They cover many well-known classical concrete function spaces having their own history. In Section 1.2 we give a corresponding short list. These two scales of spaces and their special cases attracted a lot of attention and have been treated systematically with numerous applications given. We mention in particular the following books, reflecting also the development of this theory: [Sob50], [Nik77] (first edition 1969), [Ste70], [BIN75], [Pee76], [Triα] (1978), [Triβ] (1983) and [Triγ] (1992). Special aspects but related to our intentions have been studied in [AdF03] (Sobolev spaces; first edition 1975), [Maz85] (Sobolev spaces), [Zie89] (Sobolev spaces) and [ST87] (periodic spaces, anisotropic spaces and spaces with dominating mixed smoothness). The two surveys [BKLN88] and [KuN88] cover in particular the Russian literature. More recent developments of the spaces B^s_{pq} and F^s_{pq} in the last decade may be found in [ET96], [RuS96], [AdH96], [Triδ] (1997), [Triϵ] (2001), [HeN04] and [Har06]. More detailed references especially to the original papers may be found in [Triγ], Chapter 1.

The recent theory of the above function spaces is characterised by the extensive use of building blocks such as atoms, quarks, and wavelets. Hence it seems to be appropriate to complement the above literature by some more specific references. Atomic decompositions of the spaces B^s_{pq} and F^s_{pq} go back to [FrJ85] and [FrJ90]. Descriptions are also given in [FJW91], [Tor91], [Triγ], [ET96], and [Triδ], Section 13. The theory of subatomic or quarkonial decompositions has been developed in [Triδ] and, in greater detail, in [Triϵ]. Wavelet expansions (bases or frames) are a fashionable subject, preferably with respect to L_2-spaces or L_p-spaces where $1 < p < \infty$. Other types of function spaces such as classical Sobolev or Besov spaces are also treated but not as a major topic. We refer to [Mey92], [Dau92] and [Woj97]. In this book the theory of diverse building blocks such as atoms, quarks, wavelet bases and wavelet frames, and its applications to some problems of the spaces B^s_{pq} and F^s_{pq} play a central role, both in this introductory survey and in the subsequent chapters.

1.2 Concrete spaces

The systematic study in this book begins with Chapter 2. Then we collect the notation needed in the sequel in detail. On this somewhat preliminary basis we list a few special cases of the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ without further comments. In particular we postpone the (Fourier-analytical) definition of these spaces to the following Section 1.3. Our aim is twofold. First we wish to substantiate what has been said in Section 1.1. Secondly, as far as the classical function spaces are concerned we fix our notation. Of course, $\mathbb{R}^n$ is Euclidean n-space and $L_p(\mathbb{R}^n)$ is the usual complex Lebesgue space with respect to Lebesgue measure. Otherwise we use standard notation. In case of doubt one might consult the list of symbols at the end of the book and the references given there. We will be brief. More details may be found in [Triβ], especially Section 2.2.2, pp. 35–38, and [Triγ], especially Chapter 1.

(i) Let $1 < p < \infty$. Then

$$F^0_{p2}(\mathbb{R}^n) = L_p(\mathbb{R}^n). \tag{1.2}$$

This is a Paley-Littlewood theorem, see [Triβ], Section 2.5.6, pp. 87–88.

(ii) Let $1 < p < \infty$ and $s \in \mathbb{N}_0$. Then

$$F^s_{p2}(\mathbb{R}^n) = W^s_p(\mathbb{R}^n) \tag{1.3}$$

are the *classical Sobolev spaces*, usually normed by

$$\|f \,|W^s_p(\mathbb{R}^n)\| = \left(\sum_{|\alpha| \le s} \|D^\alpha f \,|L_p(\mathbb{R}^n)\|^p \right)^{\frac{1}{p}}. \tag{1.4}$$

This generalises assertion (i). We refer again to [Triβ], Section 2.5.6, pp. 87–88.

(iii) Let $\sigma \in \mathbb{R}$. Then

$$I_\sigma \;:\quad f \mapsto \left(\langle \xi \rangle^\sigma \widehat{f} \right)^\vee, \tag{1.5}$$

with $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$, is a one-to-one map of the Schwartz space $S(\mathbb{R}^n)$ onto itself and of the space of tempered distributions $S'(\mathbb{R}^n)$ onto itself. Here $\widehat{f}$ and $f^\vee$ are the Fourier transform of f and its inverse, respectively. Then I_σ is a lift for the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ with $s \in \mathbb{R}$, $0 < p \le \infty$ ($p < \infty$ for the F-scale), $0 < q \le \infty$:

$$I_\sigma B^s_{pq}(\mathbb{R}^n) = B^{s-\sigma}_{pq}(\mathbb{R}^n) \quad \text{and} \quad I_\sigma F^s_{pq}(\mathbb{R}^n) = F^{s-\sigma}_{pq}(\mathbb{R}^n) \tag{1.6}$$

(equivalent quasi-norms), [Triβ], Section 2.3.8, pp. 58–59. In particular, let

$$H^s_p(\mathbb{R}^n) = I_{-s} L_p(\mathbb{R}^n), \qquad s \in \mathbb{R}, \quad 1 < p < \infty. \tag{1.7}$$

Then one gets by (1.2), (1.3), and (1.6),

$$H^s_p(\mathbb{R}^n) = F^s_{p2}(\mathbb{R}^n), \qquad s \in \mathbb{R}, \quad 1 < p < \infty, \tag{1.8}$$

and

$$H^s_p(\mathbb{R}^n) = W^s_p(\mathbb{R}^n) \quad \text{if} \quad s \in \mathbb{N}_0 \quad \text{and} \quad 1 < p < \infty. \tag{1.9}$$

We call $H^s_p(\mathbb{R}^n)$ *Sobolev spaces* (sometimes denoted as fractional Sobolev spaces or Bessel potential spaces) and its special cases (1.9) with (1.4) *classical Sobolev spaces.*

(iv) We denote

$$\mathcal{C}^s(\mathbb{R}^n) = B^s_{\infty\infty}(\mathbb{R}^n), \qquad s \in \mathbb{R}, \tag{1.10}$$

as *Hölder-Zygmund spaces.* Let

$$\left(\Delta^1_h f\right)(x) = f(x+h) - f(x), \quad \left(\Delta^{l+1}_h f\right)(x) = \Delta^1_h \left(\Delta^l_h f\right)(x), \tag{1.11}$$

where $x \in \mathbb{R}^n$, $h \in \mathbb{R}^n$, $l \in \mathbb{N}$, be the iterated differences in $\mathbb{R}^n$. Let $0 < s < m \in \mathbb{N}$. Then

$$\|f \,|\mathcal{C}^s(\mathbb{R}^n)\|_m = \sup_{x \in \mathbb{R}^n} |f(x)| + \sup |h|^{-s} \,|\Delta^m_h f(x)| \tag{1.12}$$

where the second supremum is taken over all $x \in \mathbb{R}^n$ and $h \in \mathbb{R}^n$ with $0 < |h| \le 1$, are equivalent norms in $\mathcal{C}^s(\mathbb{R}^n)$. For more details we refer again to [Triβ], Sections 2.2.2, 2.5.12. Hence if $s > 0$ then $\mathcal{C}^s(\mathbb{R}^n)$ are the well-known *Hölder-Zygmund spaces.* We extend this notation to all $s \in \mathbb{R}$.

(v) Assertion (iv) can be generalised as follows. Once more let $0 < s < m \in \mathbb{N}$ and $1 \le p \le \infty$, $1 \le q \le \infty$. Then

$$\|f \,|B^s_{pq}(\mathbb{R}^n)\|_m = \|f \,|L_p(\mathbb{R}^n)\| + \left(\int\limits_{|h| \le 1} |h|^{-sq} \; \|\Delta^m_h f \,|L_p(\mathbb{R}^n)\|^q \; \frac{dh}{|h|^n} \right)^{1/q} \tag{1.13}$$

(with the usual modification if $q = \infty$) are equivalent norms in $B^s_{pq}(\mathbb{R}^n)$. As for details we refer to [Triβ], Sections 2.2.2, 2.5.12. These are the *classical Besov spaces.*

Remark 1.1. There are further concrete spaces which fit in the scales $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$. For example, the (inhomogeneous) *Hardy spaces* $h_p(\mathbb{R}^n)$ with $0 < p < \infty$ can be identified with $F^0_{p2}(\mathbb{R}^n)$. Furthermore for all of the above spaces one has numerous equivalent norms and characterisations. We refer to the literature in Section 1.1 and in particular to [Triα], [Triβ], [Triγ].

1.3 The Fourier-analytical approach

Recall that this introductory first chapter should be seen in continuation of Chapter 1 in [Triγ] with the same title. We do not repeat the history presented there. Just on the contrary, we restrict ourselves to those ingredients needed later on and which are the basis of the theory of the spaces B^s_{pq} and F^s_{pq} up to recent times. In particular from now onwards we incorporate immediately distinguished results of the last decade.

We use now standard notation which will be detailed later on beginning with Chapter 2. In case of doubt one may consult the list of symbols at the end of the book and the references given there. In particular, $S(\mathbb{R}^n)$ and $S'(\mathbb{R}^n)$ are the Schwartz space of all complex-valued rapidly decreasing C^∞ functions on $\mathbb{R}^n$, and the dual space of all tempered distributions. The Fourier transform of $\varphi \in S(\mathbb{R}^n)$ is denoted by $\widehat{\varphi}$ or $F\varphi$. As usual, $\varphi^\vee$ and $F^{-1}\varphi$ stand for the inverse Fourier transform. Both F and F^{-1} are extended to $S'(\mathbb{R}^n)$ in the standard way. Let $\varphi_0 \in S(\mathbb{R}^n)$ with

$$\varphi_0(x) = 1 \text{ if } |x| \le 1 \quad \text{and} \quad \varphi_0(y) = 0 \text{ if } |y| \ge 3/2, \tag{1.14}$$

and let

$$\varphi_k(x) = \varphi_0(2^{-k}x) - \varphi_0(2^{-k+1}x), \qquad x \in \mathbb{R}^n, \quad k \in \mathbb{N}. \tag{1.15}$$

Then, since

$$1 = \sum_{j=0}^{\infty} \varphi_j(x) \qquad \text{for all} \quad x \in \mathbb{R}^n, \tag{1.16}$$

the φ_j form a dyadic resolution of unity in $\mathbb{R}^n$. Recall that $(\varphi_j \widehat{f})^\vee$ is an entire analytic function on $\mathbb{R}^n$ for any $f \in S'(\mathbb{R}^n)$. In particular, $(\varphi_j \widehat{f})^\vee(x)$ makes sense pointwise.

Definition 1.2. *Let $\varphi = \{\varphi_j\}_{j=0}^\infty$ be the dyadic resolution of unity according to (1.14)–(1.16).*

(i) *Let* $0 < p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, *and*

$$\|f \,|B^s_{pq}(\mathbb{R}^n)\|_\varphi = \left(\sum_{j=0}^{\infty} 2^{jsq} \, \|(\varphi_j \widehat{f})^\vee \,|L_p(\mathbb{R}^n)\|^q \right)^{1/q} \tag{1.17}$$

(*with the usual modification if* $q = \infty$). *Then*

$$B^s_{pq}(\mathbb{R}^n) = \left\{ f \in S'(\mathbb{R}^n) : \quad \|f \,|B^s_{pq}(\mathbb{R}^n)\|_\varphi < \infty \right\}. \tag{1.18}$$

(ii) *Let* $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, *and*

$$\|f \,|F^s_{pq}(\mathbb{R}^n)\|_\varphi = \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} |(\varphi_j \widehat{f})^\vee(\cdot)|^q \right)^{1/q} \,|L_p(\mathbb{R}^n) \right\| \tag{1.19}$$

(*with the usual modification if* $q = \infty$). *Then*

$$F^s_{pq}(\mathbb{R}^n) = \left\{ f \in S'(\mathbb{R}^n) : \quad \|f \,|F^s_{pq}(\mathbb{R}^n)\|_\varphi < \infty \right\}. \tag{1.20}$$

Remark 1.3. The history of these definitions may be found in [Triγ], Section 1.5, especially on p. 29, which will not be repeated here. Some distinguished special cases have been listed in the preceding Section 1.2. The huge corresponding literature, mostly books, may be found in Section 1.1. A systematic study of these spaces in the above generality has been given in [Triβ], [Triγ], and more recently in [Triδ], [Triϵ], including many references.

It is convenient to complement these definitions by some maximal functions. Again let $\varphi = \{\varphi_j\}_{j=0}^{\infty}$ be the above resolution of unity. Then we introduce the maximal functions

$$\left(\varphi^*_{j,a} f\right)(x) = \sup_{y \in \mathbb{R}^n} \frac{|(\varphi_j \widehat{f})^\vee(x-y)|}{1 + |2^j y|^a}, \qquad f \in S'(\mathbb{R}^n), \quad a > 0. \tag{1.21}$$

Maximal functions play a crucial role in diverse types of function spaces as demonstrated in [Ste93]. The above version and its use in connection with the spaces introduced in the definition goes back to J. Peetre, [Pee75], [Pee76]. But otherwise we refer to [Triβ], [Triγ] for history and literature. Recall that (1.21) always makes sense, accepting that the right-hand side might be infinite. More precisely: Let $\varphi \in S(\mathbb{R}^n)$ and $f \in S'(\mathbb{R}^n)$. Then for $x \in \mathbb{R}^n$,

$$\begin{aligned} (f * \varphi)(x) &= f\,(\varphi(x - \cdot)) \\ &= \int_{\mathbb{R}^n} \varphi(x-y)\, f(y)\, \mathrm{d}y = \int_{\mathbb{R}^n} \varphi(y)\, f(x-y)\, \mathrm{d}y \end{aligned} \tag{1.22}$$

is the convolution, appropriately interpreted (as far as the writing as an integral is concerned). Recall that there are numbers $c > 0$ and $N > 0$ such that

$$f * \varphi \in C^\infty(\mathbb{R}^n) \quad \text{and} \quad |(f * \varphi)(x)| \le c\,(1 + |x|^2)^N, \quad x \in \mathbb{R}^n. \tag{1.23}$$

In particular, $f * \varphi \in S'(\mathbb{R}^n)$. Furthermore,

$$(\varphi \widehat{f})^\vee(x) = (2\pi)^{-n/2}\,(f * \varphi^\vee)(x), \qquad x \in \mathbb{R}^n. \tag{1.24}$$

A few more details and in particular references to the classical literature about these fundamental properties may be found in [Triα], Section 2.2.1, p. 152. The counterparts of (1.17) and (1.19) in terms of the maximal functions in (1.21) are given by

$$\|f \,|B^s_{pq}(\mathbb{R}^n)\|_{\varphi,a} = \left(\sum_{j=0}^\infty 2^{jsq} \left\| \varphi^*_{j,a} f \,|L_p(\mathbb{R}^n)\right\|^q \right)^{1/q} \tag{1.25}$$

(with the usual modification if $q = \infty$) and

$$\|f \,|F^s_{pq}(\mathbb{R}^n)\|_{\varphi,a} = \left\| \left(\sum_{j=0}^\infty 2^{jsq} \left| (\varphi^*_{j,a} f)(\cdot) \right|^q \right)^{1/q} |L_p(\mathbb{R}^n) \right\| \tag{1.26}$$

(with the usual modification if $q = \infty$).

Theorem 1.4. *Let $\varphi = \{\varphi_j\}_{j=0}^\infty$ be the dyadic resolution of unity according to (1.14)–(1.16). Let $\varphi^*_{j,a} f$ be given by (1.21).*

(i) *Let*

$$0 < p \le \infty, \quad 0 < q \le \infty, \quad \text{and} \quad s \in \mathbb{R}. \tag{1.27}$$

Then $B^s_{pq}(\mathbb{R}^n)$ is a quasi-Banach space (Banach space if $p \ge 1$, $q \ge 1$) and it is independent of φ (equivalent quasi-norms). Let, in addition, $a > n/p$. Then

$$B^s_{pq}(\mathbb{R}^n) = \left\{ f \in S'(\mathbb{R}^n) : \|f \,|B^s_{pq}(\mathbb{R}^n)\|_{\varphi,a} < \infty \right\} \tag{1.28}$$

(equivalent quasi-norms).

(ii) *Let*

$$0 < p < \infty, \quad 0 < q \le \infty \quad \text{and} \quad s \in \mathbb{R}. \tag{1.29}$$

Then $F^s_{pq}(\mathbb{R}^n)$ is a quasi-Banach space (Banach space if $p \ge 1$, $q \ge 1$) and it is independent of φ (equivalent quasi-norms). Let, in addition, $a > \frac{n}{\min(p,q)}$. Then

$$F^s_{pq}(\mathbb{R}^n) = \left\{ f \in S'(\mathbb{R}^n) : \|f \,|F^s_{pq}(\mathbb{R}^n)\|_{\varphi,a} < \infty \right\} \tag{1.30}$$

(equivalent quasi-norms).

Remark 1.5. A proof of this theorem may be found in [Triγ], Section 2.3.2, pp. 93–96. In particular, the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ are independent of φ. This may also justify our omission of the subscript φ in (1.17) and (1.19) in what follows.

Remark 1.6. The quasi-norms in the above theorem and in the explicit norms for the concrete spaces in Section 1.2 look rather different and one may ask whether there is a unified approach. This was one of the major themes in [Triγ]. For example, the basic ingredients of the norm in (1.13) can be written as

$$|h|^{-s}\,(\Delta^m_h f)\,(x) = |h|^{-s}\,\Big(\big(e^{i\xi\cdot h}-1\big)^m\,\widehat{f}\Big)^\vee(x). \tag{1.31}$$

With $|h| \sim 2^{-j}$ one gets expressions of type $2^{js}(\varphi_j\widehat{f})^\vee$ as in Definition 1.2 but with different functions φ_j. Furthermore by (1.24) and (1.22) one has with $c = (2\pi)^{-n/2}$,

$$(\varphi_j\,\widehat{f})^\vee(x) = c\int_{\mathbb{R}^n}\varphi^\vee_j(y)\,f(x-y)\,\mathrm{d}y, \qquad j\in\mathbb{N}_0. \tag{1.32}$$

If the φ_j are given by (1.14)–(1.16) then $\varphi^\vee_j$ are analytic functions on $\mathbb{R}^n$. In particular in order to calculate (1.32) at a given point $x\in\mathbb{R}^n$ one has to know f on the whole $\mathbb{R}^n$. This is in sharp contrast to the derivatives and local differences in the classical norms in (1.4), (1.12) or (1.13), (1.31). Hence it would be desirable to shift the compactness of the support from φ_j to $\varphi^\vee_j$ in (1.32). This is possible and results in local means are discussed in the next subsection. But first we describe a rather satisfactory recent result which covers both the original approach and the just indicated local means.

We modify (1.14)–(1.16) as follows. Let $\varepsilon > 0$ and $L\in\mathbb{N}_0$. Let

$$\varphi_0\in S(\mathbb{R}^n), \qquad |\varphi_0(x)| > 0 \quad \text{if} \quad |x|\le 2\varepsilon, \tag{1.33}$$

and

$$\varphi^0\in S(\mathbb{R}^n), \quad |\varphi^0(x)|>0 \text{ if } \varepsilon/2\le|x|\le 2\varepsilon \quad \text{and} \quad (D^\alpha\varphi^0)(0)=0 \tag{1.34}$$

for $|\alpha| < L$. If $L = 0$ then the last assumption is empty. Let

$$\varphi = \{\varphi_j\}^\infty_{j=0} \quad \text{with} \quad \varphi_j(x) = \varphi^0(2^{-j}x) \text{ if } j\in\mathbb{N}. \tag{1.35}$$

Then it follows by (1.24), (1.23) that $(\varphi_j\widehat{f})^\vee$ is well defined for any $f\in S'(\mathbb{R}^n)$. It is a C^∞ function in $\mathbb{R}^n$ of at most polynomial growth. In particular, (1.21) and also (1.17), (1.19), (1.25), (1.26) make sense for any $f\in S'(\mathbb{R}^n)$ (accepting that the outcome might be infinite).

Theorem 1.7. *Let $\varepsilon > 0$ and let φ be given by* (1.33)–(1.35) *with $L\in\mathbb{N}_0$.*

(i) *Let*

$$0<p\le\infty, \quad 0<q\le\infty, \quad L>s\in\mathbb{R} \quad \textit{and} \quad a>n/p. \tag{1.36}$$

Then

$$\begin{aligned} B^s_{pq}(\mathbb{R}^n) &= \bigl\{ f \in S'(\mathbb{R}^n) \;:\; \|f\,|B^s_{pq}(\mathbb{R}^n)\|_\varphi < \infty \bigr\} \\ &= \bigl\{ f \in S'(\mathbb{R}^n) \;:\; \|f\,|B^s_{pq}(\mathbb{R}^n)\|_{\varphi,a} < \infty \bigr\} \end{aligned} \tag{1.37}$$

(*equivalent quasi-norms*).

(ii) *Let*

$$0 < p < \infty, \quad 0 < q \le \infty, \quad L > s \in \mathbb{R} \quad \textit{and} \quad a > \frac{n}{\min(p,q)}. \tag{1.38}$$

Then

$$\begin{aligned} F^s_{pq}(\mathbb{R}^n) &= \bigl\{ f \in S'(\mathbb{R}^n) \;:\; \|f\,|F^s_{pq}(\mathbb{R}^n)\|_\varphi < \infty \bigr\} \\ &= \bigl\{ f \in S'(\mathbb{R}^n) \;:\; \|f\,|F^s_{pq}(\mathbb{R}^n)\|_{\varphi,a} < \infty \bigr\} \end{aligned} \tag{1.39}$$

(*equivalent quasi-norms*).

Remark 1.8. This theorem has a little history. In [Triγ], Sections 2.4, 2.5, we studied for the F-spaces and B-spaces, respectively, to which extent the defining system φ according to (1.14)–(1.16) can be replaced by more general systems such that one gets equivalent quasi-norms in the spaces considered. Both the support conditions expressed by (1.14), (1.15) as well as the a priori assumptions $\varphi_j \in S(\mathbb{R}^n)$ are weakened in a substantial way. The replacement of $\varphi_j \in S(\mathbb{R}^n)$ by weaker assumptions is of great interest. We shift this discussion to a later subsection. But even for systems φ belonging to $S(\mathbb{R}^n)$ our restrictions in [Triγ] for L in connection with (1.34) are stronger (and less natural) than in (1.36). Furthermore we gave in [Triγ] also some characterisations of type (1.39). But there remained some gaps as far as the parameters p, q are concerned. These gaps were sealed in [BPT96] and [BPT97] including an improvement of L as it stands now in (1.36). The above version is essentially due to V.S. Rychkov in [Ry99a] who gave a streamlined proof and corrected a minor error in [BPT96]. However it should be mentioned that [BPT96], [BPT97] deal with the more general case of some weighted spaces of B^s_{pq} and F^s_{pq} type. Otherwise one may consult Remark 1.48 in Section 1.6 where we give a few additional references, especially in connection with other types of function spaces.

Remark 1.9. For given $s \in \mathbb{R}$, the two conditions for $\varphi^0 \in S(\mathbb{R}^n)$ in (1.34), (1.36), (1.38) can be reformulated as

$$|\varphi^0(x)| > 0 \text{ if } \varepsilon/2 \le |x| \le 2\varepsilon \quad \text{and} \quad (D^\alpha \varphi^0)(0) = 0 \text{ if } |\alpha| \le [s], \tag{1.40}$$

where $[s]$ is the largest integer smaller than or equal to s.

1.4 Local means

Let $B = \{y \in \mathbb{R}^n : |y| < 1\}$ be the unit ball in $\mathbb{R}^n$ and let k be a C^∞ function in $\mathbb{R}^n$ with supp $k \subset B$. Then

$$k(t,f)(x) = \int_{\mathbb{R}^n} k(y)\, f(x+ty)\,\mathrm{d}y = t^{-n} \int_{\mathbb{R}^n} k\left(\frac{y-x}{t}\right) f(y)\,\mathrm{d}y \tag{1.41}$$

with $x \in \mathbb{R}^n$ and $t > 0$ are *local means*: To calculate $k(t,f)(x)$ at $x \in \mathbb{R}^n$ one needs only the restriction of f to a ball of radius $t > 0$, centred at x. Appropriately interpreted, (1.41) makes sense for any $f \in S'(\mathbb{R}^n)$. For given $s \in \mathbb{R}$ it is assumed that the kernel k satisfies in addition for some $\varepsilon > 0$,

$$k^\vee(\xi) \neq 0 \text{ if } 0 < |\xi| \leq \varepsilon \quad \text{and} \quad (D^\alpha k^\vee)(0) = 0 \text{ if } |\alpha| \leq s. \tag{1.42}$$

(The second condition is empty if $s < 0$.) Furthermore, let k_0 be a second C^∞ function in $\mathbb{R}^n$ with supp $k_0 \subset B$ and $k_0^\vee(0) \neq 0$. Obviously, $\varphi_0 = k_0^\vee$ and $\varphi^0 = k^\vee$ satisfy (1.33) and (1.40), respectively (with a modified ε). Hence one can apply Theorem 1.7. We give an explicit formulation. The counterpart of the maximal function (1.21) is given now by

$$k^*(2^{-j},f)_a(x) = \sup_{y \in \mathbb{R}^n} \frac{|k(2^{-j},f)(x-y)|}{1+|2^j y|^a}, \quad f \in S'(\mathbb{R}^n), \quad a > 0, \tag{1.43}$$

where $j \in \mathbb{N}$, complemented by $k_0^*(1,f)_a$. Then the counterparts of the quasi-norms in Theorem 1.7 are given by

$$\begin{aligned} \|f \,|B^s_{pq}(\mathbb{R}^n)\|^{k_0,k} &= \|k_0(1,f)\,|L_p(\mathbb{R}^n)\| \\ &\quad + \left(\sum_{j=1}^\infty 2^{jsq}\,\|k(2^{-j},f)\,|L_p(\mathbb{R}^n)\|^q\right)^{1/q}, \end{aligned} \tag{1.44}$$

$$\begin{aligned} \|f \,|F^s_{pq}(\mathbb{R}^n)\|^{k_0,k} &= \|k_0(1,f)\,|L_p(\mathbb{R}^n)\| \\ &\quad + \left\| \left(\sum_{j=1}^\infty 2^{jsq}\,|k(2^{-j},f)(\cdot)|^q\right)^{1/q} |L_p(\mathbb{R}^n)\right\| \end{aligned} \tag{1.45}$$

and the corresponding expressions

$$\|f \,|B^s_{pq}(\mathbb{R}^n)\|^{k_0,k}_a \quad \text{and} \quad \|f \,|F^s_{pq}(\mathbb{R}^n)\|^{k_0,k}_a, \tag{1.46}$$

where one has to replace $k_0(1,f)$, $k(2^{-j},f)$ in (1.44) and (1.45) by $k_0^*(1,f)_a$, $k^*(2^{-j},f)_a$, respectively.

Theorem 1.10. *Let* $0 < q \le \infty$ *and* $s \in \mathbb{R}$. *Let* k_0 *and* k *be the above kernels of local means satisfying* $k_0^\vee(0) \neq 0$ *and* (1.42) *for some* $\varepsilon > 0$.

(i) *Let* $0 < p \le \infty$ *and* $a > n/p$. *Then*

$$\begin{aligned} B^s_{pq}(\mathbb{R}^n) &= \big\{ f \in S'(\mathbb{R}^n) : \ \|f \,| B^s_{pq}(\mathbb{R}^n)\|^{k_0,k} < \infty \big\} \\ &= \big\{ f \in S'(\mathbb{R}^n) : \ \|f \,| B^s_{pq}(\mathbb{R}^n)\|^{k_0,k}_a < \infty \big\} \end{aligned} \tag{1.47}$$

(*equivalent quasi-norms*).

(ii) *Let* $0 < p < \infty$ *and* $a > \frac{n}{\min(p,q)}$. *Then*

$$\begin{aligned} F^s_{pq}(\mathbb{R}^n) &= \big\{ f \in S'(\mathbb{R}^n) : \ \|f \,| F^s_{pq}(\mathbb{R}^n)\|^{k_0,k} < \infty \big\} \\ &= \big\{ f \in S'(\mathbb{R}^n) : \ \|f \,| F^s_{pq}(\mathbb{R}^n)\|^{k_0,k}_a < \infty \big\} \end{aligned} \tag{1.48}$$

(*equivalent quasi-norms*).

Proof. By (1.35), $\varphi^0 = k^\vee$, and (1.32) it follows for $j \in \mathbb{N}$,

$$(\varphi_j \widehat{f})^\vee(x) = c\, 2^{jn} \int_{\mathbb{R}^n} (\varphi^0)^\vee(2^j y)\, f(x-y)\, \mathrm{d}y = c\, k(2^{-j}, f)(x). \tag{1.49}$$

Similarly for $j = 0$ with $\varphi_0 = k_0^\vee$. Then the theorem follows from the above explanations, (1.21), and Theorem 1.7 where we used (1.40). □

Remark 1.11. The above theorem is an improvement of [Triγ], Section 2.4.6, pp. 122/123 and Section 2.5.3, pp. 138/139, for the F-spaces and B-spaces, respectively. There we proved corresponding assertions in terms of equivalent quasi-norms (and not as characterisations), including several modifications which will not be repeated here. Adapted to the above notation we needed in [Triγ] the stronger condition

$$(D^\alpha k^\vee)(0) = 0 \quad \text{if} \quad |\alpha| \le \max(s, \sigma_p) \quad \text{with} \quad \sigma_p = n\left(\frac{1}{p} - 1\right)_+, \tag{1.50}$$

compared with (1.42). In connection with the study of (fractal) measures as elements of function spaces it came out around 1995 that no assumptions of this type are needed for the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ if $s < 0$. The first explicit formulation was given in the PhD Thesis [Win95]. But it can also be found (under some additional restrictions) in [AdH96], Corollary 4.3.8, p. 102. We discussed this point in [Triϵ], p. 125. This observation has been extensively used in [Triδ] and [Triϵ] in connection with fractal measures and fractal elliptic operators. It will also play a crucial role in this book. This may justify giving an explicit formulation.

Again let k be a C^∞ function in $\mathbb{R}^n$ with $\operatorname{supp} k \subset B$ and $k^\vee(0) \neq 0$. Let

$$\|f \,|B^s_{pq}(\mathbb{R}^n)\|^k = \left(\sum_{j=0}^\infty 2^{jsq} \, \|k(2^{-j}, f) \,|L_p(\mathbb{R}^n)\|^q \right)^{1/q}, \tag{1.51}$$

$$\|f \,|F^s_{pq}(\mathbb{R}^n)\|^k = \left\| \left(\sum_{j=0}^\infty 2^{jsq} \, |k(2^{-j}, f)(\cdot)|^q \right)^{1/q} |L_p(\mathbb{R}^n) \right\| \tag{1.52}$$

and let

$$\|f \,|B^s_{pq}(\mathbb{R}^n)\|^k_a \qquad \text{and} \qquad \|f \,|F^s_{pq}(\mathbb{R}^n)\|^k_a \tag{1.53}$$

be given by (1.51), (1.52) with $k^*(2^{-j}, f)_a$ according to (1.43) in place of $k(2^{-j}, f)$.

Corollary 1.12. *Let $0 < q \leq \infty$ and $s < 0$. Let k be a C^∞ function in $\mathbb{R}^n$ with $\operatorname{supp} k \subset B$ and $k^\vee(0) \neq 0$.*

(i) *Let $0 < p \leq \infty$ and $a > n/p$. Then*

$$\begin{aligned} B^s_{pq}(\mathbb{R}^n) &= \left\{ f \in S'(\mathbb{R}^n) \,:\, \|f \,|B^s_{pq}(\mathbb{R}^n)\|^k < \infty \right\} \\ &= \left\{ f \in S'(\mathbb{R}^n) \,:\, \|f \,|B^s_{pq}(\mathbb{R}^n)\|^k_a < \infty \right\} \end{aligned} \tag{1.54}$$

(*equivalent quasi-norms*).

(ii) *Let $0 < p < \infty$ and $a > \frac{n}{\min(p,q)}$. Then*

$$\begin{aligned} F^s_{pq}(\mathbb{R}^n) &= \left\{ f \in S'(\mathbb{R}^n) \,:\, \|f \,|F^s_{pq}(\mathbb{R}^n)\|^k < \infty \right\} \\ &= \left\{ f \in S'(\mathbb{R}^n) \,:\, \|f \,|F^s_{pq}(\mathbb{R}^n)\|^k_a < \infty \right\} \end{aligned} \tag{1.55}$$

(*equivalent quasi-norms*).

Remark 1.13. This assertion follows immediately from Theorem 1.10 and $s < 0$ since one can choose $k = k_0$ in (1.42). In particular one may assume that the kernel k in (1.41) is non-negative.

Remark 1.14. In [Triγ], Sections 2.4, 2.5, we considered equivalent quasi-norms and also (under some additional assumptions) characterisations of the same type as in Theorem 1.7 but for more general basic functions φ_0 and φ^0 in (1.33) and (1.34). In particular we weakened $\varphi_0 \in S(\mathbb{R}^n)$ and $\varphi^0 \in S(\mathbb{R}^n)$ by some decay conditions at infinity. This is necessary if one wishes to incorporate more general means where (1.31) may serve as an example. It results in expressions of type (1.32) where the kernels $\varphi_j^\vee$ might have only limited smoothness. This applies in particular to the local means in (1.41) with kernels which are not C^∞. We did not follow these arguments in [Triγ] since at this time there was no interest and no use for kernels which are not C^∞. But now the situation is different. For example if one chooses for k a Daubechies wavelet then one has kernels which have necessarily only a limited smoothness. We return to this point later on in connection with wavelet bases.

But in any case it would be desirable to combine the corresponding arguments in [Triγ] with the techniques developed later on especially in [BPT96], [BPT97], [Ry99a], to develop a theory of equivalent quasi-norms and characterisations as in Theorems 1.7, 1.10 and Corollary 1.12 for non-smooth, this means not C^∞, kernels in (1.32), (1.41). Local means and the indicated more general approach have also been considered in other types of function spaces. Some corresponding references may be found in Remark 1.48 in Section 1.6.

1.5 Atoms

It is one aim of this book to characterise function spaces in terms of building blocks and to use these representations for diverse purposes. This can be done for spaces defined on $\mathbb{R}^n$, domains in $\mathbb{R}^n$, manifolds, fractals or quasi-metric spaces. We return to some of these possibilities in later chapters. But first we concentrate on spaces on $\mathbb{R}^n$, more precisely on the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ as introduced in Definition 1.2, and their representations in terms of atoms (this Section 1.5), quarks (Section 1.6), wavelet bases (Section 1.7) and wavelet frames (Section 1.8).

1.5.1 Smooth atoms

Recall that $\mathbb{Z}^n$ stands for the lattice of all points in $\mathbb{R}^n$ with integer-valued components. Furthermore, $Q_{\nu m}$ denotes the closed cube in $\mathbb{R}^n$ with sides parallel to the axes of coordinates, centred at $2^{-\nu}m$, and with side-length $2^{-\nu+1}$ where $m \in \mathbb{Z}^n$ and $\nu \in \mathbb{N}_0$. If Q is a cube in $\mathbb{R}^n$ and $r > 0$ then rQ is the cube in $\mathbb{R}^n$ concentric with Q and with side length r times the side length of Q.

Definition 1.15.

(i) *Let $K \in \mathbb{N}_0$ and $c \geq 1$. A continuous function $a : \mathbb{R}^n \to \mathbb{C}$ for which there exist all derivatives $D^\alpha a$ if $|\alpha| \leq K$ is called a 1-atom (more precisely 1_K-atom) if*

$$\operatorname{supp} a \subset c\,Q_{0m} \qquad \text{for some} \quad m \in \mathbb{Z}^n \tag{1.56}$$

and

$$|D^\alpha a(x)| \leq 1 \qquad \text{for} \quad |\alpha| \leq K. \tag{1.57}$$

(ii) *Let $s \in \mathbb{R}$, $0 < p \leq \infty$, $K \in \mathbb{N}_0$, $L \geq 0$, and $c \geq 1$. A continuous function $a : \mathbb{R}^n \to \mathbb{C}$ for which there exist all derivatives $D^\alpha a$ if $|\alpha| \leq K$ is called an (s,p)-atom (more precisely $(s,p)_{K,L}$-atom) if*

$$\operatorname{supp} a \subset c\,Q_{\nu m} \qquad \text{for some} \quad \nu \in \mathbb{N}, \quad m \in \mathbb{Z}^n, \tag{1.58}$$

$$|D^\alpha a(x)| \leq 2^{-\nu(s-n/p)+|\alpha|\nu} \qquad \text{for} \quad |\alpha| \leq K, \tag{1.59}$$

and

$$\int_{\mathbb{R}^n} x^\beta\, a(x)\, \mathrm{d}x = 0 \qquad \text{for} \quad |\beta| < L. \tag{1.60}$$

Remark 1.16. Recall $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$ where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\beta \in \mathbb{N}_0^n$. If $L = 0$ then (1.60) is empty (no condition). Obviously, (1.60) can be reformulated as

$$(D^\beta \widehat{a})(0) = 0 \qquad \text{if} \quad |\beta| < L. \tag{1.61}$$

We need some further notation. Let $0 < p \le \infty$. Then

$$\chi_{\nu m}^{(p)}(x) = 2^{(\nu-1)n/p} \text{ if } x \in Q_{\nu m} \quad \text{and} \quad \chi_{\nu m}^{(p)}(x) = 0 \text{ if } x \notin Q_{\nu m}, \tag{1.62}$$

are p-normalised characteristic functions of the above cubes $Q_{\nu m}$ with $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$.

Definition 1.17. *Let $0 < p \le \infty$, $0 < q \le \infty$,*

$$\lambda = \{\lambda_{\nu m} \in \mathbb{C} : \ \nu \in \mathbb{N}_0, \ m \in \mathbb{Z}^n\}, \tag{1.63}$$

$$\|\lambda \,|b_{pq}\| = \left(\sum_{\nu=0}^\infty \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}|^p \right)^{q/p} \right)^{1/q} \tag{1.64}$$

and

$$\|\lambda \,|f_{pq}\| = \left\| \left(\sum_{\nu=0}^\infty \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}\, \chi_{\nu m}^{(p)}(\cdot)|^q \right)^{1/q} |L_p(\mathbb{R}^n) \right\| \tag{1.65}$$

(with the usual modification if $p = \infty$ and/or $q = \infty$). Then

$$b_{pq} = \{\lambda : \ \|\lambda \,|b_{pq}\| < \infty\} \tag{1.66}$$

and

$$f_{pq} = \{\lambda : \ \|\lambda \,|f_{pq}\| < \infty\}. \tag{1.67}$$

Remark 1.18. Obviously, b_{pq} and f_{pq} are quasi-Banach spaces and $b_{pp} = f_{pp}$. We put as usual,

$$\sigma_p = n \left(\frac{1}{p} - 1 \right)_+ \qquad \text{and} \qquad \sigma_{pq} = n \left(\frac{1}{\min(p,q)} - 1 \right)_+, \tag{1.68}$$

where $c_+ = \max(c, 0)$ if $c \in \mathbb{R}$. Furthermore we indicate the location and size of a 1_K-atom or an $(s,p)_{K,L}$-atom according to Definition 1.15 by writing $a_{\nu m}$ in place of a.

Theorem 1.19.

(i) *Let $0 < p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$. Let $K \in \mathbb{N}_0$, $L \ge 0$ with*

$$K > s \qquad \textit{and} \qquad L > \sigma_p - s \tag{1.69}$$

be fixed. Then $f \in S'(\mathbb{R}^n)$ *belongs to* $B^s_{pq}(\mathbb{R}^n)$ *if, and only if, it can be represented as*

$$f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m}\, a_{\nu m}, \quad \text{unconditional convergence being in } S'(\mathbb{R}^n), \tag{1.70}$$

where for fixed $c \geq 1$, $a_{\nu m}$ *are* 1_K*-atoms* ($\nu = 0$) *or* $(s,p)_{K,L}$*-atoms* ($\nu \in \mathbb{N}$) *according to Definition* 1.15 *and Remark* 1.18, *and* $\lambda \in b_{pq}$. *Furthermore,*

$$\|f \,|B^s_{pq}(\mathbb{R}^n)\| \sim \inf \|\lambda \,|b_{pq}\| \tag{1.71}$$

are equivalent quasi-norms where the infimum is taken over all admissible representations (1.70).

(ii) *Let* $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$. *Let* $K \in \mathbb{N}_0$, $L \geq 0$ *with*

$$K > s \qquad \text{and} \qquad L > \sigma_{pq} - s \tag{1.72}$$

be fixed. Then $f \in S'(\mathbb{R}^n)$ *belongs to* $F^s_{pq}(\mathbb{R}^n)$ *if, and only if, it can be represented by* (1.70) *where now for fixed* $c \geq 1$, $a_{\nu m}$ *are* 1_K*-atoms or* $(s,p)_{K,L}$*-atoms with respect to* (1.72) *and* $\lambda \in f_{pq}$. *Furthermore,*

$$\|f \,|F^s_{pq}(\mathbb{R}^n)\| \sim \inf \|\lambda \,|f_{pq}\| \tag{1.73}$$

are equivalent quasi-norms where the infimum is taken over all admissible representations (1.70).

Remark 1.20. Atomic representations in function spaces have some history which may be found in [Triγ], Section 1.9, and which will not be repeated here. Atoms in $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ and the above theorem go essentially back to [FrJ85], [FrJ90], [FJW91] (but one may consult [Triγ], Section 1.9, for a more balanced history). In [Triδ], Section 13, we gave a new proof of Theorem 1.19, based on local means according to Section 1.4, and discussed several modifications with corresponding references. As for atomic representations in other types of function spaces one may consult Remark 1.48 in Section 1.6. But one word about the convergence of (1.70) seems to be in order. Let $\varphi \in S(\mathbb{R}^n)$ and let $\lambda \in b_{pq}$, K and L as in part (i) (covering the even stronger assumptions of part (ii)). Let the countable index set $\{\nu, m\}$ in (1.70) be numbered in any way, hence mapped one-to-one onto $\mathbb{N}$. Then it follows from [Triδ], Corollary 13.9 and its proof, pp. 81/82, that

$$\sum_{\nu,m} \lambda_{\nu,m}\, (a_{\nu,m}, \varphi) \qquad \text{converges}$$

with the same outcome for any rearrangement, called *unconditional convergence* in $S'(\mathbb{R}^n)$. Furthermore, one may ask to which extent the restrictions (1.69) for the B-spaces and (1.72) for the F-spaces are natural. Especially the q-dependence of (1.72) seems to be questionable. But one can not replace σ_{pq} in (1.72) by σ_p.

Otherwise the whole mapping theory for Calderón-Zygmund operators and pseudodifferential operators in F^s_{pq}-spaces as it may be found, for example, in [Tor91] would be independent of q. But this is not the case as it has been observed in [Wang99]. The delicate dependence of mapping properties of exotic pseudodifferential operators in $\mathbb{R}^n$ on q has been discovered quite recently in [Joh04, Joh05]. Another genuine q-dependence of characterisations of $F^s_{pq}(\mathbb{R}^n)$ will be mentioned in Remark 1.117 in connection with Theorem 1.116. A more detailed discussion is shifted to Remark 9.15.

1.5.2 Non-smooth atoms

One of the most striking applications of atomic decompositions is the study of mapping properties of, say, linear pseudo-differential operators T, hoping first that

$$T\left(\sum_\nu \sum_m \lambda_{\nu m}\, a_{\nu m}\right) = \sum_\nu \sum_m \lambda_{\nu m}\, T a_{\nu m} \tag{1.74}$$

can be justified. But usually, $Ta_{\nu m}$ preserves neither the compactness (1.58) of the support of the atom $a_{\nu m}$ nor its, (possibly) high smoothness. From this point of view it is reasonable to relax the assumptions about the compactness of the support of $a_{\nu m}$ and the smoothness of $a_{\nu m}$. Replacing (1.58) by (sufficiently strong) decay assumptions at infinity one gets *molecules*. They have been considered in detail in [FrJ90], [FJW91], [Tor91]. But we do not discuss this point here. However we need later on non-smooth atoms. A first (but for our purposes not sufficient) step is to replace the C^K-norm in (1.59) with $K > s$ according to (1.69) by a Hölder-norm $\mathcal{C}^\sigma$ with $\sigma > s$. This is also covered by the just-mentioned literature. We give a brief description following essentially [TrW96] which may also be found in [ET96], Section 2.2.3, pp. 28–32.

Let $0 < \sigma = [\sigma] + \{\sigma\}$ with $[\sigma] \in \mathbb{N}_0$ and $0 < \{\sigma\} < 1$. Then the Hölder-Zygmund space in Section 1.2(iv) can also be normed by

$$\|f\,|\mathcal{C}^\sigma(\mathbb{R}^n)\| = \sum_{|\alpha|\le[\sigma]} \sup_{x\in\mathbb{R}^n} |D^\alpha f(x)| + \sum_{|\alpha|=[\sigma]} \sup_{x\ne y} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x-y|^{\{\sigma\}}}. \tag{1.75}$$

Now the fractional version of Definition 1.15 reads as follows.

Definition 1.21.

(i) *Let* $0 < \sigma \notin \mathbb{N}$. *Then* $a : \mathbb{R}^n \to \mathbb{C}$ *is called a* 1_σ*-atom if one has* (1.56) *and*

$$a \in \mathcal{C}^\sigma(\mathbb{R}^n) \qquad \text{with} \qquad \|a\,|\mathcal{C}^\sigma(\mathbb{R}^n)\| \le 1. \tag{1.76}$$

(ii) *Let* $s \in \mathbb{R}$, $0 < p \le \infty$, $0 < \sigma \notin \mathbb{N}$ *and* $L \ge 0$. *Then* $a : \mathbb{R}^n \to \mathbb{C}$ *is called an* $(s,p)_{\sigma,L}$*-atom if one has* (1.58), (1.60), *and*

$$a \in \mathcal{C}^\sigma(\mathbb{R}^n) \qquad \text{with} \qquad \|a(2^{-\nu}\cdot)\,|\mathcal{C}^\sigma(\mathbb{R}^n)\| \le 2^{-\nu(s-n/p)}. \tag{1.77}$$

Remark 1.22. Definition 1.15 can be rephrased in the sense of the above definition, replacing $\mathcal{C}^\sigma(\mathbb{R}^n)$ by $C^K(\mathbb{R}^n)$, in obvious notation. The fractional counterpart of Theorem 1.19 reads as follows. Recall $s_+ = \max(s, 0)$.

Corollary 1.23.

(i) *Let p, q, s, L be as in Theorem* 1.19(i) *and let $s_+ < \sigma \notin \mathbb{N}$. Then $f \in S'(\mathbb{R}^n)$ belongs to $B^s_{pq}(\mathbb{R}^n)$ if, and only if, it can be represented by* (1.70) *where $a_{\nu m}$ are 1_σ-atoms respectively $(s,p)_{\sigma,L}$-atoms and $\lambda \in b_{pq}$. Furthermore,* (1.71) *are again equivalent quasi-norms.*

(ii) *Let p, q, s, L be as in Theorem* 1.19(ii)*. Let $s_+ < \sigma \notin \mathbb{N}$. Then $f \in S'(\mathbb{R}^n)$ belongs to $F^s_{pq}(\mathbb{R}^n)$ if, and only if, it can be represented by* (1.70) *where $a_{\nu m}$ are 1_σ-atoms, respectively $(s,p)_{\sigma,L}$-atoms and $\lambda \in f_{pq}$. Furthermore,* (1.73) *are again equivalent quasi-norms.*

Remark 1.24. We followed essentially [TrW96] and [ET96], Section 2.2.3. But as said the assertion itself is more or less covered by the work of Frazier and Jawerth. It was not our aim in [TrW96] to prove this assertion, but to take it as a starting point for a corresponding theory of atomic representations for spaces $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$ in irregular bounded domains in $\mathbb{R}^n$. In this context we refer also to [ET96], Section 2.5.

Whereas Definition 1.21 and Corollary 1.23 are essentially covered by the same literature which resulted in Theorem 1.19, we describe now a generalisation which will be considered in detail in Chapter 2. Let now

$$B^s_p(\mathbb{R}^n) = B^s_{pp}(\mathbb{R}^n) \qquad \text{with} \quad 0 < p \le \infty, \quad s \in \mathbb{R}. \tag{1.78}$$

In particular,

$$\mathcal{C}^s(\mathbb{R}^n) = B^s_\infty(\mathbb{R}^n), \qquad s \in \mathbb{R}, \tag{1.79}$$

are the Hölder-Zygmund spaces according to (1.10).

Definition 1.25. *Let $0 < p \le \infty$ and $\sigma_p < s < \sigma$, where σ_p is given by* (1.68)*. Then $a \in B^\sigma_p(\mathbb{R}^n)$ is called an $(s,p)^\sigma$-atom if one has* (1.56) *or* (1.58) *for some $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$ and*

$$\|a \,|B^\sigma_p(\mathbb{R}^n)\| \le 2^{\nu(\sigma - s)}. \tag{1.80}$$

Remark 1.26. We wish to compare these atoms with the atoms in the Definitions 1.15, 1.21 as used in Theorem 1.19 and Corollary 1.23. In particular, since $s > \sigma_p$ any $L \ge 0$ is admitted and we may choose $L = 0$. This gives the possibility to unify the corresponding atoms in the Definitions 1.15, 1.21 and to incorporate notationally 1_K-atoms and 1_σ-atoms. Let $\varrho = K \in \mathbb{N}_0$. Then the 1_K-atoms ($\nu = 0$) and $(s,p)_{K,0}$-atoms ($\nu \in \mathbb{N}$) according to Definition 1.15 are denoted as $(s,p)_\varrho$-atoms. Let $0 < \varrho = \sigma \notin \mathbb{N}$. Then $(s,p)_\varrho$-atoms denote the corresponding $(s,p)_{\sigma,0}$-atoms according to Definition 1.21 again incorporating 1_σ-atoms. As before we write $a_{\nu m}$ if a in the above definition is located by (1.56), (1.58).

Proposition 1.27. *Let $0 < p \le \infty$ and $\sigma_p < s < \sigma$, where σ_p is given by* (1.68).

(i) *Let $\sigma < \varrho$. Then any $(s,p)_\varrho$-atom is an $(s,p)^\sigma$-atom.*

(ii) *Let $a_{\nu m}$ be an $(s,p)^\sigma$-atom. Then*

$$\|a_{\nu m}\,|B^s_p(\mathbb{R}^n)\| \le 1 \quad \text{and} \quad \|a_{\nu m}\,|L_p(\mathbb{R}^n)\| \le 2^{-\nu s}. \tag{1.81}$$

Remark 1.28. Hence those atoms introduced in Definitions 1.15 and 1.21 to which Remark 1.26 applies are covered by the atoms according to Definition 1.25. If $s < n/p$ then $(s,p)^\sigma$-atoms might be unbounded. If $\sigma < 1/p$ then (appropriately normalised) step-functions of cubes are $(s,p)^\sigma$-atoms. We return to this subject in detail in Chapter 2 including an application of the respective theory to pointwise multipliers in the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$. However the main motivation for studying non-smooth atoms comes from the theory of function spaces on quasi-metric spaces. We return to this point in the last chapter of this book in detail and outline some basic ideas in Section 1.17. At this moment we restrict ourselves to formulate the counterparts of Theorem 1.19 and Corollary 1.23. Let $b_p = b_{pp}$ be the special sequence space according to Definition 1.17.

Theorem 1.29. *Let $0 < p \le \infty$ and $\sigma_p < s < \sigma$, where σ_p is given by* (1.68). *Then $B^s_p(\mathbb{R}^n)$ is the collection of all $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n) \cap S'(\mathbb{R}^n)$ which can be represented as*

$$f = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\, a_{\nu m} \tag{1.82}$$

where $a_{\nu m}$ for fixed $c \ge 1$ are $(s,p)^\sigma$-atoms according to Definition 1.25 *(and Remark* 1.26*) and $\lambda \in b_p$. The series on the right-hand side of* (1.82) *converges unconditionally in $S'(\mathbb{R}^n)$ and, if $p < \infty$, absolutely in some $L_r(\mathbb{R}^n)$ with $1 < r < \infty$. Furthermore,*

$$\|f\,|B^s_p(\mathbb{R}^n)\| \sim \inf \|\lambda\,|b_p\| \tag{1.83}$$

are equivalent quasi-norms where the infimum is taken over all admissible representations (1.82).

Remark 1.30. As said these problems will be studied in detail in Chapter 2. To avoid a misunderstanding we recall that the right-hand side of (1.82) is said to converge absolutely in some $L_r(\mathbb{R}^n)$ if

$$\sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}| \cdot |a_{\nu m}(x)|$$

converges in $L_r(\mathbb{R}^n)$. This is always the case if $p < \infty$ and $\lambda \in b_p$. If $p = \infty$ then one has absolute convergence in the weighted space $L_\infty(\mathbb{R}^n, w)$, normed by $\|wf\,|L_\infty(\mathbb{R}^n)\|$ with $w(x) = (1+|x|^2)^{\sigma/2}$ where $\sigma < 0$. Furthermore we remark that in case of $p \le 1$ the above theorem follows easily from Theorem 1.19, (1.81), and the triangle inequality for the p-Banach spaces $B^s_p(\mathbb{R}^n)$,

$$\|f\,|B^s_p(\mathbb{R}^n)\|^p \le \sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|^p\,\|a_{\nu m}\,|B^s_p(\mathbb{R}^n)\|^p. \tag{1.84}$$

The case $1 < p \le \infty$ is less obvious.

1.5.3 A technical modification

The sequence spaces b_{pq} and f_{pq} as introduced in Definition 1.17 play a crucial role not only in atomic decompositions of distributions as described above but also in connection with quarkonial representations, wavelet bases and wavelet frames as considered in the following Sections 1.6–1.8 and treated in detail later on. Whereas the structure of the spaces b_{pq} is rather simple, the spaces f_{pq} are more complicated. We describe two assertions which go back essentially to [FrJ90]. Let $Q_{\nu m}$ be the same cubes in $\mathbb{R}^n$ as in Section 1.5.1, centred at $2^{-\nu}m$ with sides of length $2^{-\nu+1}$ parallel to the axes of coordinates, where $\nu\in\mathbb{N}_0$ and $m\in\mathbb{Z}^n$. Let

$$E = \{E_{\nu m}\,:\,\nu\in\mathbb{N}_0,\ m\in\mathbb{Z}^n\},\qquad E_{\nu m}\subset Q_{\nu m},\quad |E_{\nu m}|\sim|Q_{\nu m}|, \tag{1.85}$$

where the equivalence constants are independent of ν and m. Let $\chi^{(p),E}_{\nu m}$ be given by (1.62) with $E_{\nu m}$ in place of $Q_{\nu m}$. We combine the replacement of $Q_{\nu m}$ by $E_{\nu m}$ in Definition 1.17 with an index-shifting $k+\mathbb{Z}^n$, where $k\in\mathbb{Z}^n$, as follows.

Definition 1.31. *Let $0<p<\infty$, $0<q\le\infty$ and E according to* (1.85). *Let*

$$\lambda = \{\lambda_{\nu,m}\in\mathbb{C}:\ \nu\in\mathbb{N}_0,\ m\in\mathbb{Z}^n\}, \tag{1.86}$$

$k\in\mathbb{Z}^n$ and

$$\|\lambda\,|f_{pq}\|^{(k)}_E = \left\| \left(\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu,m+k}\,\chi^{(p),E}_{\nu m}(\cdot)|^q\right)^{1/q} |L_p(\mathbb{R}^n)\right\|. \tag{1.87}$$

Then

$$f^{E,(k)}_{pq} = \left\{\lambda\,:\ \|\lambda\,|f_{pq}\|^{(k)}_E < \infty\right\}. \tag{1.88}$$

Remark 1.32. We generalised Definition 1.17. If $E_{\nu m} = Q_{\nu m}$ then we write $\|\lambda\,|f_{pq}\|^{(k)}$ instead of (1.87) with $\|\lambda\,|f_{pq}\|$ if, in addition, $k=0$. Furthermore if $k=0$ then we simplify (1.87) and (1.88) for arbitrary E by

$$\|\lambda\,|f_{pq}\|_E \qquad\text{and}\qquad f^E_{pq}, \tag{1.89}$$

respectively.

Proposition 1.33. *Let $0 < p < \infty$ and $0 < q \le \infty$.*

(i) *Let E be given by* (1.85)*. Then*

$$f^E_{pq} = f_{pq} \tag{1.90}$$

(*equivalent quasi-norms depending on the equivalence constants in* (1.85)).

(ii) *There are constants $a > 0$ and $c > 0$ such that for all $k \in \mathbb{Z}^n$,*

$$\|\lambda \,| f_{pq}\|^{(k)} \le c\,(1+|k|)^a \, \|\lambda \,| f_{pq}\|. \tag{1.91}$$

Remark 1.34. Part (i) seems to be a little bit surprising at the first glance. But it is a consequence of the vector-valued Hardy-Littlewood maximal inequality. Details about this technique may be found in [Triδ], p. 79, from which this assertion follows quite easily. The observation itself is due to [FrJ90], Proposition 2.7, where also a short proof is given. It is quite obvious that the index-shifting $m \to m+k$ in (1.87) maps f_{pq} onto itself. Hence it is the main point of (1.91) to compare the equivalent quasi-norms and to clarify the influence of $k \in \mathbb{Z}^n$. The number $a > 0$ depends on p and q and can be estimated by using the technique of maximal functions according to [Triδ], p. 79.

Example 1.35. We describe a simple but typical example which will be of some use for us later on. Let $0 < p < \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. Let $a_\nu(x) = a_{\nu,0}(x)$ with $\nu \in \mathbb{N}_0$ be atoms in $F^s_{pq}(\mathbb{R}^n)$ according to Theorem 1.19 with respect to the cubes $Q_{\nu,0}$ centred at the origin. Let

$$f = \sum_{\nu=0}^{\infty} \lambda_\nu \, a_\nu, \qquad \lambda_\nu \in \mathbb{C}. \tag{1.92}$$

Then

$$\|f \,| F^s_{pq}(\mathbb{R}^n)\| \le \|\lambda \,| f_{pq}\| \sim \|\lambda \,| f_{pq}\|_E \sim \left(\sum_{\nu=0}^{\infty} |\lambda_\nu|^p \right)^{1/p} \tag{1.93}$$

independently of q. This follows from (1.90) if one chooses the sets $E_\nu = E_{\nu,0}$ in (1.85) such that $E_{\nu_1} \cap E_{\nu_2} = \emptyset$ if $\nu_1 \ne \nu_2$. This is possible.

1.6 Quarks

In the preceding subsection we described atomic decompositions of functions and distributions. This powerful instrument is useful in many applications, for example in connection with mapping properties of pseudo-differential operators as briefly indicated in (1.74). According to Definitions 1.15, 1.21, 1.25 atoms are characterised in qualitative terms. But there are many problems where it would be desirable to have expansions of type (1.70) based on fixed, constructive, quantitatively determined building blocks which are independent of f. Here are a few of such problems.

- Isomorphic mappings between function spaces of type $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ on the one hand and sequence spaces of type b_{pq} and f_{pq} according to Definition 1.17 on the other hand.
- Corresponding problems for respective spaces $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$ on bounded domains Ω in $\mathbb{R}^n$ and the reduction of the study of entropy numbers of compact mappings between them to corresponding problems on the level of sequence spaces.
- Traces of spaces of type B^s_{pq} and type F^s_{pq} on $\mathbb{R}^n$ or in domains Ω on (irregular) compact sets Γ in $\mathbb{R}^n$ and the introduction of corresponding spaces of type $B^s_{pq}(\Gamma)$ in terms of constructive building blocks.

For these and other problems connected with fractal analysis we replaced atomic decompositions of type (1.70) by subatomic or quarkonial decompositions based on very simple elementary constructive building blocks which we called *quarks*. This theory started in [Triδ] with numerous applications especially to spectral theory of fractal elliptic operators. We returned to these problems in [Triϵ] in greater and more systematic detail. These considerations will be continued in the present book to some extent but now mixed with other building blocks such as *wavelet bases* and some types of *wavelet frames*, where the latter are near to quarkonial decompositions. This will be outlined and prepared in subsequent Sections 1.7, 1.8 and detailed in Chapter 3.

In Sections 2 and 3 in [Triϵ] we developed the theory of quarkonial decompositions for all spaces

$$B^s_{pq}(\mathbb{R}^n), \quad F^s_{pq}(\mathbb{R}^n) \qquad \text{with} \quad 0 < p \le \infty, \quad 0 < q \le \infty, \quad s \in \mathbb{R}, \tag{1.94}$$

($p < \infty$ for the F-spaces) and extended these considerations afterwards to corresponding spaces on (smooth) bounded domains Ω in $\mathbb{R}^n$, on Riemannian manifolds and, in particular, on arbitrary compact (fractal) sets Γ in $\mathbb{R}^n$ (occasionally under some restrictions for the spaces and their parameters). In Chapter 8 of the present book we use the theory of quarkonial decompositions for spaces of type B^s_{pq} on $\mathbb{R}^n$, and on some compact sets Γ in $\mathbb{R}^n$, to study corresponding spaces on quasi-metric spaces. We restrict ourselves now to a description of what is useful for these later purposes. This means on the one hand that we may assume that s in (1.94) is large, but on the other hand we need approximate lattices in $\mathbb{R}^n$ instead of the pure lattices $2^{-\nu}\mathbb{Z}^n$. We follow essentially the respective parts in Section 2 in [Triϵ].

Recall that

$$B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}, \qquad x \in \mathbb{R}^n, \quad r > 0, \tag{1.95}$$

is a ball centred at x and of radius r. Let

$$\{x^{\nu,m} : m \in \mathbb{Z}^n\} \subset \mathbb{R}^n \qquad \text{where} \quad \nu \in \mathbb{N}_0, \tag{1.96}$$

be a sequence of *approximate lattices* by which we mean that for some $c > 0$,

$$|x^{\nu,m_1} - x^{\nu,m_2}| \geq c\, 2^{-\nu}, \qquad \nu \in \mathbb{N}_0, \quad m_1 \neq m_2, \tag{1.97}$$

and

$$\mathbb{R}^n = \bigcup_{m\in\mathbb{Z}^n} B\left(x^{\nu,m}, 2^{-\nu-2}\right), \qquad \nu \in \mathbb{N}_0. \tag{1.98}$$

Definition 1.36. *Let* $\{\psi^{\nu,m} : m \in \mathbb{Z}^n\}$ *be subordinated resolutions of unity with respect to the above approximate lattices of* K *times differentiable functions where* $K \in \mathbb{N}$, *with*

$$\sum_{m\in\mathbb{Z}^n} \psi^{\nu,m}(x) = 1, \qquad x \in \mathbb{R}^n, \quad \nu \in \mathbb{N}_0, \tag{1.99}$$

$$\operatorname{supp} \psi^{\nu,m} \subset B\left(x^{\nu,m}, 2^{-\nu-1}\right), \qquad \nu \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n, \tag{1.100}$$

and for some $c' > 0$,

$$|D^\alpha \psi^{\nu,m}(x)| \leq c'\, 2^{\nu|\alpha|}, \qquad x \in \mathbb{R}^n, \quad \nu \in \mathbb{N}_0, \quad |\alpha| \leq K. \tag{1.101}$$

Let $s \in \mathbb{R}$, $0 < p \leq \infty$ *and* $\beta \in \mathbb{N}_0^n$ *(multi-index). Then*

$$(\beta\text{-qu})_{\nu m}(x) = 2^{-\nu(s-n/p)}\, 2^{\nu|\beta|}\, (x - x^{\nu,m})^\beta\, \psi^{\nu,m}(x), \quad x \in \mathbb{R}^n, \tag{1.102}$$

$\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, *is called an* $(s,p)_K$-β-*quark (related to the corresponding ball in* (1.98)).

Remark 1.37. Recall that x^β are the monomials as explained in Remark 1.16. One may assume that $x^{\nu,m} \sim 2^{-\nu}m$. Furthermore, for some $c > 0$, one has uniformly for all $x \in \mathbb{R}^n$, $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$ and $\beta \in \mathbb{N}_0^n$,

$$|D^\alpha (\beta\text{-qu})_{\nu m}(x)| \leq c\, 2^{-\nu(s-n/p)+\nu|\alpha|}, \tag{1.103}$$

where $|\alpha| \leq K$. Hence, ignoring the universal constant c, the $(s,p)_K$-β-quarks are $(s,p)_{K,0}$-atoms according to Definition 1.15. The above definition coincides essentially with the discussion in [Triϵ], Section 2.14, pp. 25–26.

Remark 1.38. We need the above version based on approximate lattices for our later purposes. On the other hand, the case of pure lattices deserves special attention, hence

$$x^{\nu,m} = 2^{-J-\nu}m, \qquad \nu \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n, \tag{1.104}$$

where $J \in \mathbb{N}$ is a suitable natural number. Then (1.102) can be simplified as follows. Let ψ be a, say, non-negative C^∞ function in $\mathbb{R}^n$ with

$$\operatorname{supp} \psi \subset \{y : 2|y| < 1\} \tag{1.105}$$

such that for some $J \in \mathbb{N}$,

$$\sum_{m\in\mathbb{Z}^n} \psi(x - 2^{-J}m) = 1, \qquad x \in \mathbb{R}^n. \tag{1.106}$$

Let $\psi^\beta(x) = x^\beta \psi(x)$ with $\beta \in \mathbb{N}_0^n$. Then

$$(\beta\text{-qu})_{\nu m}(x) = 2^{-\nu(s-n/p)}\, \psi^\beta(2^\nu x - 2^{-J} m), \qquad x \in \mathbb{R}^n, \tag{1.107}$$

are $(s,p)_K$-β-quarks for any $K \in \mathbb{N}$. This coincides essentially with [Triϵ], Definition 2.4.

We complement Definition 1.17 as follows. Let $0 < p \le \infty$, $0 < q \le \infty$, $\varrho \in \mathbb{R}$ and

$$\lambda = \{\lambda^\beta \,:\, \beta \in \mathbb{N}_0^n\} \quad \text{with} \quad \lambda^\beta = \{\lambda^\beta_{\nu m} \in \mathbb{C} \,:\, \nu \in \mathbb{N}_0,\ m \in \mathbb{Z}^n\}.$$

Then we put

$$\|\lambda \,|b_{pq}\|_\varrho = \sup_{\beta \in \mathbb{N}_0^n} 2^{\varrho|\beta|}\, \|\lambda^\beta \,|b_{pq}\| \tag{1.108}$$

and

$$\|\lambda \,|f_{pq}\|_\varrho = \sup_{\beta \in \mathbb{N}_0^n} 2^{\varrho|\beta|}\, \|\lambda^\beta \,|f_{pq}\|, \tag{1.109}$$

where $\|\cdot\,|b_{pq}\|$ and $\|\cdot\,|f_{pq}\|$ have been introduced in (1.64) and (1.65), respectively. The numbers σ_p and σ_{pq} have the same meaning as in (1.68). Let $\bar{p} = \max(1,p)$ where $0 < p \le \infty$ and let $L_\infty(\mathbb{R}^n, w)$ as in Remark 1.30.

Theorem 1.39.

(i) *Let $0 < p \le \infty$, $0 < q \le \infty$, $s > \sigma_p$ and let*

$$\{(\beta\text{-qu})_{\nu m} \,:\, \nu \in \mathbb{N}_0,\ m \in \mathbb{Z}^n\}$$

be a set of $(s,p)_K$-β-quarks according to Definition 1.36 with $K > s$. Let $\varrho \ge 0$. Then $B^s_{pq}(\mathbb{R}^n)$ is the collection of all $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n) \cap S'(\mathbb{R}^n)$ which can be represented as

$$f = \sum_{\beta \in \mathbb{N}_0^n} \sum_{\nu=0}^\infty \sum_{m \in \mathbb{Z}^n} \lambda^\beta_{\nu m}\, (\beta\text{-qu})_{\nu m}(x) \quad \textit{with} \quad \|\lambda \,|b_{pq}\|_\varrho < \infty. \tag{1.110}$$

The series on the right-hand side of (1.110) converges absolutely in $L_{\bar{p}}(\mathbb{R}^n)$ if $p < \infty$ and in $L_\infty(\mathbb{R}^n, w)$ where $w(x) = (1 + |x|^2)^{\sigma/2}$ with $\sigma < 0$ if $p = \infty$. Furthermore,

$$\|f \,|B^s_{pq}(\mathbb{R}^n)\| \sim \inf \|\lambda \,|b_{pq}\|_\varrho \tag{1.111}$$

are equivalent quasi-norms where the infimum is taken over all admissible representations (1.110).

(ii) *Let $0 < p < \infty$, $0 < q \le \infty$, $s > \sigma_{pq}$ and let $(\beta\text{-qu})_{\nu m}$ be the same $(s,p)_K$-β-quarks as in part (i). Let $\varrho \ge 0$. Then $F^s_{pq}(\mathbb{R}^n)$ is the collection of all $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n) \cap S'(\mathbb{R}^n)$ which can be represented as*

$$f = \sum_{\beta \in \mathbb{N}_0^n} \sum_{\nu=0}^\infty \sum_{m \in \mathbb{Z}^n} \lambda^\beta_{\nu m} (\beta\text{-qu})_{\nu m}(x) \quad \textit{with} \quad \|\lambda \,|f_{pq}\|_\varrho < \infty. \tag{1.112}$$

The series on the right-hand side of (1.112) *converges absolutely in* $L_{\bar{p}}(\mathbb{R}^n)$. *Furthermore,*

$$\| f \,| F^s_{pq}(\mathbb{R}^n)\| \sim \inf \|\lambda \,| f_{pq}\|_\varrho \tag{1.113}$$

are equivalent quasi-norms where the infimum is taken over all admissible representations (1.112).

Remark 1.40. In case of pure lattices as described in Remark 1.38 where the respective $(s,p)_K$-β-quarks can be generated from one function ψ with (1.105), (1.106), the above theorem coincides essentially with the combination of Definition 2.6 and Theorem 2.9 in [Triϵ], pp. 15/16. The extension to approximate lattices may be found in [Triϵ], Section 2.14, pp. 25–26.

Remark 1.41. We add a discussion about some technicalities in connection with the above theorem. One has by Definition 1.36,

$$|D^\alpha(\beta\text{-qu})_{\nu m}(x)| \le c\, 2^{-\nu(s-n/p)+\nu|\alpha|}\, 2^{-\varkappa|\beta|}, \qquad |\alpha| \le K, \tag{1.114}$$

for any $\varkappa$ with $0 < \varkappa < 1$ and some $c > 0$ which is independent of ν, m, β, and x. In particular, $(\beta\text{-qu})_{\nu m}$ are $(s,p)_{K,0}$ atoms according to Definition 1.15 multiplied with the factors $c\, 2^{-\varkappa|\beta|}$. Now it follows by Theorem 1.19 that any $f \in S'(\mathbb{R}^n)$ which can be represented by (1.110) belongs to $B^s_{pq}(\mathbb{R}^n)$ and

$$\| f \,| B^s_{pq}(\mathbb{R}^n)\| \le c\, \|\lambda \,| b_{pq}\|_\varrho. \tag{1.115}$$

Furthermore one gets by elementary arguments that the right-hand side of (1.110) converges absolutely in any $L_r(\mathbb{R}^n)$ with $r \ge p$ and $s - \frac{n}{p} + \frac{n}{r} > 0$. Similarly for the spaces $F^s_{pq}(\mathbb{R}^n)$. A more detailed respective discussion may be found in [Triϵ], Section 2.7, p. 14. In particular, any $f \in S'(\mathbb{R}^n)$ represented by (1.110) belongs to some $L_r(\mathbb{R}^n)$ with $1 < r \le \infty$, and hence to $L^{\mathrm{loc}}_1(\mathbb{R}^n)$. In other words, it is the main point of the above theorem to prove that any given $f \in B^s_{pq}(\mathbb{R}^n)$ or $f \in F^s_{pq}(\mathbb{R}^n)$ can be represented by (1.110) or (1.112), respectively. Furthermore one would be interested to convert these representations into *frames*, that means representations

$$f = \sum_{\beta \in \mathbb{N}^n_0} \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda^\beta_{\nu m}(f)\, (\beta\text{-qu})_{\nu m}(x), \tag{1.116}$$

where $\lambda^\beta_{\nu m}(f)$ depend linearly on f and

$$\| f \,| B^s_{pq}(\mathbb{R}^n)\| \sim \|\lambda(f) \,| b_{pq}\|_\varrho, \qquad \| f \,| F^s_{pq}(\mathbb{R}^n)\| \sim \|\lambda(f) \,| f_{pq}\|_\varrho, \tag{1.117}$$

respectively. This is possible.

Corollary 1.42. *For given* s *and* p *the system*

$$\{(\beta\text{-qu})_{\nu m} : \quad \nu \in \mathbb{N}_0,\ m \in \mathbb{Z}^n,\ \beta \in \mathbb{N}^n_0\} \tag{1.118}$$

is a frame in the spaces covered by Theorem 1.39. *In particular there are functions*

$$\Psi^{\beta}_{\nu m} \in S(\mathbb{R}^n), \qquad \nu \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n, \quad \beta \in \mathbb{N}_0^n, \tag{1.119}$$

such that

$$\lambda^{\beta}_{\nu m}(f) = \big(f, \Psi^{\beta}_{\nu m}\big), \quad \textit{dual pairing in} \quad S(\mathbb{R}^n)\text{-}S'(\mathbb{R}^n), \tag{1.120}$$

are optimal coefficients according to (1.116), (1.117).

Remark 1.43. We refer to [Triϵ], Sections 2.12, 2.16, pp. 23/24, 27. We return later on to problems of this type in Section 1.8 and in detail in Chapter 3.

Remark 1.44. We add some comments. First we remark that for given $N \in \mathbb{N}_0$ the subsystem

$$\{(\beta\text{-qu})_{\nu m} : \quad \nu - N \in \mathbb{N}_0, \ m \in \mathbb{Z}^n, \ \beta \in \mathbb{N}_0^n\} \tag{1.121}$$

of (1.118) is again a frame. This follows from (1.97), (1.98) and Definition 1.36, replacing there $\psi^{\nu+N,m}$ and $x^{\nu+N,m}$ by $\psi^{\nu,m}$ and $x^{\nu,m}$, respectively, $\nu \in \mathbb{N}_0$. For these re-named β-quarks $(\beta\text{-qu})_{\nu m}$ one gets in (1.114) the extra factor $2^{-N|\beta|}$ on the right-hand side. This explains the (at first glance somewhat curious) claim in Theorem 1.39 that $\varrho \geq 0$ can be chosen arbitrarily large (at the expense of equivalence constants which depend on ϱ). A more detailed discussion may be found in [Triϵ], Section 2.13, p. 24. Secondly we formulate one of the main tools of the proof of the above Theorem 1.39 as given in [Triϵ], pp. 15–21, which will also be of some use later on. Again let b_{pq} and f_{pq} be the sequence spaces according to Definition 1.17.

Proposition 1.45. *Let* $\{\varphi_\nu\}_{\nu=0}^{\infty}$ *be the dyadic resolution of unity according to* (1.14)–(1.16), *and let* $s \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q \leq \infty$. *Let* $J \in \mathbb{N}_0$ *and for* $f \in S'(\mathbb{R}^n)$,

$$\lambda_{\nu m} = 2^{\nu(s-n/p)}\, (\varphi_\nu\, \widehat{f})^{\vee}(2^{-\nu-J}\, m), \qquad \nu \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n. \tag{1.122}$$

There is a natural number J_0 *such that for any* $J \in \mathbb{N}$ *with* $J \geq J_0$,

$$\|f\,|B^s_{pq}(\mathbb{R}^n)\| \sim \|\lambda\,|b_{pq}\| \tag{1.123}$$

(*equivalent quasi-norms*) *and with* $p < \infty$,

$$\|f\,|F^s_{pq}(\mathbb{R}^n)\| \sim \|\lambda\,|f_{pq}\| \tag{1.124}$$

(*equivalent quasi-norms*) .

Remark 1.46. This proposition follows from the characterisations of $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ in terms of maximal functions according to (1.28) and (1.30), respectively. One can also replace the pure lattice $\{2^{-\nu-J}m\}$ by suitable approximate lattices of type (1.96)–(1.98). But we do not go into detail. The above stated case of pure lattices may be found in the literature. The B-case (including the indicated extension to approximate lattices) follows essentially from [Triβ], Sections 1.3.3, 1.3.4, pp. 19/20 and had been used in [Triδ], Section 14.15, pp. 101–104, for the proof of Theorem 1.39(i). Explicit formulations both for the B-case and the F-case may be found in [FrJ90], [FJW91], and [Tor91]. The anisotropic extensions go back to [Din95] and [Far00].

Remark 1.47. In this subsection we followed essentially [Triϵ], Section 2. An extension of the theory of quarkonial decompositions to all spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ with $s \in \mathbb{R}$ may be found in [Triϵ], Section 3. Then one needs moment conditions for the building blocks which can be incorporated in (1.107) generalising ψ^β by $(-\Delta)^L \psi^\beta$ where $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ and $L \in \mathbb{N}_0$. Another possibility comes from the interpretation of $\{(\beta\text{-qu})_{\nu m}\}$ in (1.118) and $\{\Psi^\beta_{\nu m}\}$ in (1.119) as dual frames. Using duality one can prove frame representations for the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ with $s < 0$, $1 \le p \le \infty$, $1 \le q \le \infty$, where the above frames change their roles. We refer to [Yang05]. We return to this procedure in a somewhat different context in Section 3.2.3. In [Triϵ] we extended this theory step by step to domains in $\mathbb{R}^n$, manifolds and, in particular, to fractal compact sets in $\mathbb{R}^n$. We return to some of these points later on.

Remark 1.48. In the above Sections 1.4–1.6 we developed the theory of local means, based on Theorem 1.7 in Section 1.3, atomic decompositions and quarkonial representations for the (inhomogeneous) isotropic spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ in $\mathbb{R}^n$. These tools and assertions have been generalised to several other types of spaces on $\mathbb{R}^n$. Corresponding counterparts for anisotropic spaces $B^{s,a}_{pq}(\mathbb{R}^n)$ and $F^{s,a}_{pq}(\mathbb{R}^n)$ may be found in [Din95], [Far00] and more recently in [Kyr04], [BoH05] and [Bow05]. Here $a = (a_1, \dots, a_n)$ with $a_j > 0$ and $\sum_{j=1}^n a_j = n$ is an anisotropy. We return to anisotropic spaces in Chapter 5 where one finds also the necessary definitions. Quite recently the above indicated tools and assertions for isotropic spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ have been extended to spaces with dominating mixed smoothness. We refer to [Baz03, Baz05] and [Vyb03, Vyb05a, Vyb05b, Vyb06]. Basic assertions for spaces with dominating mixed smoothness may be found in [ST87] and in the recent paper [ScS04]. In the last few years growing attention has been paid to (inhomogeneous) isotropic spaces of B-type and F-type of generalised smoothness. Replacing 2^{js} in (1.17) and (1.19) by $2^{js}\Psi(2^{-j})$ results in spaces $B^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ and $F^{(s,\Psi)}_{pq}(\mathbb{R}^n)$, respectively. Here $\Psi(2^{-j})$ is a perturbation of 2^{js}, where $\Psi(2^{-j}) = (1+j)^b$ with $b \in \mathbb{R}$ is a typical example. The corresponding theory, including local means, atomic and quarkonial decompositions may be found in [Mou99], [Mou01a], [Mou01b]. More general spaces, replacing 2^{js} in (1.17) and (1.19) by some admissible sequences σ_j and modifying in addition the underlying dyadic structure of the resolution of unity in (1.16) have been considered in [FaL01] and [FaL04], including counterparts of the characterisation in terms of Theorem 1.7 (which is the main result of these papers), local means, and atomic decompositions. We refer also to [Bri01], [Bri04].

1.7 Wavelet bases

1.7.1 Multiresolution analysis

So far we dealt in Sections 1.5 and 1.6 with atomic and quarkonial representations and related frames according to Corollary 1.42. We return to the latter subject in Section 1.8. In the present subsection we give a brief description of some well-known assertions about wavelet bases in $L_2(\mathbb{R}^n)$ and we indicate how this theory will be extended later on in Chapter 3 to all spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$. A further generalisation to anisotropic B-spaces and F-spaces will be discussed in Section 5. Then we need some details of what is called *multiresolution analysis*. This is the main reason why we give here a description of some basic ingredients of multiresolution analysis, restricting us to the bare minimum needed later on. As usual in this context we look first at the one-dimensional case.

Definition 1.49. *An (inhomogeneous) multiresolution analysis is a sequence* $\{V_j : j \in \mathbb{N}_0\}$ *of subspaces of* $L_2(\mathbb{R})$ *such that*

(i) $V_0 \subset V_1 \subset \cdots \subset V_j \subset V_{j+1} \subset \cdots ;\quad \text{span} \bigcup_{j=0}^{\infty} V_j = L_2(\mathbb{R})$,

(ii) $f \in V_0$ *if, and only if,* $f(x-m) \in V_0$ *for any* $m \in \mathbb{Z}$,

(iii) $f \in V_j$ *if, and only if,* $f(2^{-j}x) \in V_0$ *for all* $j \in \mathbb{N}$,

(iv) *there is a function* $\psi_F \in V_0$ *such that*
$$\{\psi_F(x-m) : m \in \mathbb{Z}\} \text{ is an orthonormal basis in } V_0. \tag{1.125}$$

Remark 1.50. The function ψ_F is called a *scaling function* or *father wavelet*, where the F comes from. By (iii) and (1.125) it follows that

$$\left\{ 2^{j/2}\, \psi_F(2^j x - m) : m \in \mathbb{Z} \right\}, \qquad j \in \mathbb{N}_0, \tag{1.126}$$

is an orthonormal basis in V_j. Let

$$W_j = V_{j+1} \ominus V_j; \qquad j \in \mathbb{N}_0, \quad \text{(orthogonal complement)}. \tag{1.127}$$

Then (i) can be reformulated as

$$L_2(\mathbb{R}) = V_0 \oplus \bigoplus_{j=0}^{\infty} W_j \qquad \text{(orthogonal decomposition)}. \tag{1.128}$$

It is one of the main assertions of multiresolution analysis to prove that there are functions $\psi_M \in L_2(\mathbb{R})$, called an (associated) *wavelet* or *mother wavelet*, such that

$$\{\psi_M(x-m) : m \in \mathbb{Z}\} \quad \text{is an orthonormal basis in } W_0, \tag{1.129}$$

and to construct them starting from ψ_F.

Proposition 1.51. *Let ψ_F and ψ_M be a scaling function and a wavelet according to Definition* 1.49 *and Remark* 1.50. *Then*

$$\psi^j_m(x) = \begin{cases} \psi_F(x-m) & \text{if } j=0,\ m\in\mathbb{Z}, \\ 2^{\frac{j-1}{2}}\,\psi_M(2^{j-1}x-m) & \text{if } j\in\mathbb{N},\ m\in\mathbb{Z}, \end{cases} \tag{1.130}$$

is an orthonormal basis in $L_2(\mathbb{R})$.

Proof. By (1.127), (1.129),

$$\left\{2^{j/2}\,\psi_M(2^jx-m)\ :\ m\in\mathbb{Z}\right\} \quad \text{is an orthonormal basis in } W_j. \tag{1.131}$$

Then (1.128) proves the proposition. □

Remark 1.52. The extension of the above considerations from one dimension to n dimensions follows by the standard procedures of tensor products. Let

$$\Psi_m(x) = \prod_{r=1}^n \psi_F(x_r-m_r), \qquad m\in\mathbb{Z}^n, \quad x\in\mathbb{R}^n, \tag{1.132}$$

and

$$V^n_j = \operatorname{span}\ \left\{2^{j\frac{n}{2}}\,\Psi_m(2^jx),\ m\in\mathbb{Z}^n\right\}. \tag{1.133}$$

In particular, V^n_j is spanned by the products of the bases in (1.126) of the one-dimensional version of V_j. Then one gets the n-dimensional version of Definition 1.49(i). Let

$$G = (G_1,\ldots,G_n)\in\{F,M\}^{n*}, \tag{1.134}$$

where G_r is either F or M and where * indicates that at least one of the components of G must be an M. Hence the cardinal number of $\{F,M\}^{n*}$ is 2^n-1. Let for $x\in\mathbb{R}^n$,

$$\Psi^G_m(x) = \prod_{r=1}^n \psi_{G_r}(x_r-m_r), \qquad G\in\{F,M\}^{n*}, \quad m\in\mathbb{Z}^n. \tag{1.135}$$

Let $G^0=\{F\}^n=\{(F,\ldots,F)\}$ and $G^j=\{F,M\}^{n*}$ if $j\in\mathbb{N}$.

Proposition 1.53. *Let ψ_F and ψ_M be an (one-dimensional) scaling function and a wavelet as in Proposition* 1.51. *Let Ψ_m and Ψ^G_m be given by* (1.132) *and* (1.135). *Then*

$$\Psi^{j,G}_m(x) = \begin{cases} \Psi_m(x) & \text{if } j=0,\ G\in G^0,\ m\in\mathbb{Z}^n, \\ 2^{\frac{j-1}{2}n}\,\Psi^G_m(2^{j-1}x) & \text{if } j\in\mathbb{N},\ G\in G^j,\ m\in\mathbb{Z}^n, \end{cases} \tag{1.136}$$

is an orthonormal basis in $L_2(\mathbb{R}^n)$.

Proof. By (1.127) it follows that V_{j+1}^n with $j \in \mathbb{N}_0$ can be spanned either by (1.133) with $j+1$ in place of j or by

$$2^{\frac{j}{2}\,n}\,\Psi_m(2^j x), \qquad 2^{\frac{j}{2}\,n}\,\Psi_m^G(2^j x) \tag{1.137}$$

according to (1.132), (1.135). The first functions span also V_j^n, and, hence, the latter functions span

$$W_j^n = V_{j+1}^n \ominus V_j^n, \qquad j \in \mathbb{N}_0. \tag{1.138}$$

This is the counterpart of (1.127). Now the above proposition follows by the same arguments as in Remark 1.50 and in the proof of Proposition 1.51. □

1.7.2 Haar wavelets

We illustrate the above theory by the very classical example of *Haar functions.*

Proposition 1.54. *Let*

$$\psi_M(x) = \begin{cases} 1 & \text{if } \ 0 \le x < 1/2, \\ -1 & \text{if } \ 1/2 \le x \le 1, \\ 0 & \text{if } x \notin [0,1], \end{cases} \tag{1.139}$$

and $\psi_F(x) = |\psi_M(x)|$ *(the characteristic function of the interval* $[0,1]$*). Then* ψ_F *is a scaling function and* ψ_M *is an associated wavelet according to Definition* 1.49 *and Remark* 1.50*, respectively.*

Proof. Let V_j be the subspace of $L_2(\mathbb{R})$ consisting of all step functions which are constant on the interval $[2^{-j}m, 2^{-j}(m+1)]$ with $m \in \mathbb{Z}$. Then it follows that ψ_F is a corresponding scaling function according to Definition 1.49 and that ψ_M is an associated wavelet as introduced in Remark 1.50. □

Remark 1.55. In other words, identifying ψ_F with the characteristic function of the unit interval $[0,1]$ and ψ_M with (1.139), then $\{\psi_m^j\}$ according to Proposition 1.51 is an orthonormal basis on $L_2(\mathbb{R})$. This is the classical observation of A. Haar, [Haar10]. Using Proposition 1.53 one can extend this assertion to $L_2(\mathbb{R}^n)$. The question arises whether the respective n-dimensional Haar system remains a basis in other spaces. First we recall some basic notation.

Definition 1.56. *Let* B *be a complex quasi-Banach space.*

(i) *Then* $\{b_j\}_{j=1}^\infty \subset B$ *is called a Schauder basis (or simply basis) if any* $b \in B$ *can be uniquely represented as*

$$b = \sum_{j=1}^\infty \lambda_j\, b_j, \qquad \lambda_j \in \mathbb{C}, \tag{1.140}$$

(convergence in B*).*

(ii) *A basis* $\{b_j\}_{j=1}^\infty$ *is called an unconditional basis if for any rearrangement* σ *of* $\mathbb{N}$ *(one-to-one map of* $\mathbb{N}$ *onto itself)* $\{b_{\sigma(j)}\}_{j=1}^\infty$ *is again a basis and*

$$b = \sum_{j=1}^{\infty} \lambda_{\sigma(j)}\, b_{\sigma(j)} \tag{1.141}$$

for any $b \in B$ *with* (1.140).

Remark 1.57. In case of Banach spaces there is a huge literature about unconditional convergence of series and related properties of bases and unconditional bases. A discussion in connection with wavelets and functions, especially in the spaces $L_p(\mathbb{R}^n)$ with $1 < p < \infty$ may be found in [Woj97], Section 7. For our purposes the above definition is sufficient since the spaces of type B^s_{pq} and F^s_{pq} are isomorphic to sequence spaces of type b_{pq} and f_{pq} as introduced in Definition 1.17 or subspaces of them. In particular there are no problems with rearrangement and the uniform boundedness of the operators P_J given by the partial sums of (1.140),

$$P_J b = \sum_{j=1}^{J} \lambda_j(b)\, b_j, \qquad J \in \mathbb{N}. \tag{1.142}$$

Theorem 1.58. *Let*

$$H_n = \bigl\{ \Psi^{j,G}_m \,:\, j \in \mathbb{N}_0,\ G \in G^j,\ m \in \mathbb{Z}^n \bigr\} \tag{1.143}$$

be the Haar system in $\mathbb{R}^n$ *according to* (1.136), *generated by* (1.132), (1.135) *and the one-dimensional functions* ψ_M, ψ_F *as introduced in Proposition* 1.54.

(i) *Then* H_n *is an unconditional basis in* $L_p(\mathbb{R}^n)$ *with* $1 < p < \infty$.

(ii) *Then* H_n *is a basis in* $B^s_{pq}(\mathbb{R}^n)$ *if*

$$\begin{cases} \text{either} & 1 < p < \infty, \quad \frac{1}{p} - 1 < s < \frac{1}{p}, \quad 0 < q < \infty, \\ \text{or} & \frac{n}{n+1} < p \le 1, \quad n(\frac{1}{p} - 1) < s < 1, \quad 0 < q < \infty. \end{cases} \tag{1.144}$$

(iii) *Then* H_n *is not a basis in* $B^s_{pq}(\mathbb{R}^n)$ *if*

$$\begin{cases} \text{either} & 1 < p < \infty, \quad s \notin [\frac{1}{p} - 1, \frac{1}{p}], \quad 0 < q < \infty, \\ \text{or} & \frac{n}{n+1} \le p \le 1, \quad s \notin [n(\frac{1}{p} - 1), 1], \quad 0 < q < \infty, \\ \text{or} & 0 < p < \frac{n}{n+1}, \quad s \in \mathbb{R}, \quad 0 < q < \infty. \end{cases} \tag{1.145}$$

Remark 1.59. Part (i) is an outstanding result of real analysis. A proof may be found in [Woj97], Section 8.3. It goes back to J. Marcinkiewicz, [Mar37] (one-dimensional case), who used in turn [Pal32]. We proved in [Tri73] and [Triα], Section 4.9.4, that there is a system of Haar functions H'_n which is a basis in

$$B^s_{pq}(Q) \qquad \text{if} \quad 1 < p < \infty, \quad \frac{1}{p} - 1 < s < \frac{1}{p}, \quad 1 \le q < \infty, \tag{1.146}$$

where Q is a cube in $\mathbb{R}^n$. But it comes out that the Haar functions of H_n at level j are finite linear combinations of corresponding Haar functions of H'_n at the same level j and vice versa. Hence the two systems can be transformed into each other, preserving its role as a basis or not. An extension of this assertion for the Haar system H'_n to the parameters in (1.144) and a proof that H'_n is not a basis for the parameters (1.145) may be found in [Tri78] , both for $B^s_{pq}(Q)$ and $B^s_{pq}(\mathbb{R}^n)$. This can be transferred to the system H_n. Although not explicitly stated (and not immediately covered by the proof) one can expect that H_n in $B^s_{pq}(\mathbb{R}^n)$ with (1.144) is an unconditional basis. The assertions (ii) and (iii) clarify under which circumstances H_n is a basis in $B^s_{pq}(\mathbb{R}^n)$ with exception of a few limiting cases. The middle line of (1.145) looks a little bit surprising at first glance. On the one hand it is well known that the set of all finite linear combinations of characteristic functions of rectangles with sides parallel to the axes is a subset of $B^s_{pp}(\mathbb{R}^n)$ with

$$\frac{n-1}{n} < p < 1 \quad \text{and} \quad \max(1, n(\frac{1}{p} - 1)) < s < \frac{1}{p}, \tag{1.147}$$

but on the other hand this set is not dense in these spaces and hence H_n cannot be a basis, [Tri78], Corollary 2, p. 338.

Remark 1.60. Haar functions on $\mathbb{R}$ or on the interval $I = (0, 1)$ can be used as a starting point to construct spline functions of higher order and related unconditional bases in spaces of type $B^s_{pq}(I)$ and $F^s_{pq}(I)$ and also in some generalisations to n dimensions. A brief description of some related results may be found in [Triβ], Section 2.12.3, pp. 184–187, based on [Tri81]. This theory was mainly developed in the 1970s and early 1980s by Z. Ciesielski, J. Domsta, T. Figiel, R. Ropela, P. Oswald and others. Detailed references may be found in [Triβ], Section 2.12.2. A more recent account about spline functions and spline wavelets was given in [Woj97], Section 3.3. But we will not follow this line. Our intentions are different and will be outlined in what follows.

1.7.3 Smooth wavelets

Again let $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ be the spaces introduced in Definition 1.2 and let $\mathcal{C}^\sigma(\mathbb{R}^n)$ with $\sigma > 0$ be the Hölder-Zygmund spaces according to (1.10)–(1.12). By the well-known embedding theorem

$$A^s_{pq}(\mathbb{R}^n) \hookrightarrow \mathcal{C}^\sigma(\mathbb{R}^n) \qquad \text{if} \quad 0 < \sigma < s - n/p, \tag{1.148}$$

where either $A = B$ or $A = F$, it follows that any element of these spaces has bounded classical derivatives up to order $k \in \mathbb{N}_0$ if $s > k + n/p$. Haar functions are step functions and so they are not continuous. Hence one needs smoother (one-dimensional) scaling functions and associated wavelets according to Proposition 1.51 if one wishes to extend the respective bases in Proposition 1.53 from $L_2(\mathbb{R}^n)$ to these spaces $A^s_{pq}(\mathbb{R}^n)$. This is the point where the theory of smooth wavelets comes in as developed since the middle of the 1980s.

Theorem 1.61.

(i) *There is a real scaling function* $\psi_F \in S(\mathbb{R})$ *and a real associated wavelet* $\psi_M \in S(\mathbb{R})$ *such that their Fourier transforms have compact supports,* $\widehat{\psi_F}(0) = (2\pi)^{-1/2}$,

$$\operatorname{supp} \widehat{\psi_M} \subset \left[-\frac{8}{3}\pi, -\frac{2}{3}\pi\right] \cup \left[\frac{2}{3}\pi, \frac{8}{3}\pi\right]. \qquad (1.149)$$

(ii) *For any* $k \in \mathbb{N}$ *there is a real compactly supported scaling function* $\psi_F \in C^k(\mathbb{R})$ *and a real compactly supported associated wavelet* $\psi_M \in C^k(\mathbb{R})$ *such that* $\widehat{\psi_F}(0) = (2\pi)^{-1/2}$ *and*

$$\int_{\mathbb{R}} x^l\, \psi_M(x)\, \mathrm{d}x = 0 \qquad \textit{for} \quad l = 0, \dots, k. \qquad (1.150)$$

Remark 1.62. These assertions are well-known cornerstones of the recent theory of wavelets. Part (i) goes back to Y. Meyer, [Mey86], and has been presented in detail in [Mey92] (French edition 1990). The nowadays usual approach via multiresolution analysis is due to S. Mallat, [Mal89]. Part (ii) goes back to I. Daubechies, [Dau88], and has been presented in detail in [Dau92]. The standard references for this theory are [Mey92], [Dau92] and, more recently, [Woj97]. We add a few technical explanations. Of course, $S(\mathbb{R})$ is the Schwartz space on $\mathbb{R}$. The Fourier transform on $\mathbb{R}$ is normalised by

$$\widehat{f}(\xi) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-ix\xi}\, f(x)\, \mathrm{d}x, \qquad \xi \in \mathbb{R}. \qquad (1.151)$$

In particular,

$$\int_{\mathbb{R}} \psi_F(x)\, \mathrm{d}x = 1 \qquad (1.152)$$

in both parts of the theorem. By (1.149) it follows in case of the Meyer wavelets that

$$\int_{\mathbb{R}} x^l\, \psi_M(x)\, \mathrm{d}x = 0 \qquad \text{for all} \quad l \in \mathbb{N}_0, \qquad (1.153)$$

hence one has vanishing moments of all orders, whereas for given $k \in \mathbb{N}$ one has for the corresponding Daubechies wavelets only vanishing moments up to order k. As for the details concerning the Meyer wavelets from part (i) we refer to [Woj97], p. 49 (complemented by p. 71) and p. 51, Theorem 3.4. We return to the (one-dimensional) Meyer wavelets in Section 3.1.5 making clear that they fit perfectly in the Fourier analytical approach to all spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$, especially in the version of Theorem 1.7. In case of the Daubechies wavelets we refer to [Mey92], p. 96, Theorem 3, and what follows at the end of this page concerning that ψ_F and ψ_M are real, as well as p. 108 where the cancellations are stated explicitly.

The normalising assertion (1.152) is covered by the general theory, [Woj97], p. 31. Obviously, the Meyer wavelets are analytic functions, whereas the Fourier transforms of the Daubechies wavelets are analytic.

Remark 1.63. Armed with the Meyer wavelets and the Daubechies wavelets one gets by Proposition 1.51 orthonormal bases in $L_2(\mathbb{R})$ and by Proposition 1.53 respective orthonormal bases in $L_2(\mathbb{R}^n)$. Quite obviously one may ask whether these systems remain (unconditional) bases in other spaces extending corresponding assertions for the Haar basis as described in Theorem 1.58. Recall that $H^s_p(\mathbb{R}^n)$ are the Sobolev spaces according to (1.7). It is well known that for given $k \in \mathbb{N}$ the Daubechies system is an unconditional basis in $L_p(\mathbb{R}^n)$ with $1 < p < \infty$, more generally, in

$$H^s_p(\mathbb{R}^n) \qquad \text{with} \quad 1 < p < \infty, \quad |s| < k, \tag{1.154}$$

and in the Besov spaces,

$$B^s_{pq}(\mathbb{R}^n) \qquad \text{with} \quad 1 \le p < \infty, \quad 1 \le q < \infty, \quad |s| < k. \tag{1.155}$$

We refer for details to [Mey92], Chapter 6. But corresponding assertions may also be found in [Dau92] and [Woj97]. In these books wavelets both of Meyer type and Daubechies type have also been considered in other types of spaces, including $L_1(\mathbb{R}^n)$, Hardy spaces

$$h_p(\mathbb{R}^n) = F^0_{p,2}(\mathbb{R}^n), \qquad 0 < p \le 1,$$

Hölder-Zygmund spaces $\mathcal{C}^s(\mathbb{R}^n)$ according to (1.10), spaces of BMO-type and a few other types of spaces. In Chapter 3 we return to this subject in detail. Then we start from the Daubechies wavelets according to Theorem 1.61(ii), extended to $\mathbb{R}^n$ according to Proposition 1.53, and consider these systems in all spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ covered by Definition 1.2. This approach will be based on the nature of the Daubechies wavelets which may serve simultaneously as kernels of local means and as atoms according to Sections 1.4 and 1.5, respectively. This will be complemented by corresponding assertions for the Meyer wavelets. Details will be given later on, but it seems to be reasonable to formulate some assertions in this introductory survey restricting ourselves here to the (more complicated) Daubechies wavelets.

First we adapt the sequence spaces in Definition 1.17 to the index set in connection with Proposition 1.53. Let G^j with $j \in \mathbb{N}_0$ be as there and let

$$\lambda = \left\{ \lambda^{j,G}_m \in \mathbb{C} \,:\, j \in \mathbb{N}_0,\ G \in G^j,\ m \in \mathbb{Z}^n \right\}. \tag{1.156}$$

Let

$$0 < p \le \infty, \qquad 0 < q \le \infty, \qquad s \in \mathbb{R}. \tag{1.157}$$

Let χ_{jm} be the characteristic function of the cube Q_{jm} introduced at the beginning of Section 1.5.1. Then

$$\|\lambda \,|b^s_{pq}\| = \left(\sum_{j=0}^{\infty} 2^{j(s-\frac{n}{p})q} \sum_{G\in G^j} \left(\sum_{m\in\mathbb{Z}^n} |\lambda^{j,G}_m|^p \right)^{q/p} \right)^{1/q} \tag{1.158}$$

and

$$\|\lambda \,|f^s_{pq}\| = \left\| \left(\sum_{j,G,m} 2^{jsq} \, |\lambda^{j,G}_m \, \chi_{jm}(\cdot)|^q \right)^{1/q} |L_p(\mathbb{R}^n) \right\| \tag{1.159}$$

(with the usual modifications if $p = \infty$ and/or $q = \infty$). Obviously, the index set $\{j, G, m\}$ in (1.159) is the same as in (1.158). Furthermore, b^s_{pq} and f^s_{pq} are the sequence spaces quasi-normed by (1.158) and (1.159), respectively. For brevity we write a^s_{pq} where either $a = b$ or $a = f$ and correspondingly $A^s_{pq}(\mathbb{R}^n)$ where either $A = B$ or $A = F$.

Theorem 1.64. *Let p, q, s as in* (1.157) (*with $p < \infty$ for the F-spaces*) *and let*

$$\mathbb{N} \ni k > \max\left(s, \frac{2n}{p} + \frac{n}{2} - s \right) \qquad \textit{in case of} \quad B^s_{pq}(\mathbb{R}^n) \tag{1.160}$$

and

$$\mathbb{N} \ni k > \max\left(s, \frac{2n}{\min(p,q)} + \frac{n}{2} - s \right) \qquad \textit{in case of} \quad F^s_{pq}(\mathbb{R}^n). \tag{1.161}$$

Let $\Psi^{j,G}_m$ be the n-dimensional Daubechies wavelets with smoothness k according to Proposition 1.53 *and Theorem* 1.61(ii).

(i) *Let $f \in S'(\mathbb{R}^n)$. Then $f \in A^s_{pq}(\mathbb{R}^n)$ if, and only if, it can be represented as*

$$f = \sum_{j,G,m} \lambda^{j,G}_m \, 2^{-jn/2} \, \Psi^{j,G}_m \qquad \textit{with} \quad \|\lambda \,|a^s_{pq}\| < \infty, \tag{1.162}$$

unconditional convergence being in $S'(\mathbb{R}^n)$. Furthermore the representation (1.162) *is unique,*

$$\lambda^{j,G}_m = 2^{jn/2} \, \left(f, \Psi^{j,G}_m \right), \tag{1.163}$$

and

$$I \,:\, f \to \left\{ 2^{jn/2} \left(f, \Psi^{j,G}_m \right) \right\} \tag{1.164}$$

is an isomorphic map of $A^s_{pq}(\mathbb{R}^n)$ **onto** *a^s_{pq}.*

(ii) *In addition let $p < \infty$ and $q < \infty$. Then* (1.162), (1.163) *converges unconditionally in $A^s_{pq}(\mathbb{R}^n)$ and $\{\Psi^{j,G}_m\}$ is an unconditional basis in $A^s_{pq}(\mathbb{R}^n)$.*

Remark 1.65. In Chapter 3 we return to this subject in detail. There one finds a proof of this theorem, further explanations and some consequences, following [Tri04a]. Then we deal also with a corresponding assertion for the Meyer wavelets. But a few comments seem to be in order now. Obviously, if $f \in B^s_{pq}(\mathbb{R}^n)$ in part (i), then k is restricted by (1.160) and $a^s_{pq} = b^s_{pq}$ in (1.162). Similarly if $f \in F^s_{pq}(\mathbb{R}^n)$ in part (i) then k is restricted by (1.161) and $a^s_{pq} = f^s_{pq}$ in (1.162). Let the countable index set $\{j, G, m\}$ in (1.162) be numbered in any order, in other words, mapped one-to-one onto $\mathbb{N}$, and let $\lambda \in a^s_{pq}$. Then it follows from the atomic representation Theorem 1.19 and the explanations given in Remark 1.20 that the sum in (1.162) converges in $S'(\mathbb{R}^n)$ and that the outcome is the same for any rearrangement. Hence the *unconditional convergence* stated after (1.162) is not an additional requirement but a consequence of $\lambda \in a^s_{pq}$. In addition if $\sigma < s$ then (1.162) converges locally in any space $A^\sigma_{pq}(\mathbb{R}^n)$, this means it converges in $A^\sigma_{pq}(K)$ for any ball K in $\mathbb{R}^n$. If $p < \infty$ and $q < \infty$ then one has the sharper assertion formulated in (ii). The factor $2^{-jn/2}$ in (1.162) simply means that we take away the L_2-normalisation according to (1.136). Then the outcome looks more natural. Since I in (1.164) is an isomorphism it follows that

$$\|f\,|A^s_{pq}(\mathbb{R}^n)\| \sim \|\lambda\,|a^s_{pq}\|, \tag{1.165}$$

where the components of λ are given by (1.163). If $s = 0$ and $p = q = 2$ in (1.158) or (1.159) then one gets Proposition 1.53 with respect to the n-dimensional Daubechies wavelets. If $s = 0$ and $1 < p < \infty$ then one has the Paley-Littlewood assertion

$$L_p(\mathbb{R}^n) = F^0_{p,2}(\mathbb{R}^n), \qquad 1 < p < \infty, \tag{1.166}$$

which is well reflected by (1.159) and (1.165) with $a^s_{pq} = f^0_{p,2}$. More generally one has the Paley-Littlewood assertion (1.8) for the Sobolev spaces $H^s_p(\mathbb{R}^n)$ and again (1.165) with $a^s_{pq} = f^s_{p,2}$. Our method to prove the above theorem relies on atomic decompositions, local means, duality and characterisations of the spaces $A^s_{pq}(\mathbb{R}^n)$ in terms of maximal functions. In particular the use of maximal functions spoils the parameters in the respective estimates. This is the main reason for the large smoothness k needed according to (1.160), (1.161), in contrast to the classical cases in (1.154), (1.155) where the natural restriction $k > |s|$ is sufficient. Accepting the ingredients just indicated in the proof of the theorem is surprisingly simple (one has only to struggle a little bit with the limited smoothness of the Daubechies wavelets). In [Kyr03] one finds a more direct straightforward approach, based on the techniques developed in [FrJ90], to prove expansions in (homogeneous and inhomogeneous) spaces of B^s_{pq} and F^s_{pq} type, starting from more general orthogonal and bi-orthogonal L_2-systems. Then one gets also more natural restrictions for k in (1.160) and (1.161).

Remark 1.66. As remarked above, wavelet bases for the spaces (1.154) and (1.155) are known. We gave corresponding references in Remark 1.63. In addition to the paper [Kyr03] mentioned above there are some other papers of interest in connection with the above theorem. Nearest to us are [FJW91], Section 7, [Bor95] and

[KyP01], dealing with Meyer wavelets in homogeneous B-spaces and F-spaces and perturbations of related bases, respectively. Wavelet bases of Meyer type in anisotropic spaces of B-type and F-type have been considered in [BeN93], [BeN95]. We return to this subject later on in Chapter 5. Finally we refer in this context to [RuS96] and [HeW96].

1.8 Wavelet frames

In the three preceding subsections we discussed three types of building blocks: atoms, quarks, and wavelet bases. They all have their advantages and disadvantages. We discussed this point at the beginning of Section 1.6. Returning to the questions listed there one can now say the following.

- Theorem 1.64 provides a perfect link between B-spaces and F-spaces in $\mathbb{R}^n$ on the one hand and corresponding sequence spaces on the other hand. Hence problems for function spaces can be shifted to the (easier) level of sequence spaces. But the underlying building blocks, wavelets of Daubechies type, are rather complicated and there is apparently no easy way to extend this theory from $\mathbb{R}^n$ to other structures such as domains in $\mathbb{R}^n$, fractal sets (in $\mathbb{R}^n$) or (abstract) quasi-metric spaces. Some progress has been made. We refer to [Mal99], Chapter VII, where one finds in Section 7.5 wavelets on intervals, to [CDV00], [CDV04], [Dah97], [Dah01] and in particular to [Coh03], [HoL05] for wavelets on domains and the literature given there. Nevertheless considerable additional efforts are necessary. We return later on in Section 4.2 to this subject.
- This is the point where the quarkonial decompositions according to Theorem 1.39 and Corollary 1.42 are coming in. These are frames, not bases. One loses the isomorphic map onto sequence spaces. On the other hand, the building blocks are very simple and, in particular, one needs only approximate lattices. This paves the way to extend quarkonial decompositions to domains in $\mathbb{R}^n$, their boundaries, fractals (in $\mathbb{R}^n$) and (abstract) quasi-metric spaces. This was one of the main topics in [Triϵ] and we return to this subject later on, especially in Chapter 8 when dealing with function spaces on quasi-metric spaces.
- The building blocks both of the wavelet bases and of the quarkonial decompositions are constructive. If one applies an operator T, say, a linear pseudo-differential operator, then these structures are destroyed. This is the point where the qualitatively defined atoms (and molecules) come in. We discussed this point at the beginning of Section 1.5.2.

As said it is just one of the main advantages of the quarkonial decompositions that one can work with approximate lattices in $\mathbb{R}^n$. This gives the possibility to extend this theory to other structures in a natural way. However if one wishes to stay exclusively in $\mathbb{R}^n$, then the standard lattices $2^{-\nu}\mathbb{Z}^n$ are the first choice where the quarks simplify to (1.107). One may ask for equally good information for the respective frame-generating functions $\Psi^\beta_{\nu m}$ in Corollary 1.42. This will be done in Chapter 3 and applied to develop a local smoothness theory. Then we give also some references about a local smoothness theory relying on wavelet bases. At this moment we restrict ourselves to a brief description of the outcome if one follows the indicated way, complementing the three preceding subsections. We rely mainly on [Tri03a] in the modified version detailed in Chapter 3.

First we recall and modify some previous notation. Let

$$\mathbb{R}^n_{++} = \{y \in \mathbb{R}^n : y = (y_1, \ldots, y_n),\ y_j > 0\} \tag{1.167}$$

and let k be a non-negative C^∞ function in $\mathbb{R}^n$ with

$$\operatorname{supp} k \subset \{y \in \mathbb{R}^n : |y| < 2^J\} \cap \mathbb{R}^n_{++} \tag{1.168}$$

for some $J \in \mathbb{N}$ such that

$$\sum_{m \in \mathbb{Z}^n} k(x-m) = 1 \qquad \text{where} \quad x \in \mathbb{R}^n. \tag{1.169}$$

Then

$$k^\beta(x) = (2^{-J}x)^\beta\, k(x) \ge 0 \qquad \text{if} \quad x \in \mathbb{R}^n, \quad \beta \in \mathbb{N}^n_0, \tag{1.170}$$

where x^β are the monomials as explained in Remark 1.16. We are interested in what follows only in the special case

$$B^s_p(\mathbb{R}^n) = B^s_{pp}(\mathbb{R}^n), \qquad 0 < p \le \infty, \quad s \in \mathbb{R}, \tag{1.171}$$

of the B-spaces as introduced in Definition 1.2. This justifies the simplification and modification of the quasi-norm in (1.108) by

$$\|\lambda\,|\ell_p\|_{\varrho,s} = \left(\sum_{\beta \in \mathbb{N}^n_0} \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{\varrho|\beta|p + j(s-n/p)p}\, |\lambda^\beta_{jm}|^p \right)^{1/p} \tag{1.172}$$

where $s \in \mathbb{R}$, $0 < p \le \infty$ and $\varrho \ge 0$ (with the usual modification if $p = \infty$). Here

$$\lambda = \left\{ \lambda^\beta_{jm} \in \mathbb{C} : j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n,\ \beta \in \mathbb{N}^n_0 \right\}. \tag{1.173}$$

Let

$$\omega \in S(\mathbb{R}^n), \quad \operatorname{supp} \omega \subset (-\pi,\pi)^n, \quad \omega(x) = 1 \text{ if } |x| \le 2, \tag{1.174}$$

$$\omega^\beta(x) = \frac{i^{|\beta|}\, 2^{J|\beta|}}{(2\pi)^n\, \beta!}\, x^\beta\, \omega(x) \quad \text{with} \quad x \in \mathbb{R}^n, \quad \beta \in \mathbb{N}^n_0, \tag{1.175}$$

where $|\beta| = \beta_1 + \cdots + \beta_n$, $\beta! = \beta_1! \cdots \beta_n!$ and

$$\Omega^\beta(x) = \sum_{m \in \mathbb{Z}^n} (\omega^\beta)^\vee(m)\, e^{-imx}, \qquad x \in \mathbb{R}^n. \tag{1.176}$$

Definition 1.67. *Let* φ_0 *and* φ_1 *be the same functions as in* (1.14) *and* (1.15) *and let* $\beta \in \mathbb{N}_0^n$*. Then* Φ_F^β *and* Φ_M^β *are given by their inverse Fourier transforms,*

$$(\Phi_F^\beta)^\vee(\xi) = \varphi_0(\xi)\,\Omega^\beta(\xi) \quad \text{and} \quad (\Phi_M^\beta)^\vee(\xi) = \varphi_1(\xi)\,\Omega^\beta(\xi), \tag{1.177}$$

where $\xi \in \mathbb{R}^n$*. Furthermore,*

$$\Phi_{jm}^\beta(x) = \begin{cases} \Phi_F^\beta(x-m) & \text{if } j = 0, \; m \in \mathbb{Z}^n, \\ \Phi_M^\beta(2^j x - m) & \text{if } \; j \in \mathbb{N}, \; m \in \mathbb{Z}^n, \end{cases} \tag{1.178}$$

where $\beta \in \mathbb{N}_0^n$ *and* $x \in \mathbb{R}^n$*.*

Remark 1.68. Since Ω^β in (1.176) are C^∞ functions it follows that Φ_F^β and Φ_M^β are elements of $S(\mathbb{R}^n)$. They are analytic functions. The construction (1.178), (1.177) resembles (1.136) based on Meyer wavelets according to Theorem 1.61 with a counterpart of (1.149) for Φ_M^β. The Meyer wavelets create an orthogonal basis in $L_2(\mathbb{R}^n)$ and there is a counterpart of Theorem 1.64. This cannot be expected for the system (1.178). But it comes out that these functions together with corresponding functions based on (1.170) result in frames and dual frames in some spaces $B_p^s(\mathbb{R}^n)$ which are simple and rather effective in describing global and local properties of elements in these spaces. We return to these assertions in detail in Chapter 3 and restrict ourselves here to a description of two typical results. For $f \in S'(\mathbb{R}^n)$ we put

$$\lambda_{jm}^\beta(f) = 2^{jn}\left(f, \Phi_{jm}^\beta\right), \qquad j \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n, \quad \beta \in \mathbb{N}_0^n, \tag{1.179}$$

which resembles (1.163), $\bar{p} = \max(1,p)$, $\sigma_p = n(\frac{1}{p} - 1)_+$ and

$$B_p^+(\mathbb{R}^n) = \bigcup_{s > \sigma_p} B_p^s(\mathbb{R}^n) \qquad \text{where} \quad 0 < p \le \infty. \tag{1.180}$$

Theorem 1.69. *Let* $0 < p \le \infty$*,* $s > \sigma_p$ *and* $\varrho \ge 0$*. Let* k *and* k^β *be as in* (1.168)–(1.170).

(i) *Then* $f \in S'(\mathbb{R}^n)$ *is an element of* $B_p^s(\mathbb{R}^n)$ *if, and only if, it can be represented as*

$$f = \sum_{\beta,j,m} \lambda_{jm}^\beta\, k^\beta(2^j x - m), \qquad x \in \mathbb{R}^n, \tag{1.181}$$

with $\|\lambda\,|\ell_p\|_{\varrho,s} < \infty$ *and absolute convergence in* $L_{\bar{p}}(\mathbb{R}^n)$*. Furthermore,*

$$\|f\,|B_p^s(\mathbb{R}^n)\| \sim \inf \|\lambda\,|\ell_p\|_{\varrho,s}, \tag{1.182}$$

where the infimum is taken over all admissible representations (1.181)*.*

(ii) *Let* $\lambda_{jm}^\beta(f)$ *be given by* (1.179)*. Then any* $f \in B_p^+(\mathbb{R}^n)$ *can be represented as*

$$f = \sum_{\beta,j,m} \lambda_{jm}^\beta(f)\, k^\beta(2^j x - m), \qquad x \in \mathbb{R}^n, \tag{1.183}$$

with absolute convergence in $L_{\bar{p}}(\mathbb{R}^n)$. Furthermore,

$$B^s_p(\mathbb{R}^n) = \big\{ f \in B^+_p(\mathbb{R}^n) \,:\, \|\lambda(f)\,|\ell_p\|_{\varrho,s} < \infty \big\} \tag{1.184}$$

(*equivalent quasi-norms*).

Remark 1.70. Details are shifted to Chapter 3. But a few comments seem to be in order. The absolute convergence in (1.181) in $L_{\bar{p}}(\mathbb{R}^n)$ is not an additional requirement but a consequence of $\|\lambda\,|\ell_p\|_{\varrho,s} < \infty$. The same comment applies to (1.183). This justifies the abbreviation

$$\sum_{\beta,j,m} = \sum_{\beta\in\mathbb{N}^n_0} \sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \tag{1.185}$$

above and in what follows. Quite obviously, these are representations of type (1.162), (1.163), but instead of a basis we have now two dual frame systems which are remarkably independent of each other.

To formulate the dual assertions we need the local means according to (1.41) with respect to the above kernels k^β, hence

$$k^\beta(t,f)(x) = \int_{\mathbb{R}^n} k^\beta(y)\, f(x+ty)\,\mathrm{d}y, \qquad t>0, \quad x\in\mathbb{R}^n, \tag{1.186}$$

and in particular,

$$k^\beta_{jm}(f) = k^\beta\left(2^{-j},f\right)(2^{-j}m), \qquad j\in\mathbb{N}_0, \quad m\in\mathbb{Z}^n, \tag{1.187}$$

where $f\in S'(\mathbb{R}^n)$. Let

$$\|k(f)\,|\ell_p\|_s = \Bigg(\sum_{\beta,j,m} 2^{j(s-n/p)p}\,|k^\beta_{jm}(f)|^p \Bigg)^{1/p} \tag{1.188}$$

(with the usual modification if $p=\infty$) be a special case of (1.172) with $\varrho = 0$, denoted now by $\|\lambda\,|\ell_p\|_s$. Recall that $\mathcal{C}^s(\mathbb{R}^n) = B^s_{\infty\infty}(\mathbb{R}^n)$. We put

$$\mathcal{C}^{-\infty}(\mathbb{R}^n) = \bigcup_{s<0} \mathcal{C}^s(\mathbb{R}^n). \tag{1.189}$$

We have by elementary embedding

$$\bigcup_{s<0} B^s_p(\mathbb{R}^n) = B^{-\infty}_p(\mathbb{R}^n) \subset \mathcal{C}^{-\infty}(\mathbb{R}^n). \tag{1.190}$$

Theorem 1.71. *Let* $1 < p \le \infty$ *and* $s < 0$.

(i) *Then* $f \in S'(\mathbb{R}^n)$ *is an element of* $B^s_p(\mathbb{R}^n)$ *if, and only if, it can be represented as*

$$f = \sum_{\beta,j,m} \lambda^\beta_{jm}\, \Phi^\beta_{jm} \tag{1.191}$$

with Φ^β_{jm} *as in Definition* 1.67 *and* $\|\lambda\,|\ell_p\|_s < \infty$, *unconditional convergence being in* $S'(\mathbb{R}^n)$. *Furthermore,*

$$\|f\,|B^s_p(\mathbb{R}^n)\| \sim \inf \|\lambda\,|\ell_p\|_s, \tag{1.192}$$

where the infimum is taken over all admissible representations (1.191).

(ii) *Let* $k^\beta_{jm}(f)$ *be given by* (1.187). *Then any* $f \in \mathcal{C}^{-\infty}(\mathbb{R}^n)$ *can be represented as*

$$f = \sum_{\beta,j,m} k^\beta_{jm}(f)\, \Phi^\beta_{jm}, \tag{1.193}$$

unconditional convergence being in $S'(\mathbb{R}^n)$. *Furthermore,*

$$B^s_p(\mathbb{R}^n) = \big\{ f \in \mathcal{C}^{-\infty}(\mathbb{R}^n) \;:\; \|k(f)\,|\ell_p\|_s < \infty \big\} \tag{1.194}$$

(*equivalent norms*).

Remark 1.72. To avoid any misunderstanding we remark that ω, φ_0, φ_1, and hence Φ^β_{jm}, are fixed. We refer for details to Chapter 3. Then we prove the above theorem by dualising Theorem 1.69 with $1 \le p < \infty$ and we discuss in detail how to extract local assertions from representations of type (1.193).

1.9 Envelopes

The comprehensive theory of envelopes of function spaces began with Part II of the Habilitationsschrift of D.D. Haroske [Har02], based on the underlying report [Har01]. It is the rather final characterisation of growth, continuity and differentiability of functions considered as elements of some function spaces. Of special interest are delicate limiting situations. On the other hand the respective inequalities have a very long and substantial history since the early 1960s. Many mathematicians both from the East and the West contributed to this distinguished field of research within the framework of the theory of function spaces, quite often parallel to each other and unaware of each other's work. We presented this theory in Chapter II of [Triϵ] based on [Har01],[Har02], the immense literature spanning almost 40 years and some of our own contributions. In particular we tried to report on the history of this subject in a (hopefully) balanced way. This will not be repeated here. The reason for returning to this subject is at least twofold. First, we said nothing about this substantial subject in the historically-oriented Chapter 1 in [Triγ] with the same title as the present introductory chapter and so it seems to be reasonable to seal this gap now. Secondly, although envelopes do not play a central role in the present book, they and their underlying inequalities will be of some use occasionally. This might explain that we restrict ourselves to some outstanding assertions mostly formulated in terms of inequalities, referring for the full machinery to [Har01], [Har02] and [Triϵ], Chapter 2. But we complement the literature by some more recent titles. The most comprehensive treatment of the theory of envelopes in function spaces and an updated version of the state of the art may be found in the recent book by D.D. Haroske [Har06].

1.9.1 Sharp embeddings

First we collect some notation and describe some embeddings and inequalities on which the recent theory of envelopes rests. We follow [Triϵ]. As there we restrict ourselves to the sub-critical, the critical, and the super-critical case for the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$, somewhat in contrast to [Har01], [Har02], [Har06] where a wider point of view had been adopted.

First we recall that

$$\omega(f,t) = \sup_{x\in\mathbb{R}^n, |h|\le t} |f(x+h) - f(x)| \quad \text{and} \quad \widetilde{\omega}(f,t) = \frac{\omega(f,t)}{t} \tag{1.195}$$

with $t > 0$, are the usual moduli of continuity and the divided moduli of continuity, respectively. Then $\mathrm{Lip}(\mathbb{R}^n)$ is the collection of all complex-valued functions on $\mathbb{R}^n$ such that

$$\|f\,|\mathrm{Lip}(\mathbb{R}^n)\| = \sup_{x\in\mathbb{R}^n} |f(x)| + \sup_{0<t<1} \widetilde{\omega}(f,t) < \infty. \tag{1.196}$$

Furthermore $C(\mathbb{R}^n)$ is the space of all complex-valued, bounded, uniformly continuous functions on $\mathbb{R}^n$ normed by

$$\|f\,|C(\mathbb{R}^n)\| = \sup_{x\in\mathbb{R}^n} |f(x)|, \tag{1.197}$$

whereas

$$C^1(\mathbb{R}^n) = \left\{ f \in C(\mathbb{R}^n) \;:\; \frac{\partial f}{\partial x_j} \in C(\mathbb{R}^n) \text{ with } j = 1,\ldots,n \right\} \tag{1.198}$$

is the obviously normed space of differentiable functions. By the mean value theorem, $C^1(\mathbb{R}^n)$ is a closed subspace of $\mathrm{Lip}(\mathbb{R}^n)$.

Theorem 1.73.

(i) (Sub-critical case) *Let*

$$1 < r < \infty, \quad s > 0, \quad s - n/p = -n/r \quad \textit{and} \quad 0 < q \le \infty, \tag{1.199}$$

(*the dashed line in Figure* 1.9.1). *Then*

$$B^s_{pq}(\mathbb{R}^n) \hookrightarrow L_r(\mathbb{R}^n) \quad \textit{if, and only if,} \quad 0 < q \le r, \tag{1.200}$$

and

$$F^s_{pq}(\mathbb{R}^n) \hookrightarrow L_r(\mathbb{R}^n) \quad \textit{for all} \quad 0 < q \le \infty. \tag{1.201}$$

(ii) (Critical case) *Let*

$$0 < p < \infty, \quad 0 < q \le \infty, \quad s = n/p. \tag{1.202}$$

Then

$$B^{n/p}_{pq}(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n) \quad \textit{if, and only if,} \quad 0 < p < \infty, \quad 0 < q \le 1, \tag{1.203}$$

and

$$F^{n/p}_{pq}(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n) \quad \textit{if, and only if,} \quad 0 < p \le 1, \quad 0 < q \le \infty. \tag{1.204}$$

One can replace $C(\mathbb{R}^n)$ *by* $L_\infty(\mathbb{R}^n)$ *in* (1.203) *and* (1.204).

(iii) (Super-critical case) *Let*

$$0 < p < \infty, \quad 0 < q \le \infty, \quad s = 1 + n/p, \tag{1.205}$$

(*the dotted line in Figure* 1.9.1). *Then*

$$B^{1+n/p}_{pq}(\mathbb{R}^n) \hookrightarrow C^1(\mathbb{R}^n) \quad \textit{if, and only if,} \quad 0 < p < \infty, \quad 0 < q \le 1, \tag{1.206}$$

and

$$F^{1+n/p}_{pq}(\mathbb{R}^n) \hookrightarrow C^1(\mathbb{R}^n) \quad \textit{if, and only if,} \quad 0 < p \le 1, \quad 0 < q \le \infty. \tag{1.207}$$

One can replace $C^1(\mathbb{R}^n)$ *by* $\mathrm{Lip}(\mathbb{R}^n)$ *in* (1.206), (1.207).

Remark 1.74. The above theorem coincides essentially with Theorem 11.4 in [Triϵ], pp. 170/171. In [Triϵ], Sections 11.3–11.5, pp. 169–173, one finds respective discussions and references which will not be repeated here. Roughly speaking, the if-parts have been known for some time. The only-if-parts, further discussions and further sharp embeddings may be found in [SiT95], complemented in [Triϵ], proof of Theorem 11.4, as far as part (iii) is concerned. Short descriptions of results of this type may also be found in [RuS96], Section 2.2, and in [ET96], Section 2.3.3. As for the general embedding theory (for different metrics and traces) we refer to [Triα] (classical spaces) and to [Triβ] (the above spaces in full generality).

It is quite clear that in the critical case special attention should be given to those spaces which are not covered by (1.203) and (1.204), hence $B^{n/p}_{pq}(\mathbb{R}^n)$ with $q > 1$ and $F^{n/p}_{pq}(\mathbb{R}^n)$ with $p > 1$, in particular if $B^{n/p}_{pq}(\mathbb{R}^n)$ are classical Besov spaces, hence $1 \le p \le \infty$, and if

$$H^{n/p}_p(\mathbb{R}^n) = F^{n/p}_{p,2}(\mathbb{R}^n), \qquad 1 < p < \infty, \tag{1.208}$$

are Sobolev spaces according to (1.8) with the classical Sobolev spaces (1.9) as a subclass,

$$H^k_p(\mathbb{R}^n) = W^k_p(\mathbb{R}^n), \qquad 1 < p < \infty, \quad \mathbb{N} \ni k = n/p. \tag{1.209}$$

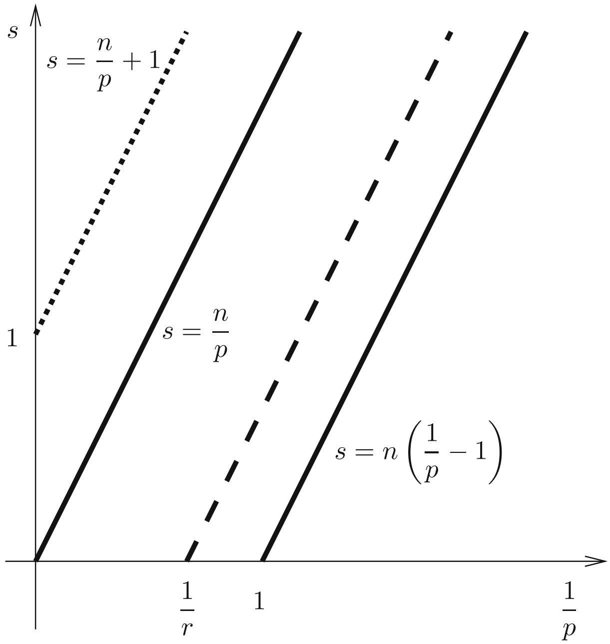

Figure 1.9.1

Similar conditions hold for the super-critical case with $1 + n/p$ in place of n/p in the B-spaces, F-spaces, and the Sobolev spaces according to (1.208), (1.209). In the last forty years many mathematicians contributed to this field of research. We tried to present in [Triϵ], Section 11.8, pp. 177–181, a balanced history of this subject which will not be repeated here. But we give a brief description of some basic notation and distinguished (historical) assertions paving the way for the theory of envelopes.

Let $f \in L_r(\mathbb{R}^n)$ with $0 < r < \infty$. Then the distribution function $\mu_f(\lambda)$ and the decreasing (this means non-increasing) rearrangement f^* of f are given by

$$\mu_f(\lambda) = |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}|, \qquad \lambda \geq 0, \tag{1.210}$$

and

$$f^*(t) = \inf\{\lambda : \mu_f(\lambda) \leq t\}, \qquad t \geq 0. \tag{1.211}$$

It is well known that one can use the rearrangement f^* of f for functions on $\mathbb{R}^n$, on (bounded) domains in $\mathbb{R}^n$, and on (finite) intervals on $\mathbb{R}$ to introduce several refinements of the respective Lebesgue spaces L_r, such as the Lorentz spaces $L_{r,u}$ with $0 < r < \infty$, $0 < u \leq \infty$, the Zygmund spaces $L_r(\log L)_a$ with $0 < r < \infty$, $a \in \mathbb{R}$, and, combining these two types of refinements, the Lorentz-Zygmund spaces $L_{r,u}(\log L)_a$. This will not be done here. The standard references are [BeS88] and (especially in connection with the combined Lorentz-Zygmund spaces) [BeR80].

Short descriptions, further properties and references may be found in [Har02], [Triϵ], Section 11.6, pp. 174–176, and in the recent book [Har06]. We refer also to [DeL93] and [EdE04], Section 3, for the general background and to [ET96], Sections 2.6.1, 2.6.2, pp. 65–75, for some representation theorems which will be recalled later on in detail when needed. Here we describe refined embeddings and related envelopes in terms of sharp inequalities. Let $\widetilde{\omega}(f,t)$ be the divided differences as introduced in (1.195).

Theorem 1.75. (Classical refinements) *Let* $0 < \varepsilon < 1$.

(i) (Sub-critical case) *Let*

$$s > 0, \qquad 1 < p < \infty, \qquad s - n/p = -n/r < 0 \tag{1.212}$$

(*the dashed line in Figure* 1.9.1). *Then there is a constant* $c > 0$ *such that for all* $f \in H^s_p(\mathbb{R}^n)$,

$$\left(\int_0^\varepsilon \left(t^{1/r} f^*(t) \right)^p \frac{\mathrm{d}t}{t} \right)^{1/p} \le c \, \| f \, | H^s_p(\mathbb{R}^n) \|, \tag{1.213}$$

for all $f \in B^s_{pq}(\mathbb{R}^n)$ *with* $1 \le q < \infty$,

$$\left(\int_0^\varepsilon \left(t^{1/r} f^*(t) \right)^q \frac{\mathrm{d}t}{t} \right)^{1/q} \le c \, \| f \, | B^s_{pq}(\mathbb{R}^n) \|, \tag{1.214}$$

and for all $f \in B^s_{p\infty}(\mathbb{R}^n)$,

$$\sup_{0<t<\varepsilon} t^{1/r} f^*(t) \le c \, \| f \, | B^s_{p\infty}(\mathbb{R}^n) \|. \tag{1.215}$$

(ii) (Critical case) *Let* $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$. *Then there is a constant* $c > 0$ *such that for all* $f \in H^{n/p}_p(\mathbb{R}^n)$,

$$\sup_{0<t<\varepsilon} \frac{f^*(t)}{|\log t|^{1/p'}} \le c \, \| f \, | H^{n/p}_p(\mathbb{R}^n) \| \tag{1.216}$$

and

$$\left(\int_0^\varepsilon \left(\frac{f^*(t)}{|\log t|} \right)^p \frac{\mathrm{d}t}{t} \right)^{1/p} \le c \, \| f \, | H^{n/p}_p(\mathbb{R}^n) \|. \tag{1.217}$$

(iii) (Super-critical case) *Let*

$$1 < p < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad 1 + n/p = k \in \mathbb{N}. \tag{1.218}$$

Then there is a constant $c > 0$ *such that for all* $f \in W^k_p(\mathbb{R}^n)$,

$$\sup_{0<t<\varepsilon} \frac{\widetilde{\omega}(f,t)}{|\log t|^{1/p'}} \le c \, \| f \, | W^k_p(\mathbb{R}^n) \|. \tag{1.219}$$

Remark 1.76. Recall that $H^s_p(\mathbb{R}^n)$ with $1 < p < \infty$ are the Sobolev spaces according to (1.8), (1.9) with the classical Sobolev spaces as a subclass. We compare the above theorem with Theorem 1.73. But first we remark that the above inequalities measure the unboundedness and the non-differentiability in the sub-critical, critical, and super-critical case, respectively. This justifies the restriction of the t-variable to $0 < t < \varepsilon < 1$, where $\varepsilon < 1$ is immaterial, but convenient in connection with $|\log t|$. To get a better understanding of what is going on we first remark that the infimum in (1.211) is a minimum and one can choose $\lambda = f^*(t)$ in (1.210), (1.211). In other words, for any $t > 0$ there is a measurable set M_t with $|M_t| \le t$ such that

$$|f(x)| \le f^*(t) \qquad \text{if} \quad x \in \mathbb{R}^n \backslash M_t. \tag{1.220}$$

The divided differences $\widetilde{\omega}(f,t)$ according to (1.195) are equivalent to a decreasing function, for example to its rearrangement $\widetilde{\omega}^*$. This is not obvious but one finds corresponding remarks in [Triϵ], p. 196, with a reference to [DeL93], Chapter 2, §6, pp. 40–44. Hence the inequalities (1.216) and (1.219) are of the same nature. To simplify the comparison of the above theorem with Theorem 1.73 we assume

$$|\text{supp } f| \le \varepsilon < 1. \tag{1.221}$$

Then we have in the sub-critical cases (1.213), (1.214) with (1.212),

$$\|f\,|L_r(\mathbb{R}^n)\| \le c \left(\int_0^\varepsilon \left(t^{1/r} f^*(t) \right)^p \frac{\mathrm{d}t}{t} \right)^{1/p} \le c\, \|f\,|H^s_p(\mathbb{R}^n)\|, \tag{1.222}$$

and for $1 \le q \le r$,

$$\|f\,|L_r(\mathbb{R}^n)\| \le c \left(\int_0^\varepsilon \left(t^{1/r} f^*(t) \right)^q \frac{\mathrm{d}t}{t} \right)^{1/q} \le c\|f\,|B^s_{pq}(\mathbb{R}^n)\|, \tag{1.223}$$

where the first inequalities in (1.222), (1.223) follow from $p < r$ and $q \le r$, and well-known monotonicity assertions for the Lorentz-(quasi-)norms in the middle of the last two inequalities. Recall that

$$\|f\,|L_r(\mathbb{R}^n)\| = \left(\int_0^\varepsilon f^{*r}(t)\, \mathrm{d}t \right)^{1/r}. \tag{1.224}$$

In other words, (1.213) and (1.214) are refinements of (1.201) (with $H^s_p = F^s_{p,2}$) and of (1.200). In addition one has now also sharp embeddings (1.214), (1.215) in the cases $q > r$, not covered by (1.200). In the critical case one gets

$$\sup_{0<t<\varepsilon} \frac{f^*(t)}{|\log t|^{1/p'}} \le c \left(\int_0^\varepsilon \left(\frac{f^*(t)}{|\log t|} \right)^p \frac{\mathrm{d}t}{t} \right)^{1/p} \le c'\, \|f\,|H^{n/p}_p(\mathbb{R}^n)\|, \tag{1.225}$$

where again the first inequality is a monotonicity assertion. We refer for proofs and detailed discussions to [Triϵ], Section 12, and, as far as (1.225) is concerned, to Example 2 on p. 187. In particular, (1.217) is a refinement of (1.216) and both complement (1.204) (with $H^s_p = F^s_{p,2}$). Finally, (1.219) is a classical complement of (1.207) with $W^k_p = F^k_{p,2}$ and $\mathrm{Lip}(\mathbb{R}^n)$ in place of $C^1(\mathbb{R}^n)$.

Remark 1.77. In contrast to Theorem 1.73 which reflects the respective situation around 1995, mainly based on [SiT95] (especially as far as the only-if-parts are concerned) we listed in the above theorem classical inequalities. Each of these inequalities has a long history which we tried to describe in a balanced way in [Triϵ], Section 11.8, pp. 177–181, which we do not repeat here in detail. There one finds also in Theorem 11.7 on p. 176/177 further classical inequalities. In particular, the above theorem is covered by [Triϵ], Theorem 11.7. But it seems to be reasonable to justify the above selection of inequalities by adding at least a few quotations referring the interested reader for greater detail to [Triϵ]. The refinements (1.213)–(1.215) came into being in the middle of the 1960s in close connection with real interpolation which is especially well adapted to Lorentz spaces defined in terms of the left-hand sides of (1.213)–(1.215). This started with [Pee66], [Str67]. Beginning with the early 1970s many Russian mathematicians contributed to this field. The history of (1.216) is especially interesting and also a little bit controversial. The left-hand side of (1.216) is finite for functions f on the interval $[0, \varepsilon]$ if, and only if,

$$\int_0^\varepsilon \exp\left\{\lambda^{p'}\,|f(t)|^{p'}\right\}\,\mathrm{d}t < \infty \qquad \text{for some} \quad \lambda > 0 \tag{1.226}$$

(exponential Orlicz space). In this version, (1.216) is due to Strichartz, [Str72]. The case of classical Sobolev spaces $W^k_p(\mathbb{R}^n)$ with $k = n/p$ and in particular $1 = n/p$ had been treated before by Trudinger, [Tru67]. This paper by Trudinger made problems of this type widely known and influenced further developments. The elegant reformulation (1.216) came later on and is covered by [BeS88]. However it should be mentioned that corresponding assertions had also been known in the Russian literature. We refer especially to [Yud61] and [Poh65]. According to (1.225) the assertion (1.217) is sharper than (1.216). This improvement goes back to [Hans79] and [BrW80]. Finally we remark that (1.219) is due to [BrW80]. We mentioned only a few outstanding papers. A more detailed and more balanced account may be found in [Triϵ], Section 11.8, pp. 177–181. In addition we refer to the surveys [KaL87], [Liz86], [Kol89], [Kol98] which describe especially what has been done in the Russian literature.

1.9.2 The critical case

So far we have described in Theorems 1.73 and 1.75 the roots of what follows: Sharp embeddings with $L_r(\mathbb{R}^n)$, $C(\mathbb{R}^n)$, $C^1(\mathbb{R}^n)$ as admitted target spaces, and classical refinements in terms of rearrangements, respectively. We follow now [Triϵ] in a simplified way concentrating mostly on sharp inequalities. First we deal with the critical case, interested in those spaces which are not covered by Theorem 1.73(ii).

Theorem 1.78. *Let* $0 < \varepsilon < 1$ *and* $0 < u \le \infty$. *Let* $\varkappa$ *be a decreasing positive function on* $(0, \varepsilon]$.

(i) *Let* $0 < p < \infty$, $1 < q \le \infty$, *and* $\frac{1}{q} + \frac{1}{q'} = 1$. *Then there is a constant* $c > 0$ *such that for all* $f \in B^{n/p}_{pq}(\mathbb{R}^n)$,

$$\left(\int_0^{\varepsilon} \left(\frac{\varkappa(t)\, f^*(t)}{|\log t|^{1/q'}} \right)^u \frac{\mathrm{d}t}{t|\log t|} \right)^{1/u} \le c \| f \,| B^{n/p}_{pq}(\mathbb{R}^n)\| \tag{1.227}$$

if, and only if, simultaneously $q \le u \le \infty$ *and* $\varkappa$ *is bounded.*

(ii) *Let* $1 < p < \infty$, $0 < q \le \infty$ *and* $\frac{1}{p} + \frac{1}{p'} = 1$. *Then there is a constant* $c > 0$ *such that for all* $f \in F^{n/p}_{pq}(\mathbb{R}^n)$,

$$\left(\int_0^{\varepsilon} \left(\frac{\varkappa(t)\, f^*(t)}{|\log t|^{1/p'}} \right)^u \frac{\mathrm{d}t}{t|\log t|} \right)^{1/u} \le c \| f \,| F^{n/p}_{pq}(\mathbb{R}^n)\| \tag{1.228}$$

if, and only if, simultaneously $p \le u \le \infty$ *and* $\varkappa$ *is bounded.*

Remark 1.79. First we remark that (1.227) with $u = \infty$ must be interpreted as

$$\sup_{0<t<\varepsilon} \frac{\varkappa(t)\, f^*(t)}{|\log t|^{1/q'}} \le c\, \| f \,| B^{n/p}_{pq}(\mathbb{R}^n)\|, \tag{1.229}$$

similarly (1.228). To avoid any misunderstanding we explain in detail what is meant by part (i) (and similarly by part (ii)). If $u < q$ then (1.227) is false for $\varkappa = 1$ (and hence for any positive decreasing $\varkappa$ on $(0, \varepsilon]$). If $q \le u$, then (1.227) is valid for $\varkappa = 1$, but it is false for any unbounded decreasing positive function $\varkappa$ on $(0, \varepsilon]$. The left-hand side of (1.227) is monotone with respect to u, in particular,

$$\begin{aligned}
\sup_{0<t<\varepsilon} \frac{f^*(t)}{|\log t|^{1/q'}} &\le c \left(\int_0^{\epsilon} \left(\frac{f^*(t)}{|\log t|^{1/q'}} \right)^u \frac{\mathrm{d}t}{t|\log t|} \right)^{1/u} \\
&\le c' \left(\int_0^{\varepsilon} \left(\frac{f^*(t)}{|\log t|} \right)^q \frac{\mathrm{d}t}{t} \right)^{1/q} \\
&\le c''\, \| f \,| B^{n/p}_{pq}(\mathbb{R}^n)\|
\end{aligned} \tag{1.230}$$

for any u with $q \le u \le \infty$. Hence one gets the sharpest assertion if one chooses $u = q$. Similarly for part (ii) of the theorem,

$$\sup_{0<t<\varepsilon} \frac{f^*(t)}{|\log t|^{1/p'}} \le c \left(\int_0^\varepsilon \left(\frac{f^*(t)}{|\log t|} \right)^p \frac{\mathrm{d}t}{t} \right)^{1/p} \le c \, \| f \, | F^{n/p}_{pq}(\mathbb{R}^n) \|. \tag{1.231}$$

Since $H^s_p(\mathbb{R}^n) = F^s_{p,2}(\mathbb{R}^n)$ for $1 < p < \infty$, one obtains (1.225) as a special case of (1.231). A complete proof of the above theorem may be found in [Triϵ], Section 13. These rather decisive assertions have several forerunners in the 1990s, including some of our own contributions which we collected in [Triϵ], Section 13.5, pp. 214–215, together with other related papers and some additional comments. This will not be repeated here.

Remark 1.80. However we wish to explain where the notion *envelope* comes from. Let $A^s_{pq}(\mathbb{R}^n)$ be either $B^s_{pq}(\mathbb{R}^n)$ or $F^s_{pq}(\mathbb{R}^n)$ with

$$1 < r < \infty, \qquad s > 0, \qquad s - \frac{n}{p} = -\frac{n}{r}, \qquad 0 < q \le \infty, \tag{1.232}$$

(sub-critical case) or $A^s_{pq}(\mathbb{R}^n) = B^{n/p}_{pq}(\mathbb{R}^n)$ with $0 < p < \infty$, $1 < q \le \infty$ (part (i) of the theorem) or $A^s_{pq}(\mathbb{R}^n) = F^{n/p}_{pq}(\mathbb{R}^n)$ with $1 < p < \infty$, $0 < q \le \infty$ (part (ii) of the theorem). Then

$$\mathcal{E}_G A^s_{pq}(t) = \sup \left\{ f^*(t) \,:\, \| f \, | A^s_{pq}(\mathbb{R}^n) \| \le 1 \right\}, \quad 0 < t < \varepsilon, \tag{1.233}$$

is called the *growth envelope function*, where again we assume that $0 < \varepsilon < 1$. It is quite clear that some additional considerations are necessary, for example the influence of equivalent normings. Then one ends up with continuous representatives of corresponding equivalence classes. We refer for details to [Har01], [Har02], [Triϵ], Section 12, and in the recent book [Har06]. Ignoring these formalisations it follows almost immediately from Theorem 1.78 and the explanations given in Remark 1.79 that

$$\mathcal{E}_G B^{n/p}_{pq}(t) = |\log t|^{1/q'}, \qquad 0 < p < \infty, \quad 1 < q \le \infty, \tag{1.234}$$

and

$$\mathcal{E}_G F^{n/p}_{pq}(t) = |\log t|^{1/p'}, \qquad 1 < p < \infty, \quad 0 < q \le \infty. \tag{1.235}$$

This corresponds to the left-hand sides of (1.230) and (1.231), respectively. The indicated monotonicity of the integrals in (1.227) suggests the following refinement: The couple

$$\mathfrak{E}_G B^{n/p}_{pq} = \left(\mathcal{E}_G B^{n/p}_{pq}(\cdot\,), u \right), \qquad 0 < p < \infty, \quad 1 < q \le \infty, \tag{1.236}$$

is called the *growth envelope* of $B^{n/p}_{pq}(\mathbb{R}^n)$ with (1.234) if u is the infimum of all numbers v with $0 < v \le \infty$ for which there is a constant $c_v > 0$ such that

$$\left(\int_0^\varepsilon \left(\frac{f^*(t)}{|\log t|^{1/q'}} \right)^v \frac{\mathrm{d}t}{t|\log t|} \right)^{1/v} \le c_v \, \|f \,|B^{n/p}_{pq}(\mathbb{R}^n)\| \tag{1.237}$$

for all $f \in B^{n/p}_{pq}(\mathbb{R}^n)$ (with the indicated modification if $v = \infty$). By the above Theorem 1.78 and the explanations given in Remark 1.79 one has

$$\mathfrak{E}_G B^{n/p}_{pq} = \left(|\log t|^{1/q'}, q \right), \qquad 0 < p < \infty, \quad 1 < q \le \infty, \tag{1.238}$$

and the above infimum is a minimum. Similarly one gets

$$\mathfrak{E}_G F^{n/p}_{pq} = \left(|\log t|^{1/p'}, p \right), \qquad 1 < p < \infty, \quad 0 < q \le \infty. \tag{1.239}$$

As for technical explanations and an incorporation in a more general measure-theoretical point of view we refer to [Har01], [Har02], [Triϵ], Sections 12 and 13, and [Har06].

1.9.3 The super-critical case

Comparing the parts (ii) and (iii) of Theorem 1.73 one gets the impression that one obtains the super-critical case by lifting from the critical one. This is largely correct but requires a lot of work in detail and is by no means obvious. This is also well reflected by the parts (ii) and (iii) of Theorem 1.75 as far as the known classical refinements are concerned. Whereas (1.219) with the restriction (1.218) might be considered as a special case of the lifting of (1.216), there is apparently no lifted counterpart of (1.216) in full generality and of (1.217) up to the 1980s. Again we restrict ourselves here to a simplified report, referring for explanations, proofs, further discussions, and additional references to [Har01], [Har02], [Triϵ], Section 14, and [Har06]. Recall that the divided differences $\widetilde{\omega}(f,t)$ as introduced in (1.195) are essentially decreasing, in particular the corresponding rearrangement $\widetilde{\omega}^*(f,t)$ is equivalent to $\widetilde{\omega}(f,t)$, hence $\widetilde{\omega}(f,t) \sim \widetilde{\omega}^*(f,t)$ for $0 < t < \varepsilon < 1$. Another candidate for lifting of $f^*(t)$ is the rearrangement $|\nabla f|^*(t)$ of the gradient ∇f of f. Both work.

Theorem 1.81. *Let $0 < \varepsilon < 1$ and $0 < u \le \infty$. Let $\varkappa$ be a decreasing positive function on $(0,\varepsilon]$.*

(i) *Let $0 < p < \infty$, $1 < q \le \infty$, and $\frac{1}{q} + \frac{1}{q'} = 1$. Then there is a constant $c > 0$ such that for all $f \in B^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)$,*

$$\left(\int_0^\varepsilon \left(\frac{\varkappa(t)\, \widetilde{\omega}(f,t)}{|\log t|^{1/q'}} \right)^u \frac{\mathrm{d}t}{t|\log t|} \right)^{1/u} \le c \, \|f \,|B^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)\| \tag{1.240}$$

if, and only if, simultaneously $q \le u \le \infty$ *and* $\varkappa$ *is bounded. Similarly there is a constant* $c > 0$ *such that for all* $f \in B^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)$,

$$\left(\int_0^\varepsilon \left(\frac{\varkappa(t)\, |\nabla f|^*(t)}{|\log t|^{1/q'}} \right)^u \frac{\mathrm{d}t}{t|\log t|} \right)^{1/u} \le c\, \|f\,|B^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)\| \tag{1.241}$$

if, and only if, simultaneously $q \le u \le \infty$ *and* $\varkappa$ *is bounded.*

(ii) *Let* $1 < p < \infty$, $0 < q \le \infty$ *and* $\frac{1}{p} + \frac{1}{p'} = 1$. *Then there is a constant* $c > 0$ *such that for all* $f \in F^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)$,

$$\left(\int_0^\varepsilon \left(\frac{\varkappa(t)\, \widetilde{\omega}(f,t)}{|\log t|^{1/p'}} \right)^u \frac{\mathrm{d}t}{t|\log t|} \right)^{1/u} \le c\, \|f\,|F^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)\| \tag{1.242}$$

if, and only if, simultaneously $p \le u \le \infty$ *and* $\varkappa$ *is bounded. Similarly there is a constant* $c > 0$ *such that for all* $f \in F^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)$,

$$\left(\int_0^\varepsilon \left(\frac{\varkappa(t)\, |\nabla f|^*(t)}{|\log t|^{1/p'}} \right)^u \frac{\mathrm{d}t}{t|\log t|} \right)^{1/u} \le c\, \|f\,|F^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)\| \tag{1.243}$$

if, and only if, simultaneously $p \le u \le \infty$ *and* $\varkappa$ *is bounded.*

Remark 1.82. This theorem is very similar to Theorem 1.78 and one has the same interpretations as in Remark 1.79, now with $\widetilde{\omega}(f,t)$ and $|\nabla f|^*(t)$ in place of $f^*(t)$ and with respect to the spaces $B^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)$ and $F^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)$. The counterparts of (1.230) and (1.231) are now

$$\sup_{0<t<\varepsilon} \frac{\widetilde{\omega}(f,t)}{|\log t|^{1/q'}} \le c \left(\int_0^\varepsilon \left(\frac{\widetilde{\omega}(f,t)}{|\log t|} \right)^q \frac{\mathrm{d}t}{t} \right)^{1/q} \le c'\, \|f\,|B^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)\| \tag{1.244}$$

and

$$\sup_{0<t<\varepsilon} \frac{\widetilde{\omega}(f,t)}{|\log t|^{1/p'}} \le c \left(\int_0^\varepsilon \left(\frac{\widetilde{\omega}(f,t)}{|\log t|} \right)^p \frac{\mathrm{d}t}{t} \right)^{1/p} \le c'\, \|f\,|F^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)\| \tag{1.245}$$

and the same inequalities with $|\nabla f|^*$ in place of $\widetilde{\omega}(f,t)$. In particular, the last inequality covers the classical refinements (1.219) with (1.218). Furthermore, the inequalities with $|\nabla f|^*(t)$ are sharper than the corresponding assertions with $\widetilde{\omega}(f,t)$: If $q \le u \le \infty$, then

$$\begin{aligned} \left(\int_0^\varepsilon \left(\frac{\widetilde{\omega}(f,t)}{|\log t|^{1/q'}} \right)^u \frac{\mathrm{d}t}{t|\log t|} \right)^{1/u} &\le c \left(\int_0^\varepsilon \left(\frac{|\nabla f|^*(t)}{|\log t|^{1/q'}} \right)^u \frac{\mathrm{d}t}{t|\log t|} \right)^{1/u} \\ &\le c'\, \|f\,|B^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)\| \end{aligned} \tag{1.246}$$

for some $c > 0$, $c' > 0$ and all $f \in B^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)$. If $p < \infty$ and $p \le u \le \infty$, then

$$\left(\int_0^\varepsilon \left(\frac{\widetilde{\omega}(f,t)}{|\log t|^{1/p'}} \right)^u \frac{\mathrm{d}t}{t|\log t|} \right)^{1/u} \le c \left(\int_0^\varepsilon \left(\frac{|\nabla f|^*(t)}{|\log t|^{1/p'}} \right)^u \frac{\mathrm{d}t}{t|\log t|} \right)^{1/u} \le c' \| f \,| F^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n) \| \tag{1.247}$$

for some $c > 0$, $c' > 0$ and all $f \in F^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)$. A complete proof of the theorem and of the above inequalities and further explanations may be found in [Triϵ], Section 14.

Remark 1.83. There is the following counterpart of the growth envelope described in Remark 1.80. Let $A^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)$ be either $B^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)$ with $0 < p < \infty$, $1 < q \le \infty$, or $F^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)$ with $1 < p < \infty$, $0 < q \le \infty$. Then

$$\mathcal{E}_C A^{1+\frac{n}{p}}_{pq}(t) = \sup \left\{ \widetilde{\omega}(f,t) \;:\; \|f \,| A^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)\| \le 1 \right\}, \quad 0 < t < \varepsilon, \tag{1.248}$$

is called the *continuity envelope function*, where again we assume that $0 < \varepsilon < 1$. As in Remark 1.80 we refer for a more detailed discussion to [Har01], [Har02], [Triϵ], Section 12, and [Har06]. It follows almost immediately from Theorem 1.81 and Remark 1.82 that

$$\mathcal{E}_C B^{1+\frac{n}{p}}_{pq}(t) = |\log t|^{1/q'}, \qquad 0 < p < \infty, \quad 1 < q \le \infty, \tag{1.249}$$

and

$$\mathcal{E}_C F^{1+\frac{n}{p}}_{pq}(t) = |\log t|^{1/p'}, \qquad 1 < p < \infty, \quad 0 < q \le \infty. \tag{1.250}$$

This corresponds to the left-hand sides of (1.244) and (1.245). Again we have a monotonicity of the integrals in (1.240)–(1.243) with respect to u. This suggests the following refinement: The couple

$$\mathfrak{E}_C B^{1+\frac{n}{p}}_{pq} = \left(\mathcal{E}_C B^{1+\frac{n}{p}}_{pq}(\cdot), u \right), \qquad 0 < p < \infty \quad 1 < q \le \infty, \tag{1.251}$$

is called the *continuity envelope* of $B^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)$ with (1.249) if u is the infimum of all numbers v with $0 < v \le \infty$ for which there is a constant c_v such that

$$\left(\int_0^\varepsilon \left(\frac{\widetilde{\omega}(f,t)}{|\log t|^{1/q'}} \right)^v \frac{\mathrm{d}t}{t|\log t|} \right)^{1/v} \le c_v \, \|f \,| B^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)\| \tag{1.252}$$

for all $f \in B^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)$ (with the indicated modification if $v = \infty$). By the above Theorem 1.81 and the above explanations one has

$$\mathfrak{E}_C B^{1+\frac{n}{p}}_{pq} = \left(|\log t|^{1/q'}, q \right), \qquad 0 < p < \infty, \quad 1 < q \le \infty, \tag{1.253}$$

and the above infimum is a minimum. Similarly one gets

$$\mathfrak{E}_C F_{pq}^{1+\frac{n}{p}} = \Big(|\log t|^{1/p'}, p \Big), \qquad 1 < p < \infty, \quad 0 < q \le \infty. \tag{1.254}$$

Finally we remark that one can replace $\widetilde{\omega}(f,t)$ in (1.248) by $|\nabla f|^*(t)$ with the same outcome in (1.249)–(1.254).

1.9.4 The sub-critical case

The sub-critical case is characterised by (1.199), the dashed line in Figure 1.9.1. So far we have the sharp embeddings (1.200), (1.201) and the classical refinements described in Theorem 1.75. Otherwise we follow the scheme of the two preceding Sections 1.9.2 and 1.9.3.

Theorem 1.84. *Let* $0 < \varepsilon < 1$ *and* $0 < u \le \infty$. *Let* $\varkappa$ *be a decreasing positive function on* $(0, \varepsilon]$. *Let*

$$1 < r < \infty, \quad s > 0, \quad s - \frac{n}{p} = -\frac{n}{r} \quad \textit{and} \quad 0 < q \le \infty, \tag{1.255}$$

(*the dashed line in Figure* 1.9.1).

(i) *Then there is a constant* $c > 0$ *such that for all* $f \in B^s_{pq}(\mathbb{R}^n)$,

$$\left(\int_0^\varepsilon \Big(\varkappa(t)\, t^{1/r}\, f^*(t) \Big)^u \frac{\mathrm{d}t}{t} \right)^{1/u} \le c\, \| f \,|B^s_{pq}(\mathbb{R}^n)\| \tag{1.256}$$

if, and only if, simultaneously $q \le u \le \infty$ *and* $\varkappa$ *is bounded.*

(ii) *Then there is a constant* $c > 0$ *such that for all* $f \in F^s_{pq}(\mathbb{R}^n)$,

$$\left(\int_0^\varepsilon \Big(\varkappa(t)\, t^{1/r}\, f^*(t) \Big)^u \frac{\mathrm{d}t}{t} \right)^{1/u} \le c\, \| f \,|F^s_{pq}(\mathbb{R}^n)\| \tag{1.257}$$

if, and only if, simultaneously $p < \infty$, $p \le u \le \infty$ *and* $\varkappa$ *is bounded.*

Remark 1.85. The technical explanations are the same as in Remark 1.79 in connection with Theorem 1.78. This applies to the counterpart of (1.229) and to the monotonicity of the integrals in (1.256), (1.257) with respect to u. We have in particular,

$$\sup_{0<t<\varepsilon} t^{1/r}\, f^*(t) \le c \left(\int_0^\varepsilon \Big(t^{1/r} f^*(t) \Big)^q \frac{\mathrm{d}t}{t} \right)^{1/q} \le c'\, \| f \,|B^s_{pq}(\mathbb{R}^n)\| \tag{1.258}$$

and

$$\sup_{0<t<\varepsilon} t^{1/r} f^*(t) \le c \left(\int_0^\varepsilon \left(t^{1/r} f^*(t) \right)^p \frac{\mathrm{d}t}{t} \right)^{1/p} \le c' \, \|f \,| F^s_{pq}(\mathbb{R}^n)\|. \tag{1.259}$$

These inequalities cover in particular (1.200), (1.201), and also (1.213)–(1.215).

Remark 1.86. Let $\mathcal{E}_G A^s_{pq}(t)$ be the *growth envelope function* according to (1.233) with (1.232) (which coincides with (1.255)). Then

$$\mathcal{E}_G A^s_{pq}(t) = t^{-1/r}, \qquad 0 < t < \varepsilon. \tag{1.260}$$

As before, the couple

$$\mathfrak{E}_G A^s_{pq} = \left(\mathcal{E}_G A^s_{pq}(\cdot\,), u \right), \qquad \text{with} \quad (1.255), \tag{1.261}$$

is called the *growth envelope* of $A^s_{pq}(\mathbb{R}^n)$ if u is the infimum of all numbers v with $0 < v \le \infty$ for which there is a constant $c_v > 0$ such that

$$\left(\int_0^\varepsilon \left(t^{1/r} f^*(t) \right)^v \frac{\mathrm{d}t}{t} \right)^{1/v} \le c_v \, \|f \,| A^s_{pq}(\mathbb{R}^n)\| \tag{1.262}$$

for all $f \in A^s_{pq}(\mathbb{R}^n)$. By the above theorem and the monotonicity of the integrals in (1.262) with respect to v it follows that

$$\mathfrak{E}_G B^s_{pq} = \left(t^{-1/r}, q \right) \qquad \text{and} \qquad \mathfrak{E}_G F^s_{pq} = \left(t^{-1/r}, p \right). \tag{1.263}$$

We refer to [Triϵ], Section 15, for complete proofs and further inequalities.

1.9.5 Some generalisations and further references

In the present book envelopes will play only a marginal role. As had been said the whole theory started in a systematic way and from a wider point of view in [Har01], [Har02]. Based on these papers, some of our own contributions, and numerous forerunners, we presented this theory in detail in [Triϵ], Chapter II, restricted to the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$, and to the three distinguished cases considered: sub-critical, critical, super-critical. So far we gave only very few references, mostly in Remark 1.77 in connection with the classical refinements described in Theorem 1.75. In [Triϵ] we collected in a (as we hope) balanced and rather comprehensive way the corresponding literature. We refer to the Sections 11.8, 13.5, 14.7 and 15.5 in [Triϵ]. Furthermore there is a close connection between growth envelopes and Hardy inequalities. We do not give a description of this subject here and refer to [Triϵ], Section 16, where one finds also the corresponding literature. However we wish to complement the references given in [Triϵ] by some more recent papers especially in connection with more general spaces. Very first comments may be found in [Triϵ], Sections 17.2–17.4. Detailed descriptions of the state of the art and many references may be found in the book [Har06] and in [Har05].

The classical assertion (1.219) and the sharp inequalities (1.244), (1.245) suggest that we generalise the spaces $\mathrm{Lip}(\mathbb{R}^n)$ normed according to (1.196) by spaces $\mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)$ with $\alpha \geq 0$, normed by

$$\|f\,|\mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)\| = \sup_{x\in\mathbb{R}^n} |f(x)| + \sup_{0<t<\varepsilon} \frac{\widetilde{\omega}(f,t)}{|\log t|^\alpha}, \tag{1.264}$$

where $0 < \varepsilon < 1$. Here $\widetilde{\omega}(f,t)$ are the divided differences according to (1.195). Then one gets as a consequence of Theorem 1.81 and Remark 1.82,

$$B^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n) \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n) \qquad \text{if, and only if,} \quad \alpha \geq 1/q'\,, \tag{1.265}$$

and

$$F^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n) \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n) \qquad \text{if, and only if,} \quad \alpha \geq 1/p'\,, \tag{1.266}$$

under the same restrictions for p and q as in Theorem 1.81. One may ask what happens if one replaces $\mathbb{R}^n$ in (1.265) and (1.266) by a bounded domain (compactness of the embeddings, approximation numbers, entropy numbers etc.). The relevant papers are [EdH99], [EdH00], [Har00].

In this book we are mainly interested in the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ as introduced in Definition 1.2 (occasionally we deal with respective spaces on domains, anisotropic and weighted spaces). But for some thirty years there has been a growing interest in spaces of generalised smoothness. We refer to [Mer84], [CoF88] and the surveys [KaL87], [Liz86], [KuN88] reflecting the extensive Russian literature up to the end of the 1980s. More detailed references especially in connection with the work of M.L. Goldman and G.A. Kaljabin may be found in [FaL01], [FaL04]. As for spaces of variable smoothness one may also consult [Bes05] and the references given there. We concentrate here on those spaces of generalised smoothness which are directly related to our approach and to the theory of envelopes.

A positive decreasing or increasing function Ψ on the interval $(0,1]$ is called admissible if

$$\Psi(2^{-j}) \sim \Psi(2^{-2j}), \qquad j \in \mathbb{N}_0. \tag{1.267}$$

One may typically think of

$$\Psi(t) = (1 + |\log t|)^b\,, \qquad t \in (0,1], \quad b \in \mathbb{R}. \tag{1.268}$$

Then the spaces $B^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ and $F^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ generalise the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ as introduced in Definition 1.2 for the same parameters s, p, q as there and with $2^{js}\Psi(2^{-j})$ in place of 2^{js}, hence for the corresponding B-spaces,

$$\|f\,|B^{(s,\Psi)}_{pq}(\mathbb{R}^n)\|_\varphi = \left(\sum_{j=0}^\infty 2^{jsq}\Psi(2^{-j})^q\,\|(\varphi_j\widehat{f})^\vee\,|L_p(\mathbb{R}^n)\|^q\right)^{1/q}, \tag{1.269}$$

and similarly for the F-spaces. The interest in these spaces comes from fractal geometry when generalising so-called d-sets by (d,Ψ)-sets and playing the corresponding fractal drums. Then one needs the above spaces of generalised smoothness. The first step was done in [EdT98]. But the complete theory of the spaces $B^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ and $F^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ is due to S. Moura in connection with her dissertation, [Mou99], [Mou01a], [Mou01b]. In recent times both growth envelopes and continuity envelopes for these spaces have been considered. We refer to [CaM04a], [CaM04b], [HaM04] and [CaH04]. Sharp (local and global) embeddings for (fractional) logarithmic Sobolev spaces and logarithmic Besov spaces may be found in [OpT03] and [GuO05]. This has been generalised in [GuO06] to spaces $B^{s,b}_{pq}$ where b is a slowly varying function.

One can replace 2^{js} in (1.17), (1.19) or $2^{js}\Psi(2^{-j})$ in (1.269) by more general sequences $\boldsymbol{\sigma} = (\sigma_j)_{j=0}^{\infty}$ of positive numbers σ_j such that for some $d_0 > 0$ and $d_1 > 0$,

$$d_0\sigma_j \le \sigma_{j+1} \le d_1\sigma_j, \qquad j \in \mathbb{N}_0. \tag{1.270}$$

Hence the corresponding spaces $B^{\sigma}_{pq}(\mathbb{R}^n)$ are quasi-normed by

$$\|f\,|B^{\boldsymbol{\sigma}}_{pq}(\mathbb{R}^n)\|_{\varphi} = \Bigg(\sum_{j=0}^{\infty} \sigma_j\,\|(\varphi_j\widehat{f})^{\vee}\,|L_p(\mathbb{R}^n)\|^q\Bigg)^{1/q}. \tag{1.271}$$

Similarly for the F-spaces. The interest in these generalisations comes again from fractal geometry, so-called h-sets and related function spaces. We refer to [Bri04]. But we return later on to this subject in greater detail. Growth envelopes for these spaces have been studied in [BrM03]. So far the original spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ according to Definition 1.2 but also the generalisations according to (1.269) and (1.271) are based on the dyadic resolutions of unity, (1.14)–(1.16). Let $N = (N_j)_{j=0}^{\infty}$ be a sequence of positive numbers such that there are two numbers $1 < \lambda_0 \le \lambda_1$ with

$$\lambda_0 N_j \le N_{j+1} \le \lambda_1 N_j, \qquad j \in \mathbb{N}_0. \tag{1.272}$$

Then it makes sense to replace 2^k in (1.15) by N_k, hence,

$$\varphi_j(x) = \varphi_0(N_j^{-1}x) - \varphi_0(N_{j-1}^{-1}x), \qquad x \in \mathbb{R}^n, \quad j \in \mathbb{N}. \tag{1.273}$$

One has again (1.16). Identifying φ_j in (1.271) with the functions φ_j from (1.273) one gets spaces $B^{\sigma,N}_{pq}(\mathbb{R}^n)$ and their counterparts $F^{\sigma,N}_{pq}(\mathbb{R}^n)$. This is a somewhat rough description. For greater details and more precise formulations we refer to [FaL01], [FaL04]. There one finds many references, especially to the Russian school and, in particular, to the work by M.L. Goldman and G.A. Kaljabin. The interest in this generalisation comes from the classical theory and the study of modified B-spaces, where for example, $|h|^{-s}|\Delta^m_h f|$ in (1.13) is replaced by $\lambda(|h|)\,|\Delta^m_h f|$ for suitable functions λ. This results on the Fourier side in the replacement of 2^j by N_j with (1.272). The comprehensive papers [FaL01], [FaL04] are largely self-contained surveys about these spaces $B^{\sigma,N}_{pq}(\mathbb{R}^n)$ and $F^{\sigma,N}_{pq}(\mathbb{R}^n)$ on an advanced level covering Littlewood-Paley assertions, duality, atomic representations and local means. This has been completed recently ind [Mou05]. Growth envelopes for these spaces have been considered in [CaF05] and [CaL05].

Quite obviously, envelopes produce very sharp embedding assertions, for example, of type (1.265), (1.266). Furthermore when one restricts these spaces to bounded domains Ω, hence $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$, then envelopes can also be used to say something about compactness, as well as on the related approximation numbers and entropy numbers. We refer to [CaH03], [HaM04], [CaH04]. Finally we wish to mention that function spaces of variable and generalised smoothness play a crucial role in probability theory and the theory of stochastic processes. The relevant key words are *continuous negative definite functions, Bernstein functions, Dirichlet forms* (on fractal sets) and *Markov processes*. We refer in this context to [FaJ01, FJS01a, FJS01b, FaL04, KnZ06] and the books by N. Jacob [Jac01, Jac02, Jac05].

1.10 A digression: How to measure compactness in quasi-Banach spaces

Later on we study in detail compact embeddings between function spaces defined on bounded domains in $\mathbb{R}^n$, or between corresponding weighted spaces on $\mathbb{R}^n$. Furthermore we continue our considerations in [Triδ] and [Triϵ] on fractal elliptic operators in $\mathbb{R}^n$, concentrating on the distribution of corresponding eigenvalues and properties of related eigenfunctions. Both topics are based on the abstract theory of entropy numbers and approximation numbers of compact mappings between quasi-Banach spaces and their relations to spectral theory. We give a brief description of what is needed later on, fixing also some more general notation.

A *quasi-norm* on a complex linear space B is a map $\| \cdot \,|B\|$ from B to the non-negative reals $\mathbb{R}_+$ such that

$$\|x\,|B\| = 0 \qquad \text{if, and only if,} \quad x = 0, \tag{1.274}$$

$$\|\lambda x\,|B\| = |\lambda|\,\|x\,|B\| \qquad \text{for all } \lambda \in \mathbb{C} \text{ and all } x \in B, \tag{1.275}$$

and there is a constant C such that for all $x \in B$ and $y \in B$,

$$\|x+y\,|B\| \le C\,(\|x\,|B\| + \|y\,|B\|). \tag{1.276}$$

Plainly, $C \ge 1$. If $C = 1$ is allowed then $\| \cdot \,|B\|$ is a norm in B. As usual, B is called a *quasi-Banach space* if every Cauchy sequence with respect to $\| \cdot \,|B\|$ is a convergent sequence.

Given any $p \in (0,1]$, a *p-norm* on a complex linear space B is a map $\| \cdot \,|B\| \mapsto \mathbb{R}_+$ which satisfies (1.274), (1.275), and instead of (1.276),

$$\|x+y\,|B\|^p \le \|x\,|B\|^p + \|y\,|B\|^p \qquad \text{for} \quad x \in B, \quad y \in B. \tag{1.277}$$

Any p-norm is a quasi-norm. Two quasi-norms $\| \cdot \,|B\|_1$ and $\| \cdot \,|B\|_2$ on B are said to be *equivalent* if there is a constant $c \ge 1$ such that for all $x \in B$,

$$c^{-1}\,\|x\,|B\|_1 \le \|x\,|B\|_2 \le c\,\|x\,|B\|_1. \tag{1.278}$$

If $\| \cdot |B\|_1$ is a quasi-norm on B, then there exists $p \in (0,1]$ and a p-norm $\| \cdot |B\|_2$ on B which is equivalent to $\| \cdot |B\|_1$. We refer to [Kon86], p. 47 or [DeL93], p. 20, for a proof of this non-obvious assertion.

Let A, B be quasi-Banach spaces and let $T : A \mapsto B$ be linear. Then T will be called *bounded* or *continuous* if

$$\|T\| = \sup \{ \|Ta \,|B\| \,:\, a \in A,\ \|a \,|A\| \le 1 \} < \infty. \tag{1.279}$$

The family of all such operators will be denoted by $L(A,B)$ or $L(A)$ if $A = B$. We use the notation

$$T : A \hookrightarrow B \quad \text{if, and only if,} \quad T \in L(A,B). \tag{1.280}$$

Otherwise we use standard notation naturally extended from Banach spaces to quasi-Banach spaces. Let

$$U_A = \{ a \in A \,:\, \|a \,|A\| \le 1 \} \tag{1.281}$$

be the unit ball in the quasi-Banach space A. An operator $T \in L(A,B)$ is called *compact* if TU_A is precompact in B. The classical Riesz theory for compact operators $T \in L(B)$ can be extended from Banach spaces to quasi-Banach spaces. We refer to [ET96], pp. 3–7. In particular, let $T \in L(B)$ be compact and let $\sigma(T)$ be its spectrum.

Then $\sigma(T) \backslash \{0\}$ consists of an at most countably infinite number of eigenvalues of finite algebraic multiplicity which accumulate only at the origin.

Of interest is the distribution of the eigenvalues of compact operators. This is closely related to two distinguished sequences of numbers: entropy numbers and approximation numbers.

Definition 1.87. *Let A, B be quasi-Banach spaces and let $T \in L(A,B)$. Let U_A be the unit ball in A according to* (1.281).

(i) *Then for all $k \in \mathbb{N}$ the* kth *entropy number $e_k(T)$ of T is defined as the infimum of all $\varepsilon > 0$ such that*

$$T(U_A) \subset \bigcup_{j=1}^{2^{k-1}} (b_j + \varepsilon U_B) \quad \textit{for some} \quad b_1, \ldots, b_{2^{k-1}} \in B. \tag{1.282}$$

(ii) *Then for all $k \in \mathbb{N}$ the* kth *approximation number $a_k(T)$ of T is defined by*

$$a_k(T) = \inf \{ \|T - L\| \,:\, L \in L(A,B), \quad \operatorname{rank} L < k \}, \tag{1.283}$$

where $\operatorname{rank} L$ *is the dimension of the range of L.*

Remark 1.88. Both entropy numbers and approximation numbers in Banach spaces have a long and substantial history. We do not go into detail here and refer to [Pie80], [Kon86], [Pie87], [EdE87], [CaS90], [LGM96], [ET96]. We followed here [ET96] where the corresponding theory had been extended from Banach spaces to quasi-Banach spaces. We list a few properties needed later on restricting us to the bare minimum and referring again for detailed proofs to [ET96].

Proposition 1.89. *Let A, B, C be complex quasi-Banach spaces, let $S \in L(A,B)$, $T \in L(A,B)$ and $R \in L(B,C)$. Let h_k be either the entropy numbers e_k or the approximation numbers a_k.*

(i) *Then*

$$\|T\| \geq h_1(T) \geq h_2(T) \geq \cdots \tag{1.284}$$

with $\|T\| = a_1(T)$. Furthermore, T is compact if, and only if, $e_k(T) \to 0$ for $k \to \infty$.

(ii) *For all $k \in \mathbb{N}$, $l \in \mathbb{N}$,*

$$h_{k+l-1}(R \circ S) \leq h_k(R)\, h_l(S). \tag{1.285}$$

(iii) *If B is a p-Banach space, $0 < p \leq 1$, then for all $k \in \mathbb{N}$ and $l \in \mathbb{N}$,*

$$h^p_{k+l-1}(S+T) \leq h^p_k(S) + h^p_l(T). \tag{1.286}$$

Remark 1.90. Proofs may be found in [ET96]. To avoid a misunderstanding: h_k in (1.284)–(1.286) is either always e_k or always a_k (no mixed inequalities). In (1.284) one has $\|T\| = a_1(T)$, but not necessarily $\|T\| = e_1(T)$ in case of quasi-Banach spaces.

The following interpolation result for entropy numbers will be of some service for us. We assume that the reader is familiar with basic assertions for real interpolation as it may be found in [Triα, Section 1.3, pp. 23–27] or [Triβ, Section 2.4.1, pp. 62–64].

Proposition 1.91.

(i) *Let A be a quasi-Banach space and let $\{B_0, B_1\}$ be an interpolation couple of quasi-Banach spaces. Let $0 < \theta < 1$ and let B_θ be a quasi-Banach space such that*

$$B_0 \cap B_1 \hookrightarrow B_\theta \hookrightarrow B_0 + B_1 \qquad (\textit{naturally quasi-normed}) \tag{1.287}$$

and

$$\|b\,|B_\theta\| \leq \|b\,|B_0\|^{1-\theta}\, \|b\,|B_1\|^\theta \quad \textit{for all } b \in B_0 \cap B_1. \tag{1.288}$$

Let $T \in L(A, B_0 \cap B_1)$. Then there is a number $c > 0$ such that for all $k \in \mathbb{N}$,

$$e_{2k}(T : A \hookrightarrow B_\theta) \leq c\, e_k^{1-\theta}(T : A \hookrightarrow B_0) \cdot e_k^\theta(T : A \hookrightarrow B_1). \tag{1.289}$$

(ii) *Let* $\{A_0, A_1\}$ *be an interpolation couple of quasi-Banach spaces and let* B *be a quasi-Banach space. Let* $0 < \theta < 1$, $0 < q \le \infty$ *and* $A = (A_0, A_1)_{\theta,q}$. *Let* $T : A_0 + A_1 \hookrightarrow B$ *be linear such that its restriction to* A_0 *and* A_1 *are continuous. Then its restriction to* A *is also continuous and there is a number* $c > 0$ *such that for all* $k \in \mathbb{N}$,

$$e_{2k}(T : A \hookrightarrow B) \le c\, e_k^{1-\theta}(T : A_0 \hookrightarrow B) \cdot e_k^{\theta}(T : A_1 \hookrightarrow B). \qquad (1.290)$$

Remark 1.92. We refer to [ET96], Section 1.3.3, pp. 15–18, for a sharper assertion where we cared also for some constants. It goes back to [HaT94a]. In case of Banach spaces it has some history where one finds the necessary references in [ET96], p. 13.

Theorem 1.93. *Let* $T \in L(B)$ *be compact and let* $\{\lambda_k(T)\}_{k\in\mathbb{N}}$ *be the sequence of all non-zero eigenvalues of* T, *repeated according to algebraic multiplicity and ordered so that*

$$|\lambda_1(T)| \ge |\lambda_2(T)| \ge \cdots > 0. \qquad (1.291)$$

(If T *has only* M *eigenvalues then we put* $\lambda_m(T) = 0$ *if* $m > M$.*)*

(i) *Then*

$$|\lambda_k(T)| \le \sqrt{2}\, e_k(T), \qquad k \in \mathbb{N}. \qquad (1.292)$$

(ii) *If, in addition,* $B = H$ *is a Hilbert space, and if* T *is self-adjoint, then*

$$|\lambda_k(T)| = a_k(T), \qquad k \in \mathbb{N}. \qquad (1.293)$$

Remark 1.94. A proof of the well-known assertion (1.293) may be found in [EdE87], Theorem II.5.10, p. 91. Recall that (1.292) is Carl's inequality proved for Banach spaces in a wider context in [Carl81]. The alternative proof given in [CaT80] was extended in [ET96], Section 1.3.4, pp. 18–21, to quasi-Banach spaces.

1.11 Function spaces on domains

In this subsection we collect some properties of function spaces of type $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$ on bounded domains Ω in $\mathbb{R}^n$ and prepare our later more specific considerations in Chapter 4. We are mainly interested in bounded Lipschitz domains. But first we deal with those assertions which remain valid also for more general bounded domains.

1.11.1 General domains: definitions, embeddings

We fix some notation. Let Ω be an arbitrary domain in $\mathbb{R}^n$. Domain means open set. Then $L_p(\Omega)$ with $0 < p \le \infty$ is the quasi-Banach space of all complex-valued

Lebesgue measurable functions in Ω such that

$$\|f\,|L_p(\Omega)\| = \left(\int_\Omega |f(x)|^p \, \mathrm{d}x \right)^{1/p} \tag{1.294}$$

(with the usual modification if $p = \infty$) is finite. As usual, $D(\Omega) = C_0^\infty(\Omega)$ stands for the collection of all complex-valued infinitely differentiable functions in $\mathbb{R}^n$ with compact support in Ω. Let $D'(\Omega)$ be the dual space of distributions on Ω. Let $g \in S'(\mathbb{R}^n)$. Then we denote by $g|\Omega$ its restriction to Ω,

$$g|\Omega \in D'(\Omega): \quad (g|\Omega)(\varphi) = g(\varphi) \quad \text{for} \quad \varphi \in D(\Omega). \tag{1.295}$$

Let $A^s_{pq}(\mathbb{R}^n)$ be the spaces introduced in Definition 1.2 where either $A = B$ or $A = F$.

Definition 1.95. *Let Ω be a domain in $\mathbb{R}^n$. Let $s \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q \leq \infty$ (with $p < \infty$ for the F-spaces).*

(i) *Then $A^s_{pq}(\Omega)$ is the collection of all $f \in D'(\Omega)$ such that there is a $g \in A^s_{pq}(\mathbb{R}^n)$ with $g|\Omega = f$. Furthermore,*

$$\|f\,|A^s_{pq}(\Omega)\| = \inf \|g\,|A^s_{pq}(\mathbb{R}^n)\|, \tag{1.296}$$

where the infimum is taken over all $g \in A^s_{pq}(\mathbb{R}^n)$ such that its restriction $g|\Omega$ to Ω coincides in $D'(\Omega)$ with f.

(ii) *Then $\mathring{A}^s_{pq}(\Omega)$ is the completion of $D(\Omega)$ in $A^s_{pq}(\Omega)$.*

(iii) *Then $\widetilde{A}^s_{pq}(\Omega)$ is the collection of all $f \in D'(\Omega)$ such that there is a*

$$g \in A^s_{pq}(\mathbb{R}^n) \quad \textit{with} \quad g|\Omega = f \quad \textit{and} \quad \operatorname{supp} g \subset \overline{\Omega}. \tag{1.297}$$

Furthermore, $\|f\,|\widetilde{A}^s_{pq}(\Omega)\|$ is given by (1.296) with $\widetilde{A}^s_{pq}(\Omega)$ in place of $A^s_{pq}(\Omega)$ where the infimum is taken over all g with (1.297).

(iv) *Let $\widetilde{A}^s_{pq}(\overline{\Omega})$ be the closed subspace of $A^s_{pq}(\mathbb{R}^n)$ given by*

$$\widetilde{A}^s_{pq}(\overline{\Omega}) = \{f \in A^s_{pq}(\mathbb{R}^n) : \operatorname{supp} f \subset \overline{\Omega}\}. \tag{1.298}$$

Remark 1.96. By standard arguments it follows that all these spaces are quasi-Banach spaces (Banach spaces if $p \geq 1$ and $q \geq 1$). Spaces on domains have been considered from the very beginning of the theory of function spaces and almost all books and papers listed in the introductory Section 1.1 deal also with spaces on domains. In [Triϵ, Sections 5 and 6] we studied spaces of the above type in detail under the assumption that Ω is a bounded C^∞ domain. We relied there on [Tri99]. The related classical theory may be found in [Triα, Chapter 4]. In [Tri02a] we extended this theory to bounded Lipschitz domains. We return to this subject in Section 1.11.6 below. As far as classical Sobolev spaces $W^m_p(\Omega)$ with $1 < p < \infty$,

$m \in \mathbb{N}$, and special Besov spaces $B^s_{pp}(\Omega)$ with $1 < p < \infty$ and $0 < s \neq \mathbb{N}$ in non-smooth domains are concerned we refer also to [Gri85]. Otherwise in what follows we rely partly on [Tri02a] concentrating us first on the spaces $A^s_{pq}(\Omega)$ and shifting more detailed assertions for the spaces in the parts (ii)–(iv) of the above definition to Section 1.11.6.

Some assertions for the spaces $A^s_{pq}(\mathbb{R}^n)$ can be extended to the spaces $A^s_{pq}(\Omega)$ as an immediate consequence of the above definition. This applies in particular to embeddings. We mention a few of them.

(i) Let $s \in \mathbb{R}$, $0 < p < \infty$, and q, u, $v \in (0, \infty]$. Then

$$B^s_{pu}(\Omega) \hookrightarrow F^s_{pq}(\Omega) \hookrightarrow B^s_{pv}(\Omega) \tag{1.299}$$

if, and only if, $0 < u \leq \min(p, q)$ and $\max(p, q) \leq v \leq \infty$.

(ii) Let

$$s \in \mathbb{R}, \quad 0 < p_0 < p < p_1 \leq \infty, \quad s_0 - \frac{n}{p_0} = s - \frac{n}{p} = s_1 - \frac{n}{p_1} \tag{1.300}$$

and q, u, $v \in (0, \infty]$. Then

$$B^{s_0}_{p_0 u}(\Omega) \hookrightarrow F^s_{pq}(\Omega) \hookrightarrow B^{s_1}_{p_1 v}(\Omega) \tag{1.301}$$

if, and only if, $0 < u \leq p \leq v \leq \infty$ with $p < \infty$.

(iii) Let

$$s_0 \in \mathbb{R}, \quad 0 < p_0 < p_1 < \infty, \quad s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}, \tag{1.302}$$

and u, $v \in (0, \infty]$. Then

$$F^{s_0}_{p_0 u}(\Omega) \hookrightarrow F^{s_1}_{p_1 v}(\Omega). \tag{1.303}$$

The 'if'-parts of these embeddings with $\mathbb{R}^n$ in place of Ω may be found in [Triβ], pp. 47, 129, 131, and in [Jaw77], [Fra86a] as far as (1.301) is concerned. As for the respective 'only-if'-parts we refer to [SiT95]. A short description and further embeddings may also be found in [ET96], Section 2.2.3, pp. 43–45. These assertions extend immediately from $\mathbb{R}^n$ to Ω.

1.11.2 General domains: entropy numbers

If Ω is a bounded domain, then one may ask whether the embeddings mentioned in Section 1.11.1 are even compact. But this is not so in the listed limiting cases. However the situation is much better if one replaces the equalities in (1.300) by strict inequalities. Recall that e_k are the entropy numbers according to Definition 1.87.

Theorem 1.97. *Let* Ω *be an arbitrary bounded domain in* $\mathbb{R}^n$. *Let* p_0, p_1, q_0, $q_1 \in (0,\infty]$, *and*

$$-\infty < s_1 < s_0 < \infty, \qquad s_0 - \frac{n}{p_0} > s_1 - \frac{n}{p_1}. \tag{1.304}$$

Then the embedding

$$id: \qquad B^{s_0}_{p_0 q_0}(\Omega) \hookrightarrow B^{s_1}_{p_1 q_1}(\Omega) \tag{1.305}$$

is compact and

$$e_k(id) \sim k^{-\frac{s_0-s_1}{n}}, \qquad k \in \mathbb{N}. \tag{1.306}$$

Remark 1.98. This theorem has a substantial history which may be found in [ET96], Section 3.3.5, pp. 126–128, which will not be repeated here. The first complete proof for bounded C^∞ domains may be found in [ET96], Section 3.3, pp. 105–118, based on [EdT89], [EdT92]. An extension of this assertion to more general bounded domains was given in [TrW96] and in [ET96], Section 3.5, pp. 151–152. The above version coincides with [Triδ], Theorem 23.2, p. 186. We use $\sim$ here and in the sequel,

$$a(x) \sim b(x) \qquad \text{or} \qquad a_k \sim b_k \tag{1.307}$$

for two positive functions $a(x)$ and $b(x)$ or for two sequences of positive numbers a_k and b_k with k in some countable index set if there are two positive numbers c and C such that

$$c\, a(x) \le b(x) \le C\, a(x) \qquad \text{or} \qquad c\, a_k \le b_k \le C\, a_k \tag{1.308}$$

for all admitted variables x and k. By (1.299) one can replace B in (1.305), (1.306) by F. It is well known that (1.304) is also necessary for compact embeddings. One may consult the beginning of the proof of Proposition 6.29 below.

1.11.3 General domains: atoms

So far Definition 1.95 and Theorem 1.97 apply to arbitrary bounded domains. But this generality is somewhat doubtful. For example, let g be a continuous function on $\mathbb{R}^n$. Then, of course, $f = g|\Omega$ can be extended continuously to $\overline{\Omega}$. Coming that way, isolated points, or surfaces of lower dimensions within the open interior of $\overline{\Omega}$ are completely ignored. If g is a singular distribution then the situation might be a little bit more delicate. Nevertheless it seems to be reasonable to assume that Ω can be recovered by taking the interior (inner points) of its closure, hence

$$\Omega = \text{int}\,(\overline{\Omega}). \tag{1.309}$$

We refer to [TrW96] for further details and discussions. It was the main aim of this paper to give intrinsic characterisations of the spaces introduced in Definition

1.95 by using atoms in domains under minimal assumptions for Ω. This can be done for the spaces

$$B^s_{pq}(\Omega); \quad 0 < p \le \infty, \quad 0 < q \le \infty, \quad s > \sigma_p = n\left(\frac{1}{p} - 1\right)_+, \tag{1.310}$$

only assuming that the bounded domain Ω satisfies in addition (1.309). If $s \le \sigma_p$ or in case of the spaces $F^s_{pq}(\Omega)$ one needs a few additional mild restrictions. We refer to [TrW96]. We presented this theory also in [ET96], Section 2.5, in detail, but without proofs. We do not repeat this description here, but restrict ourselves to the spaces in (1.310). This illustrates how well this theory fits in the scheme of atomic decompositions according to Section 1.5. First we introduce some relevant notation. For this purpose we adapt Definition 1.21 and the sequence spaces b_{pq} according to Definition 1.17 appropriately.

Now we always assume that Ω is a bounded domain in $\mathbb{R}^n$ satisfying (1.309). Let $0 < \sigma = [\sigma] + \{\sigma\}$ with $[\sigma] \in \mathbb{N}_0$ and $0 < \{\sigma\} < 1$. According to Definition 1.95 the Hölder-Zygmund space $\mathcal{C}^\sigma(\Omega)$ is the restriction of $\mathcal{C}^\sigma(\mathbb{R}^n) = B^\sigma_{\infty\infty}(\mathbb{R}^n)$ on Ω, where one may assume that the latter is normed by (1.75). In particular any $f \in \mathcal{C}^\sigma(\Omega)$ has classical derivatives $D^\alpha f$ for $|\alpha| \le [\sigma]$ in Ω which can be extended continuously to $\overline{\Omega}$ and

$$\sum_{|\alpha|\le[\sigma]} \|D^\alpha f \,|L_\infty(\Omega)\| + \sum_{|\alpha|=[\sigma]} \sup \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x-y|^{\{\sigma\}}} \le \|f \,|\mathcal{C}^\sigma(\Omega)\| \tag{1.311}$$

is finite, where the supremum is taken over all $x \in \overline{\Omega}$, $y \in \overline{\Omega}$ with $x \neq y$. In general the two norms in (1.311) need not to be equivalent (correcting a corresponding assertion in [TrW96, p. 658] and [ET96, pp. 60/61]). But $\mathcal{C}^\sigma(\Omega)$ can be normed intrinsically in the framework of Whitney extensions,[JoW84, pp. 22,44/45], [Ste70, p. 173]. Let

$$\Omega^\nu = \left\{x \in \mathbb{R}^n : 2^{-\nu}x \in \Omega\right\}, \qquad \nu \in \mathbb{N}_0, \tag{1.312}$$

be the dilated domain Ω. Let $Q_{\nu m}$ and $c\,Q_{\nu m}$ with $c > 1$ be the same cubes in $\mathbb{R}^n$ as at the beginning of Section 1.5.1.

Definition 1.99. *Let Ω be a bounded domain in $\mathbb{R}^n$ satisfying* (1.309).

(i) *Let $0 < p \le \infty$, $0 < q \le \infty$, $c > 1$. Let*

$$\lambda = \{\lambda_{\nu m} \in \mathbb{C} : \nu \in \mathbb{N}_0, \ m \in \mathbb{Z}^n \text{ with } cQ_{\nu m} \cap \overline{\Omega} \neq \emptyset\}, \tag{1.313}$$

and

$$\|\lambda \,|b_{pq}(\Omega)\| = \left(\sum_{\nu=0}^\infty \left(\sum_{m\in\mathbb{Z}^n}^{\nu,\Omega} |\lambda_{\nu m}|^p\right)^{q/p}\right)^{1/q} \tag{1.314}$$

where the summation is restricted to the couples (ν, m) according to (1.313). *Then*

$$b_{pq}(\Omega) = \{\lambda : \|\lambda \,|b_{pq}(\Omega)\| < \infty\}. \tag{1.315}$$

(ii) *Let* $s \in \mathbb{R}$, $0 < p \le \infty$, $c > 1$, *and* $0 < \sigma \notin \mathbb{N}$. *Then* $a : \overline{\Omega} \to \mathbb{C}$ *is called an* $(s,p)_\sigma$*-atom if*

$$\operatorname{supp} a \subset c\,Q_{\nu m} \cap \overline{\Omega} \quad \textit{for some} \quad \nu \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n, \tag{1.316}$$

with $cQ_{\nu m} \cap \overline{\Omega} \neq \emptyset$ *and*

$$a \in \mathcal{C}^\sigma(\Omega) \quad \textit{with} \quad \|a(2^{-\nu}\cdot)\,|\mathcal{C}^\sigma(\Omega^\nu)\| \le 2^{-\nu(s-\frac{n}{p})}. \tag{1.317}$$

Remark 1.100. Part (i) is the counterpart of (1.64), (1.66). There is an obvious counterpart $f_{pq}(\Omega)$ of f_{pq}. Part (ii) is the adequate modification of Definition 1.21 without moment conditions. As before we write $a_{\nu m}$ if the $(s,p)_\sigma$-atom is supported by (1.316).

Theorem 1.101. *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *satisfying* (1.309). *Let* $0 < p \le \infty$, $0 < q \le \infty$, *and*

$$\sigma_p < s < \sigma \notin \mathbb{N}. \tag{1.318}$$

Then $B^s_{pq}(\Omega)$ *is the collection of all* $f \in L_1(\Omega)$ *which can be represented as*

$$f = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n}^{\nu,\Omega} \lambda_{\nu m}\, a_{\nu m}, \qquad \lambda \in b_{pq}(\Omega), \tag{1.319}$$

where $a_{\nu m}$ *are* $(s,p)_\sigma$*-atoms according to Definition* 1.99 *for fixed* $c > 1$. *The series on the right-hand side of* (1.319) *converges absolutely in* $L_1(\Omega)$. *Furthermore,*

$$\|f\,|B^s_{pq}(\Omega)\| \sim \inf \|\lambda\,|b_{pq}(\Omega)\| \tag{1.320}$$

are equivalent quasi-norms where the infimum is taken over all admissible representations (1.319).

Remark 1.102. This follows from the presentation given in [ET96], pp. 63–65, and the underlying paper [TrW96]. It is the direct counterpart both of Corollary 1.23 and Theorem 1.29. The absolute convergence in $L_1(\Omega)$ can be replaced by the absolute convergence in $L_r(\Omega)$ with $1 < r \le \infty$ and $s - \frac{n}{p} > -\frac{n}{r}$. The above theorem covers the spaces in (1.310). In the case of $B^s_{pq}(\Omega)$ with $s \le \sigma_p$ or the F-spaces one has similar assertions. Now one needs mild additional assumptions about Ω and the atoms, satisfying some moment conditions. We refer for details to [TrW96] and the report about these results given in [ET96], Section 2.5. In all cases the mild additional conditions for Ω are of such a type that they are satisfied for *bounded Lipschitz domains* which we are going to define now.

1.11.4 Lipschitz domains: definitions

Let $2 \le n \in \mathbb{N}$. Then

$$\mathbb{R}^{n-1} \ni x' \mapsto h(x') \in \mathbb{R} \tag{1.321}$$

is called a Lipschitz function (on $\mathbb{R}^{n-1}$) if there is a number $c > 0$ such that

$$|h(x') - h(y')| \le c\,|x' - y'| \qquad \text{for all} \quad x' \in \mathbb{R}^{n-1}, \quad y' \in \mathbb{R}^{n-1}. \tag{1.322}$$

Definition 1.103. *Let $n \in \mathbb{N}$.*

(i) *A special Lipschitz domain in $\mathbb{R}^n$ with $n \geq 2$ is the collection of all points $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$ such that*

$$h(x') < x_n < \infty, \tag{1.323}$$

where $h(x')$ is a Lipschitz function according to (1.321), (1.322).

(ii) *A bounded Lipschitz domain in $\mathbb{R}^n$ with $n \geq 2$ is a bounded domain Ω in $\mathbb{R}^n$ where the boundary $\partial\Omega$ can be covered by finitely many open balls B_j in $\mathbb{R}^n$ where $j = 1, \ldots, J$, centred at $\partial\Omega$ such that*

$$B_j \cap \Omega = B_j \cap \Omega_j \qquad \text{with} \quad j = 1, \ldots, J, \tag{1.324}$$

where Ω_j are rotations of suitable special Lipschitz domains in $\mathbb{R}^n$.

(iii) *A bounded Lipschitz domain in the real line $\mathbb{R}$ is the interior of a finite union of disjoint bounded closed intervals.*

Remark 1.104. Of course we always assume that bounded Lipschitz domains are not empty. We describe briefly the so-called localisation method. If Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, then $\partial\Omega$ can be covered by J balls $B_j = B(x^j, r)$ centred at $x^j \in \partial\Omega$ and of radius $r > 0$. Let $\{\varphi_j\}_{j=1}^J \subset S(\mathbb{R}^n)$ with $0 \leq \varphi_j(x) \leq 1$ be a subordinated resolution of unity, hence

$$\operatorname{supp} \varphi_j \subset B_j \,, \qquad \sum_{j=1}^J \varphi_j(x) = 1 \text{ in a neighborhood of } \partial\Omega. \tag{1.325}$$

Let $A^s_{pq}(\Omega)$ be a space according to Definition 1.95. Then $\varphi_j f \in A^s_{pq}(\Omega)$ if $f \in A^s_{pq}(\Omega)$ and there is a number $c > 0$ such that

$$\|\varphi_j f \,|A^s_{pq}(\Omega)\| \leq c \, \|f \,|A^s_{pq}(\Omega)\| \qquad \text{for all} \quad f \in A^s_{pq}(\Omega). \tag{1.326}$$

This follows from the corresponding property for the spaces $A^s_{pq}(\mathbb{R}^n)$ and Definition 1.95. Many properties for spaces of type A^s_{pq} in domains can be reduced with the help of this localisation method to local investigations. This applies in particular to the above special Lipschitz domains and bounded Lipschitz domains. But it is quite clear that it works also more generally.

1.11.5 Lipschitz domains: extension

Let Ω be an (arbitrary) domain in $\mathbb{R}^n$. Then by Definition 1.95 the *restriction operator* re,

$$\text{re}\,(g) = g|\Omega : \qquad S'(\mathbb{R}^n) \mapsto D'(\Omega) \tag{1.327}$$

generates a linear and bounded operator

$$\text{re} : \quad A^s_{pq}(\mathbb{R}^n) \hookrightarrow A^s_{pq}(\Omega) \tag{1.328}$$

for all admitted $A = B$, $A = F$ and s, p, q. Of course, (1.328) is the restriction of (1.327) to $A^s_{pq}(\mathbb{R}^n)$. But as usual this will not be indicated by additional marks. This tacit agreement applies also to other related operators such as extension operators or identification operators acting between diverse spaces. Simply by definition there is a (non-linear) bounded extension operator from $A^s_{pq}(\Omega)$ into $A^s_{pq}(\mathbb{R}^n)$. This is sufficient to prove Theorem 1.97 for arbitrary bounded domains. But for many other problems it is desirable to know whether there is a linear and bounded *extension operator* ext,

$$\mathrm{ext} : \quad A^s_{pq}(\Omega) \hookrightarrow A^s_{pq}(\mathbb{R}^n) \tag{1.329}$$

such that

$$\mathrm{re} \circ \mathrm{ext} = \mathrm{id} \qquad (\text{identity in } A^s_{pq}(\Omega)). \tag{1.330}$$

This is the so-called *extension problem.* If Ω is a bounded C^∞ domain in $\mathbb{R}^n$ then this problem has been solved in a satisfactory way in [Triγ], Section 4.5, pp. 221–227. There one finds also the necessary references and historical comments, including our own contributions. It is the first step to use the localisation method as described briefly in Remark 1.104 and to reduce afterwards the problem via local C^∞ diffeomorphic maps $y = \psi(x)$ to the half-space $\mathbb{R}^n_+$, to construct there explicitly linear extension operators from $\mathbb{R}^n_+$ to $\mathbb{R}^n$, and to return afterwards to the bounded C^∞ domain. According to [Triγ], Corollary 4.5.2, p. 225, one finds in this way for any $\varepsilon > 0$ a common (linear) extension operator ext^ε,

$$\mathrm{ext}^\varepsilon : \begin{cases} B^s_{pq}(\Omega) \hookrightarrow B^s_{pq}(\mathbb{R}^n), & |s| < \varepsilon^{-1}, \ \varepsilon < p \le \infty, \ 0 < q \le \infty, \\ F^s_{pq}(\Omega) \hookrightarrow F^s_{pq}(\mathbb{R}^n), & |s| < \varepsilon^{-1}, \ \varepsilon < p < \infty, \ \varepsilon < q \le \infty. \end{cases} \tag{1.331}$$

Here *common extension operator* means that this operator is defined on the union of all admitted spaces such that its restriction to one of these spaces is a linear and bounded operator according to (1.331). But this method does not work in this generality if Ω is a (special or bounded) Lipschitz domain. One has the localisation method as described in Remark 1.104, but the reduction of the extension problem to $\mathbb{R}^n_+$ by diffeomorphic maps $y = \psi(x)$ works only under severe restrictions for s, p, q. In general one needs other arguments. On the other hand, the extension problem for bounded Lipschitz domains attracted a lot of attention since the 1960s. By Calderón's extension method (1960/61), combined with some interpolation, one gets for bounded Lipschitz domains Ω in $\mathbb{R}^n$ the following assertion: For any $N \in \mathbb{N}$, there is a common extension operator ext^N for all spaces

$$H^s_p(\Omega) \text{ and } B^s_{pq}(\Omega), \quad 0 < s < N, \ 1 < p < \infty, \ 0 < q \le \infty. \tag{1.332}$$

We refer to [Triα], Section 4.2.3, pp. 312–315, where one finds also the necessary references. This result was extended by E.M. Stein in [Ste70], p. 181, combined with some interpolation, by constructing a common extension operator ext^∞ for all spaces

$$H^s_p(\Omega) \text{ and } B^s_{pq}(\Omega), \quad s > 0, \ 1 < p < \infty, \ 0 < q \le \infty. \tag{1.333}$$

It will be of interest for us later on that Stein's method works also for the Sobolev spaces

$$W_1^k(\Omega) \quad \text{and} \quad W_\infty^k(\Omega) \qquad \text{with} \quad k \in \mathbb{N}_0. \tag{1.334}$$

Afterwards, G.A. Kaljabin proved in [Kal85a], Theorem 1, that Stein's extension operator ext^∞ is also a common extension operator for all spaces

$$F^s_{pq}(\Omega), \qquad s>0, \quad 1<p<\infty, \quad 1<q<\infty, \tag{1.335}$$

(and also for more general spaces). We refer also to [Kal83] and [KaL87]. The final step is due to V.S. Rychkov in [Ry99b]. We call an extension operator *universal* if it is a common linear extension operator for all spaces $A^s_{pq}(\Omega)$ according to Definition 1.95.

Theorem 1.105. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Then there is a universal extension operator.*

Remark 1.106. This universal extension operator is explicitly constructed in [Ry99b] with [Ry98] as a forerunner. It is based on a Calderón reproducing formula. This gives the possibility to decide intrinsically to which space $A^s_{pq}(\Omega)$ a given element $f \in D'(\Omega)$ or $f \in \mathrm{re}\, S'(\mathbb{R}^n)$ belongs: For the related quasi-norms one needs only the knowledge of f in Ω (and not of some $g \in S'(\mathbb{R}^n)$ with $g|\Omega = f$). We return to the problem of intrinsic descriptions later on.

1.11.6 Lipschitz domains: subspaces

We return now to the spaces introduced in Definition 1.95(ii)–(iv) under the assumption that Ω is a bounded Lipschitz domain according to Definition 1.103. We collect a few properties which will be useful later on, referring for proofs and further details to [Tri02a].

(i) *Let $0<p\le\infty$ ($p<\infty$ for the F-spaces), $0<q\le\infty$ and*

$$\max\Big(\frac{1}{p}-1, n(\frac{1}{p}-1)\Big) < s < \infty. \tag{1.336}$$

Then

$$\widetilde{A}^s_{pq}(\Omega) = \widetilde{A}^s_{pq}(\overline{\Omega}), \tag{1.337}$$

and (*hence*)

$$\|f\,|\widetilde{A}^s_{pq}(\Omega)\| = \|f\,|A^s_{pq}(\mathbb{R}^n)\|, \quad f\in A^s_{pq}(\mathbb{R}^n), \quad \mathrm{supp}\, f \subset \overline{\Omega}. \tag{1.338}$$

(ii) *Let*

$$0<p<\infty, \quad 0<q<\infty, \quad \max\Big(\frac{1}{p}-1, n(\frac{1}{p}-1)\Big) < s < \frac{1}{p} \tag{1.339}$$

and $q \ge \min(p,1)$ for the F-spaces. Then

$$\mathring{A}^s_{pq}(\Omega) = A^s_{pq}(\Omega) = \widetilde{A}^s_{pq}(\Omega). \tag{1.340}$$

We refer to [Tri02a, Proposition 3.1]. Under the additional assumption that Ω is a bounded C^∞ domain in $\mathbb{R}^n$, $0 < p < \infty$, $0 < q < \infty$,

$$s > n\left(\frac{1}{p} - 1\right)_+ \qquad \text{and} \qquad s - \frac{1}{p} \notin \mathbb{N}_0 \tag{1.341}$$

one has

$$\mathring{A}^s_{pq}(\Omega) = \widetilde{A}^s_{pq}(\Omega). \tag{1.342}$$

This has been stated in [Triϵ, 5.22, p. 69] with references to the original papers. But it is a tricky subject. Assertions of this type for classical Sobolev-Besov spaces may be found in [Triα, Section 4.3.2]. It is not possible to extend (1.341) to $s - 1/p \in \mathbb{N}_0$, [Triα, Section 4.3.2], [Triϵ, 5.23, p. 70]. There are many good reasons to consider for bounded Lipschitz domains Ω,

$$\left\{\bar{A}^s_{pq}(\Omega),\ 1 < p < \infty,\ 1 < q < \infty\right\} \tag{1.343}$$

with

$$\bar{A}^s_{pq}(\Omega) = \begin{cases} \widetilde{A}^s_{pq}(\Omega) & \text{if } s > \frac{1}{p} - 1, \\ A^s_{pq}(\Omega) & \text{if } s < \frac{1}{p}, \end{cases} \tag{1.344}$$

as the right substitute of the scales of spaces $A^s_{pq}(\mathbb{R}^n)$. The overlap in (1.344) is justified by (1.339), (1.340). We refer for further details to [Tri02a, Section 3].

1.11.7 Lipschitz domains: approximation numbers

Under the hypotheses of Theorem 1.97 we have the equivalence (1.306) for the entropy numbers of the compact embedding (1.305). One may ask what can be said for other quantities measuring compactness. Of interest for us are approximation numbers as introduced in Definition 1.87. But now one needs additional information about the underlying bounded domain. If Ω is a bounded C^∞ domain then one has a final answer with the exception of a few limiting cases. This can be extended to bounded Lipschitz domains. Recall that $b_+ = \max(b, 0)$ if $b \in \mathbb{R}$. Furthermore if $1 \le p \le \infty$ then p' is given by $\frac{1}{p} + \frac{1}{p'} = 1$. If $0 < p < 1$ then we put $p' = \infty$.

Theorem 1.107. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Let p_0, p_1, q_0, $q_1 \in (0, \infty]$ and*

$$-\infty < s_1 < s_0 < \infty, \qquad s_0 - \frac{n}{p_0} > s_1 - \frac{n}{p_1}. \tag{1.345}$$

Then the embedding

$$\mathrm{id} : \quad B^{s_0}_{p_0 q_0}(\Omega) \hookrightarrow B^{s_1}_{p_1 q_1}(\Omega) \tag{1.346}$$

is compact. Let

$$\begin{aligned} \delta_+ &= s_0 - s_1 - n\left(\frac{1}{p_0} - \frac{1}{p_1}\right)_+, \\ \lambda &= \frac{s_0 - s_1}{n} - \max\left(\frac{1}{2} - \frac{1}{p_1}, \frac{1}{p_0} - \frac{1}{2}\right). \end{aligned} \tag{1.347}$$

Then for $k \in \mathbb{N}$,

$$a_k(\mathrm{id}) \sim k^{-\delta_+/n} \quad \textit{if} \quad \begin{cases} \textit{either} & 0 < p_0 \le p_1 \le 2, \\ \textit{or} & 2 \le p_0 \le p_1 \le \infty, \\ \textit{or} & 0 < p_1 \le p_0 \le \infty, \end{cases} \tag{1.348}$$

$$a_k(\mathrm{id}) \sim k^{-\lambda} \qquad \textit{if} \quad 0 < p_0 < 2 < p_1 < \infty, \quad \lambda > 1/2, \tag{1.349}$$

and

$$a_k(\mathrm{id}) \sim k^{-\frac{\delta_+}{n}\cdot \frac{\min(p_0',p_1)}{2}} \quad \textit{if} \quad 0 < p_0 < 2 < p_1 < \infty, \quad \lambda < 1/2. \tag{1.350}$$

Proof. If Ω is a bounded C^∞ domain in $\mathbb{R}^n$ then the above assertions are covered by [ET96], Section 3.3.4, p. 119, and [Cae98], Theorem 3.1, p. 387, where the latter paper deals with the sophisticated case (1.350). Let Ω be a Lipschitz domain and let Ω_0 and Ω_1 be two open balls with

$$\overline{\Omega_0} \subset \Omega \qquad \text{and} \qquad \overline{\Omega} \subset \Omega_1. \tag{1.351}$$

Let ext_Ω be the extension operator according to Theorem 1.105, multiplied with a suitable cut-off function such that

$$\mathrm{ext}_\Omega \; : \qquad B^s_{pq}(\Omega) \hookrightarrow B^s_{pq}(\Omega_1). \tag{1.352}$$

Similarly ext_{Ω_0} with respect to Ω_0. Let $\mathrm{id}_\Omega = \mathrm{id}$ be given by (1.346), similarly id_{Ω_0} and id_{Ω_1}. Then one has in obvious notation

$$\mathrm{id}_\Omega = \mathrm{re}_\Omega \circ \mathrm{id}_{\Omega_1} \circ \mathrm{ext}_\Omega \tag{1.353}$$

and

$$\mathrm{id}_{\Omega_0} = \mathrm{re}_{\Omega_0} \circ \mathrm{id}_\Omega \circ \mathrm{ext}_{\Omega_0}. \tag{1.354}$$

By Proposition 1.89 it follows that

$$c\, a_k(\mathrm{id}_{\Omega_0}) \le a_k(\mathrm{id}_\Omega) \le c'\, a_k(\mathrm{id}_{\Omega_1}), \qquad k \in \mathbb{N}, \tag{1.355}$$

for some $c > 0$ and $c' > 0$. Now the above assertions for $a_k(\mathrm{id}_\Omega)$ follow from the corresponding assertions for bounded C^∞ domains. □

Remark 1.108. With the exception of $p_1 = \infty$ and $\lambda = 1/2$ in (1.349), (1.350) the theorem covers all cases. We return to a special case of the above theorem later on in Chapter 4 when comparing sampling numbers introduced there with approximation numbers and entropy numbers. Since all assertions in the above theorem are independent of q_0 and q_1 it follows by (1.299) that one can replace in the above theorem B by F (with $p_0 < \infty$ and $p_1 < \infty$). In particular the above theorem holds also for the Sobolev spaces $H^s_p(\Omega) = F^s_{p,2}(\Omega)$, extended to $p \le 1$.

Remark 1.109. The above proof extends a known assertion for bounded C^∞ domains to bounded Lipschitz domains by using (1.353), (1.354), and (1.355) where the latter is based on Proposition 1.89. But there are many other numbers measuring the compactness of mappings, including embedding operators of type (1.346), at least in the case of Banach spaces, for example Kolmogorov numbers, Gelfand numbers, Weyl numbers etc. One may consult the literature mentioned in Remark 1.88. One has to check whether these numbers can be extended to quasi-Banach function spaces of the above type, in particular to the embedding (1.346) and whether one has counterparts of Proposition 1.89. Then one can extend by the above arguments corresponding assertions for bounded C^∞ domains to bounded Lipschitz domains.

1.11.8 Lipschitz domains: interpolation

We assume that the reader is familiar with the basic assertions of interpolation theory. Let $\{A_0, A_1\}$ be an interpolation couple of complex quasi-Banach spaces. Then

$$(A_0, A_1)_{\theta,q}, \qquad 0 < \theta < 1, \qquad 0 < q \le \infty, \tag{1.356}$$

denotes, as usual, the real interpolation method based on Peetre's K-functional. Let

$$[A_0, A_1]_\theta, \qquad 0 < \theta < 1, \tag{1.357}$$

be the classical complex interpolation method as introduced by A.P. Calderón which is restricted to complex Banach spaces. Basic assertions about these interpolation methods may be found in [Triα] and [BeL76]. One may also consult [Triβ], Section 2.4, and [Triγ], Section 1.6. What follows might be considered as a continuation of the last two references. Let $I(A_0, A_1)$ be either the real interpolation method for quasi-Banach spaces or the classical complex interpolation method for Banach spaces. If $A(\mathbb{R}^n)$ is a given B-space or F-space according to Definition 1.2 then $A(\Omega)$ stands for its restriction to Ω as introduced in Definition 1.95.

Theorem 1.110. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Let I be either the real interpolation method for quasi-Banach spaces or the classical complex interpolation method for Banach spaces. Let $A_0(\mathbb{R}^n)$ and $A_1(\mathbb{R}^n)$ be two admitted spaces according to Definition* 1.2 *such that*

$$I\left(A_0(\mathbb{R}^n), A_1(\mathbb{R}^n)\right) = A(\mathbb{R}^n) \tag{1.358}$$

results in a space $A(\mathbb{R}^n)$ again covered by Definition 1.2. *Then*

$$I\left(A_0(\Omega), A_1(\Omega)\right) = A(\Omega). \tag{1.359}$$

Proof. Step 1. To make clear what is going on it is sufficient to deal with an example. Let

$$A_k(\mathbb{R}^n) = B^{s_k}_{pq_k}(\mathbb{R}^n), \qquad \text{where} \quad k = 0 \text{ or } k = 1, \tag{1.360}$$

with p, q_0, $q_1 \in (0,\infty]$ and $-\infty < s_0 < s_1 < \infty$. Let

$$0 < \theta < 1, \quad 0 < q \le \infty \quad \text{and} \quad s = (1-\theta)s_0 + \theta s_1. \tag{1.361}$$

Then, according to [Triβ], Theorem 2.4.2, p. 64,

$$\big(B^{s_0}_{pq_0}(\mathbb{R}^n), B^{s_1}_{pq_1}(\mathbb{R}^n) \big)_{\theta,q} = B^s_{pq}(\mathbb{R}^n). \tag{1.362}$$

Hence $A(\mathbb{R}^n) = B^s_{pq}(\mathbb{R}^n)$ in (1.358). We wish to prove that this interpolation formula remains valid with Ω in place of $\mathbb{R}^n$, this means

$$B_\theta(\Omega) = B^s_{pq}(\Omega), \tag{1.363}$$

where we put temporarily

$$\big(B^{s_0}_{pq_0}(\Omega), B^{s_1}_{pq_1}(\Omega) \big)_{\theta,q} = B_\theta(\Omega). \tag{1.364}$$

Let re and ext be the restriction operator and extension operator according to Theorem 1.105 and (1.330). Denoting the right-hand side of (1.362) by $B_\theta(\mathbb{R}^n)$ it follows by the interpolation property for the spaces on $\mathbb{R}^n$ and on Ω that

$$\begin{aligned} \|f \,|B_\theta(\Omega)\| &= \|\mathrm{re} \circ \mathrm{ext}\, f \,|B_\theta(\Omega)\| \\ &\le c \,\|\mathrm{ext}\, f \,|B_\theta(\mathbb{R}^n)\| \le c' \,\|f \,|B_\theta(\Omega)\|. \end{aligned} \tag{1.365}$$

Hence

$$\|f \,|B_\theta(\Omega)\| \sim \|\mathrm{ext}\, f \,|B^s_{pq}(\mathbb{R}^n)\| \sim \|f \,|B^s_{pq}(\Omega)\| \tag{1.366}$$

and (1.363). Then (1.364) proves (1.362) with Ω in place of $\mathbb{R}^n$.

Step 2. In the same way one gets (1.359) from (1.358), the extension property and the interpolation property. □

We fix a few examples.

Corollary 1.111. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Let $0 < \theta < 1$.*

(i) *Let q_0, q_1, $q \in (0,\infty]$. Let $-\infty < s_0 < s_1 < \infty$ and*

$$s = (1-\theta)s_0 + \theta s_1. \tag{1.367}$$

If $0 < p \le \infty$, then

$$\big(B^{s_0}_{pq_0}(\Omega), B^{s_1}_{pq_1}(\Omega) \big)_{\theta,q} = B^s_{pq}(\Omega) \tag{1.368}$$

and if $0 < p < \infty$, then

$$\big(F^{s_0}_{pq_0}(\Omega), F^{s_1}_{pq_1}(\Omega) \big)_{\theta,q} = B^s_{pq}(\Omega). \tag{1.369}$$

(ii) *Let p_0, p_1, $q_0, q_1 \in (1,\infty)$, $s_0 \in \mathbb{R}$, $s_1 \in \mathbb{R}$ and*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \tag{1.370}$$

Let s be given by (1.367). *Then*

$$\left[B^{s_0}_{p_0q_0}(\Omega), B^{s_1}_{p_1q_1}(\Omega)\right]_\theta = B^s_{pq}(\Omega) \tag{1.371}$$

and

$$\left[F^{s_0}_{p_0q_0}(\Omega), F^{s_1}_{p_1q_1}(\Omega)\right]_\theta = F^s_{pq}(\Omega). \tag{1.372}$$

Proof. This follows from Theorem 1.110 and the corresponding formulas with $\mathbb{R}^n$ in place of Ω according to [Triβ], Theorem 2.4.2, p. 64 for part (i) and [Triα], Theorem 2.4.1, p. 182, and Theorem 2.4.2/1, pp. 184–185 for part (ii). □

Remark 1.112. There are many other interpolation formulas for spaces on $\mathbb{R}^n$ in [Triβ], Section 2.4 and [Triα], Section 2.4. If they fit in the scheme of Theorem 1.110 then they can be carried over from $\mathbb{R}^n$ to Ω. If Ω is a bounded C^∞ domain then this has been done in detail in [Triα], Section 4.3.1, p. 317 and in [Triβ], Section 3.3.6, p. 204. But it is also clear that one has for the complex method some difficulties to incorporate limiting cases. For example, according to [Triα], p. 182, one can extend (1.371) (first on $\mathbb{R}^n$ and then on Ω) if $1 \le q_0 < \infty$, $1 \le q_1 \le \infty$, but not if $q_0 = q_1 = \infty$.

Remark 1.113. In contrast to the real method, the complex method cannot be extended immediately from Banach spaces to all quasi-Banach spaces on an abstract level. But several attempts have been made to circumvent the arising obstacles by modifying the method and restricting the considerations from the very beginning to the spaces $A^s_{pq}(\mathbb{R}^n)$ and $A^s_{pq}(\Omega)$. We refer to [Triβ], Section 2.4 (spaces on $\mathbb{R}^n$) and Section 3.3.6 (spaces on bounded C^∞ domains) for references and for our own contributions. The outcome looks perfect, extending (1.371), (1.372) with $\mathbb{R}^n$ in place of Ω to all parameters admitted for these spaces. However there is a drawback since the interpolation property is not satisfied automatically but must be checked from case to case (as we did in the applications in [Triβ]). A new attempt had been made in [FrJ90], §8, where one finds also further references. But the most promising extension of Calderón's original complex interpolation method from complex Banach spaces to the bigger class of so-called A-convex (analytically convex) complex quasi-Banach spaces is due to O. Mendez and M. Mitrea, [MeM00]. Restricted to this subclass of quasi-Banach spaces the interpolation property is always valid. It is one of the main aims of [MeM00], Section 4, to prove that all spaces

$$A^s_{pq}(\mathbb{R}^n), \qquad s \in \mathbb{R}, \quad 0 < p \le \infty, \quad 0 < q \le \infty, \tag{1.373}$$

with $p < \infty$ if $A = F$, are A-convex quasi-Banach spaces. Let us use the notation (1.357) also for this extended method. Again let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and let ext be the universal extension operator according to Theorem 1.105. Then we have (1.329), (1.330) for all spaces and

$$P = \mathrm{ext} \circ \mathrm{re} : \qquad A^s_{pq}(\mathbb{R}^n) \hookrightarrow A^s_{pq}(\mathbb{R}^n) \tag{1.374}$$

is a universal projector of $A^s_{pq}(\mathbb{R}^n)$ onto a complemented subspace of $A^s_{pq}(\mathbb{R}^n)$ denoted by $PA^s_{pq}(\mathbb{R}^n)$, and ext is an isomorphic map of $A^s_{pq}(\Omega)$ onto this subspace. Then it follows from [MeM00] and the corresponding arguments in [Tri02a], pp. 488/89, that $A^s_{pq}(\Omega)$ are also A-convex quasi-Banach spaces to which the complex interpolation method can be extended preserving the interpolation property. Hence assertions proved in [MeM00] for spaces on $\mathbb{R}^n$ can be extended to corresponding spaces on Ω.

Proposition 1.114. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Let $0 < \theta < 1$. Let p_0, p_1, $q_0, q_1 \in (0,\infty)$, $s_0 \in \mathbb{R}$ and $s_1 \in \mathbb{R}$. Let s, p, q be given as in* (1.367), (1.370)*, respectively. Then*

$$\left[F^{s_0}_{p_0q_0}(\Omega), F^{s_1}_{p_1q_1}(\Omega)\right]_\theta = F^s_{pq}(\Omega). \tag{1.375}$$

If, in addition $s_0 \neq s_1$, then

$$\left[B^{s_0}_{p_0q_0}(\Omega), B^{s_1}_{p_1q_1}(\Omega)\right]_\theta = B^s_{pq}(\Omega). \tag{1.376}$$

Proof. The corresponding assertions with $\mathbb{R}^n$ in place of Ω are covered by [MeM00], Theorem 11, p. 520. Since the interpolation property is available one can now argue in the same way as in the proof of Theorem 1.110. □

Remark 1.115. One can include a few but not all limiting cases where $q_0 = \infty$ or $q_1 = \infty$ and, in case of the B-spaces, $p_0 = \infty$ or $p_1 = \infty$. We refer to [MeM00] for details.

1.11.9 Characterisations by differences

First we recall a few known characterisations of some spaces on $\mathbb{R}^n$ in terms of differences and describe afterwards the modifications if $\mathbb{R}^n$ is replaced by bounded Lipschitz domains.

Let $\Delta^M_h f$ be the differences in $\mathbb{R}^n$ according to (1.11), where $M \in \mathbb{N}$. Then for $0 < u \le \infty$,

$$d^M_{t,u}f(x) = \left(t^{-n} \int\limits_{|h|\le t} |\Delta^M_h f(x)|^u \, \mathrm{d}h \right)^{1/u}, \quad x \in \mathbb{R}^n, \quad t > 0, \tag{1.377}$$

(with the usual modification if $u = \infty$) are ball means. Let $\overline{p} = \max(1,p)$ and

$$\|f|L_{\overline{p}}(\mathbb{R}^n)\|^* = \|f|L_p(\mathbb{R}^n)\| + \|f|L_{\overline{p}}(\mathbb{R}^n)\|.$$

Theorem 1.116.

(i) *Let*

$$0 < p \le \infty, \quad 0 < q \le \infty, \quad n\left(\frac{1}{p} - 1\right)_+ < s < M \in \mathbb{N}. \tag{1.378}$$

Then $B^s_{pq}(\mathbb{R}^n)$ *is the collection of all* $f \in L_{\overline{p}}(\mathbb{R}^n)$ *such that*

$$\|f\,|L_{\overline{p}}(\mathbb{R}^n)\|^* + \left(\int_0^1 t^{-sq} \sup_{|h|\le t} \|\Delta^M_h f\,|L_p(\mathbb{R}^n)\|^q \frac{\mathrm{d}t}{t}\right)^{1/q} < \infty \tag{1.379}$$

(*modification if* $q=\infty$) *in the sense of equivalent quasi-norms.*

(ii) *Let* $1 \le r \le \infty$, $0 < u \le r$ *and*

$$0 < p \le \infty, \quad 0 < q \le \infty, \quad n\left(\frac{1}{p} - \frac{1}{r}\right)_+ < s < M \in \mathbb{N}. \tag{1.380}$$

Then $B^s_{pq}(\mathbb{R}^n)$ *is the collection of all* $f \in L_{\max(p,r)}(\mathbb{R}^n)$ *such that*

$$\|f\,|L_{\overline{p}}(\mathbb{R}^n)\|^* + \left(\int_0^1 t^{-sq} \|d^M_{t,u} f\,|L_p(\mathbb{R}^n)\|^q \frac{\mathrm{d}t}{t}\right)^{1/q} < \infty \tag{1.381}$$

(*modification if* $q=\infty$) *in the sense of equivalent quasi-norms.*

(iii) *Let* $1 \le r \le \infty$, $0 < u \le r$ *and*

$$0 < p < \infty, \quad 0 < q \le \infty, \quad n\left(\frac{1}{\min(p,q)} - \frac{1}{r}\right)_+ < s < M \in \mathbb{N}. \tag{1.382}$$

Then $F^s_{pq}(\mathbb{R}^n)$ *is the collection of all* $f \in L_{\max(p,r)}(\mathbb{R}^n)$ *such that*

$$\|f\,|L_{\overline{p}}(\mathbb{R}^n)\|^* + \left\| \left(\int_0^1 t^{-sq}\, d^M_{t,u} f(\cdot)^q \frac{\mathrm{d}t}{t}\right)^{1/q} |L_p(\mathbb{R}^n)\right\| < \infty \tag{1.383}$$

(*modification if* $q=\infty$) *in the sense of equivalent quasi-norms.*

Remark 1.117. Part (i) is covered by Theorem and Remark 3 in [Triβ], pp. 110, 113, and an embedding theorem as far as the replacement of $\|f\,|L_p(\mathbb{R}^n)\|$ in [Triβ] by $\|f\,|L_{\overline{p}}(\mathbb{R}^n)\|^*$ in the case of $p<1$ is concerned. Otherwise it is an extension and modification of the classical assertions (iv) and (v) in Section 1.2. The above parts (ii) and (iii) are covered by [Triγ], Theorem 3.5.3, p. 194, where again the replacement of $\|f\,|L_p(\mathbb{R}^n)\|$ by $\|f\,|L_{\overline{p}}(\mathbb{R}^n)\|^*$ in case of $p<1$ is immaterial and covered by embedding. Similarly as in Remark 1.20 in connection with atoms one may ask to which extent the conditions (1.380) for the B-spaces and (1.382) for the F-spaces are natural. This applies especially to the q-dependence of (1.382). But it has been proved recently in [ChS05, Section 6, Remark] that in case of $0<q<p<\infty$ the condition (1.382) cannot be replaced by (1.380). We refer also to Remark 9.15.

We wish to carry over these characterisations to a bounded Lipschitz domain Ω. For this purpose we have to adapt the differences $\Delta^M_h f$ and the ball means (1.377) to Ω. Again let $\Delta_h^M f$ be the differences in $\mathbb{R}^n$ according to (1.11) where $M \in \mathbb{N}$ and $h \in \mathbb{R}^n$. Let for $x \in \Omega$,

$$\left(\Delta_{h,\Omega}^M f\right)(x) = \begin{cases} \left(\Delta_h^M f\right)(x) & \text{if } x + lh \in \Omega \text{ for } l = 0, \ldots, M, \\ 0 & \text{otherwise.} \end{cases} \tag{1.384}$$

To modify the ball means in (1.377) we introduce for $M \in \mathbb{N}$, $t > 0$ and $x \in \Omega$,

$$V^M(x,t) = \{h \in \mathbb{R}^n : |h| < t \text{ and } x + \tau h \in \Omega \text{ for } 0 \le \tau \le M\}. \tag{1.385}$$

This is the maximal open subset of a ball of radius t, centred at the origin, star-shaped with respect to the origin, such that $x + MV^M(x,t) \subset \Omega$. Then for $0 < u \le \infty$,

$$d_{t,u}^{M,\Omega} f(x) = \left(t^{-n} \int\limits_{h \in V^M(x,t)} |(\Delta_h^M f)(x)|^u \, \mathrm{d}h \right)^{1/u}, \quad x \in \Omega, \quad t > 0, \tag{1.386}$$

(with the usual modification if $u = \infty$) is the substitute of (1.377). It coincides with [Triγ], Definition 3.5.2, p. 193 (now for bounded Lipschitz domains). Again let $\overline{p} = \max(p,1)$.

Theorem 1.118. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$.*

(i) *Let p, q, s, M as in (1.378). Then $B_{pq}^s(\Omega)$ is the collection of all $f \in L_{\overline{p}}(\Omega)$ such that*

$$\|f \,|L_{\overline{p}}(\Omega)\| + \left(\int_0^1 t^{-sq} \sup_{|h| \le t} \|\Delta_{h,\Omega}^M f \,|L_p(\Omega)\|^q \frac{\mathrm{d}t}{t} \right)^{1/q} < \infty \tag{1.387}$$

(modification if $q = \infty$) in the sense of equivalent quasi-norms.

(ii) *Let $1 \le u \le r \le \infty$ and let p, q, s, M be as in (1.380). Then $B_{pq}^s(\Omega)$ is the collection of all $f \in L_{\max(p,r)}(\Omega)$ such that*

$$\|f \,|L_{\overline{p}}(\Omega)\| + \left(\int_0^1 t^{-sq} \|d_{t,u}^{M,\Omega} f \,|L_p(\Omega)\|^q \frac{\mathrm{d}t}{t} \right)^{1/q} < \infty \tag{1.388}$$

(modification if $q = \infty$) in the sense of equivalent quasi-norms.

(iii) *Let $1 \le u \le r \le \infty$ and let p, q, s, M be as in (1.382). Then $F_{pq}^s(\Omega)$ is the collection of all $f \in L_{\max(p,r)}(\Omega)$ such that*

$$\|f \,|L_{\overline{p}}(\Omega)\| + \left\| \left(\int_0^1 t^{-sq} \, d_{t,u}^{M,\Omega} f(\cdot)^q \frac{\mathrm{d}t}{t} \right)^{1/q} |L_p(\Omega) \right\| < \infty \tag{1.389}$$

(modification if $q = \infty$) in the sense of equivalent quasi-norms.

Proof. Step 1. Since Ω is bounded it follows by Hölders inequality that

$$\|f\,|L_{\overline{p}}(\Omega)\| \sim \|f\,|L_{\overline{p}}(\Omega)\|^* = \|f\,|L_{\overline{p}}(\Omega)\| + \|f\,|L_p(\Omega)\|$$

for the counterpart of $\|f\,|L_{\overline{p}}(\mathbb{R}^n)\|^*$ used in Theorem 1.116. Otherwise a proof of part (i) may be found in [Dis03a]. We refer also to [Dis03b].

Step 2. Let

$$\|f\,|F^s_{pq}(\Omega)\|_{u,M} \qquad \text{and} \qquad \|f\,|F^s_{pq}(\mathbb{R}^n)\|_{u,M} \tag{1.390}$$

be the quasi-norms in (1.389) and (1.383), respectively. Let $f \in F^s_{pq}(\Omega)$. Then by Definition 1.95 and the equivalent quasi-norm (1.383) there is an element $g \in F^s_{pq}(\mathbb{R}^n)$ with $g|\Omega = f$ such that

$$\|f\,|F^s_{pq}(\Omega)\|_{u,M} \le \|g\,|F^s_{pq}(\mathbb{R}^n)\|_{u,M} \le c\,\|f\,|F^s_{pq}(\Omega)\| \tag{1.391}$$

where $c > 0$ is independent of f. Similarly for $B^s_{pq}(\Omega)$.

Step 3. As for the converse we rely on the intrinsic characterisation of $F^s_{pq}(\Omega)$ in Lipschitz domains in terms of local means according to [Ry99b], Theorem 3.2, p. 251. As for the kernels of these local means one may choose the distinguished kernels constructed in [Triγ], Section 3.3.2, especially formula (10) on p. 175, which can be estimated from above by

$$c\,t^{-n} \int\limits_{h \in V^M(x,t)} |\Delta^M_h f(x)|\,\mathrm{d}h. \tag{1.392}$$

However these arguments are very much the same as in [Dis03a], which covers also these estimates although not stated explicitly. Since $u \ge 1$ it follows by Hölder's inequality that (1.392) can be estimated from above by $d^{M,\Omega}_{t,u} f$. This proves the converse of (1.391). Similarly for $B^s_{pq}(\Omega)$. □

Remark 1.119. Characterisations of the classical Besov spaces $B^s_{pq}(\Omega)$ with $1 \le p \le \infty$, $1 \le q \le \infty$, $s > 0$, in smooth bounded domains Ω in terms of the differences $\Delta^M_{h,\Omega}$ have been considered extensively from the very beginning of the theory of these spaces. Detailed descriptions of the results and the history may be found in [Nik77] (first edition 1969), [BIN75], [Triα], Sections 4.4.1, 4.4.2, pp. 321–324, and [Triγ], Section 1.10.3, pp. 72–73. There one finds also many references to the relevant literature, including extensions of this theory to more general domains, especially bounded Lipschitz domains. We refer in particular to [Mur71]. As it has been mentioned, part (i) of the above theorem is due to S. Dispa, [Dis03a]. But there is an alternative way to justify assertions of this type. The Besov spaces on bounded Lipschitz domains Ω in $\mathbb{R}^n$ (and even on more general domains) as considered in [DeS93] are defined as subspaces of $L_p(\Omega)$ with $0 < p \le \infty$ in terms of the above differences $\Delta^M_{h,\Omega}$. But if $s > n(\frac{1}{p} - 1)_+$ then these spaces coincide with the Besov spaces as considered here. This results in characterisations as in part (i) of the above theorem. Comparing parts (ii) and (iii) of the above theorem with the respective parts in Theorem 1.116 we have now the additional restriction $u \geq 1$. This comes from the reduction of these assertions to (1.392) and the application of Hölder's inequality with respect to u. In case of bounded C^∞ domains Ω in $\mathbb{R}^n$ we proved in [Triγ], Section 5.2.2, p. 245, the above parts (ii) and (iii) by other means, where the restriction $u \geq 1$ was not needed. Hence in this case one has a full counterpart of parts (ii) and (iii) of Theorem 1.116. Otherwise these assertions have their forerunners in the case of Banach spaces, hence if $p \geq 1$, $q \geq 1$. Some references may be found in [Triγ], Section 10.4, pp. 73–75. We mention in particular the work of the Russian school, [Kal83], [Kal85a], [Kal85b], [Kal88], [KaL87], [Bes90].

1.11.10 Lipschitz domains: Sobolev and Hölder-Zygmund spaces

For sake of completeness and for the possibility of later quotations we recall two very classical distinguished assertions. First we extend the definitions of the classical Sobolev spaces and Hölder-Zygmund spaces as mentioned in Section 1.2(ii) and (iv) from $\mathbb{R}^n$ to domains.

Definition 1.120. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$.*

(i) *Let $1 \leq p \leq \infty$, $k \in \mathbb{N}$, and*

$$\| f \,| W_p^k(\Omega) \| = \sum_{|\alpha| \leq k} \| D^\alpha f \,| L_p(\Omega) \|. \tag{1.393}$$

Then

$$W_p^k(\Omega) = \left\{ f \in L_p(\Omega) \,:\, \| f \,| W_p^k(\Omega) \| < \infty \right\}. \tag{1.394}$$

(ii) *Let $0 < s < M \in \mathbb{N}$ and let*

$$\| f \,| \mathcal{C}^s(\Omega) \|_M = \| f \,| L_\infty(\Omega) \| + \sup |h|^{-s} \, |(\Delta^M_{h,\Omega} f)(x)| \tag{1.395}$$

where $\Delta^M_{h,\Omega} f$ are the differences according to (1.384) *and where the supremum is taken over all $x \in \Omega$ and $h \in \mathbb{R}^n$ with $0 < |h| \leq 1$. Then*

$$\mathcal{C}^s(\Omega) = \{ f \in L_\infty(\Omega) \,:\, \| f \,| \mathcal{C}^s(\Omega) \|_M < \infty \}. \tag{1.396}$$

Remark 1.121. These are the well-known Sobolev spaces $W_p^k(\Omega)$ and Hölder-Zygmund spaces $\mathcal{C}^s(\Omega)$ on domains. They are Banach spaces, where $\mathcal{C}^s(\Omega)$ is independent of M (equivalent norms). The latter is well known but also a consequence of what follows. If $s \notin \mathbb{N}$ then the left-hand side of (1.311) is another equivalent norm. According to (1.3), (1.4) and (1.10) one has

$$W_p^k(\mathbb{R}^n) = F^k_{p,2}(\mathbb{R}^n) \qquad \text{where} \quad k \in \mathbb{N} \quad \text{and} \quad 1 < p < \infty, \tag{1.397}$$

and

$$\mathcal{C}^s(\mathbb{R}^n) = B^s_{\infty\infty}(\mathbb{R}^n), \qquad s > 0. \tag{1.398}$$

Obviously, $W_p^k(\mathbb{R}^n)$, including now $p = 1$ and $p = \infty$, is defined as in Section 1.2(ii), (1.4).

Theorem 1.122. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$.*

(i) *Let $1 < p < \infty$, $k \in \mathbb{N}$, and $s > 0$. Then*

$$W_p^k(\Omega) = F_{p,2}^k(\Omega) \tag{1.399}$$

and

$$\mathcal{C}^s(\Omega) = B_{\infty\infty}^s(\Omega), \tag{1.400}$$

where $F_{p,2}^k(\Omega)$ and $\mathcal{C}^s(\Omega)$ are the restrictions of the corresponding spaces on $\mathbb{R}^n$ to Ω according to Definition 1.95.

(ii) *There is a common linear extension operator for all spaces $W_p^k(\Omega)$ and $\mathcal{C}^s(\Omega)$ with $1 \le p \le \infty$, $k \in \mathbb{N}$, and $s > 0$.*

Proof. One obtains (1.400) as an immediate consequence of Theorem 1.118 and (1.395). Furthermore, (1.399) with (1.397) is a very classical famous result. A short proof and references may be found in [Triα], Section 4.2.4, p. 316. Finally Stein's extension operator in [Ste70], VI, §3, Theorem 5 on p. 181, can be taken as a common extension operator for all spaces in part (ii). □

1.11.11 General domains: sharp embeddings and envelopes

Let Ω be a domain in $\mathbb{R}^n$ and let $A_{pq}^s(\Omega)$ be the spaces as introduced in Definition 1.95. One can extend this method of definition of spaces by restriction to the spaces at the beginning of Section 1.9.1. One gets $C(\Omega)$, $C^1(\Omega)$, and $\mathrm{Lip}(\Omega)$. Then Theorem 1.73 remains valid if one replaces there $\mathbb{R}^n$ by Ω. This is almost obvious for the if-part. The proof of the sharpness, the only-if-part, in Theorem 1.73 is a local matter and applies to any domain Ω. We refer to [SiT95]. Similarly one can replace $\mathbb{R}^n$ by Ω in Theorem 1.75 (classical refinements) and in Theorems 1.78 (critical case), 1.81 (super-critical case) and 1.84 (sub-critical case). In all these cases the sharpness (only-if-parts) follows from local arguments. We refer to the corresponding sections in [Triϵ]. One can also formalise these assertions by introducing corresponding growth envelopes and continuity envelopes. Let

$$\mathcal{E}_{G,\Omega} A_{pq}^s(t) = \sup\left\{ f^*(t) \,:\, \| f \,| A_{pq}^s(\Omega)\| \le 1 \right\}, \qquad 0 < t < \varepsilon, \tag{1.401}$$

be the counterpart of $\mathcal{E}_G A_{pq}^s(t)$ in (1.233) for the same spaces as there. Let

$$\mathfrak{E}_{G,\Omega} A_{pq}^s = \left(\mathcal{E}_{G,\Omega} A_{pq}^s(\cdot\,), u \right) \tag{1.402}$$

be the corresponding growth envelopes, the Ω-counterparts of the related growth envelopes as considered in Remarks 1.80 and 1.86. Then it follows that

$$\mathfrak{E}_{G,\Omega} A_{pq}^s = \mathfrak{E}_G A_{pq}^s \tag{1.403}$$

with the specifications (1.238), (1.239) and (1.263). Similarly one gets in the super-critical case in obvious notation

$$\mathfrak{E}_{C,\Omega} A_{pq}^{1+\frac{n}{p}} = \mathfrak{E}_C A_{pq}^{1+\frac{n}{p}} \tag{1.404}$$

with the specifications (1.253), (1.254).

1.12 Fractal measures

1.12.1 An introduction to the non-smooth

To measure the smooth, means to measure the rough, or the fractality of objects to put it in more fashionable terms. In the remaining subsections of this chapter we describe some aspects of fractal measures and fractal elliptic operators, related fractal characteristics in $\mathbb{R}^n$ from the point of view of function spaces, and glance at function spaces on quasi-metric spaces. As always in this introductory chapter we collect some basic assertions accompanied by corresponding references preparing our later more detailed considerations and describe a few distinguished assertions. Quite understandably we wish to present an approach to fractal analysis via function spaces. But it seems to be reasonable first to set the stage by saying a few words about the whole subject from a broader point of view including some historical comments and quotations. We follow partly [Tri02b].

A.S. Besicovitch developed (together with only a few co-workers) over more than forty years, from the early 1920s up to the late 1960s,

The Geometry of Sets of Points

(title of a book planned by him which remained unfinished when he died in 1970). On this background a new branch of mathematics emerged in the last 25 years, called nowadays *fractal geometry* as it may be found in the monographs [Fal85], [Fal90], [Mat95], [Fal97] (to mention only a few). The word *fractal* was coined by B.B. Mandelbrot in 1975. In contrast to Besicovitch, who developed a pure inner-mathematical theory of non-smooth structures (complementing differential analysis and geometry) Mandelbrot propagated the idea that many objects in nature have a fractal structure which cannot be described appropriately in terms of (differentiable) analysis and geometry. In particular he suggested the notion of *self-similarity*, which found its rigorous mathematical definition in [Hut81] (1981). Together with its affine generalisation IFS (*Iterated Function System*) it is a corner stone of the recent fractal geometry and fractal analysis. Many attempts have been made to say under which circumstances a set (maybe a set of points in $\mathbb{R}^n$) should be called *fractal*. But there is no satisfactory definition, and *fractal* is now widely accepted as a somewhat vague synonym for *non-smooth* (either the objects themselves or the instruments admitted to deal with them). One has to say from case to case which specific assumptions are made.

In the last decade, and in particular in the last few years, there are many new developments in fractal geometry and now also in fractal analysis. But so far no dominating directions and techniques emerged. Asking for adequate geometries *completely distinct from Euclidean geometry* one finds in [Sem01], Preface:

They suggest rather strong shifts in outlook, for what kind of geometries are really around, what one might look for, how one might work with them, and so on.

This has been complemented by [DaS97], Preface:

The subject remains a wilderness, with no central zone, and many paths to try. The lack of main roadways is also one of the attractions of the subject.

So far self-similarity and iterated function systems are outstanding tools in the recent fractal analysis although it would be desirable to find more general approaches. In [Kig01], Introduction, p. 5, one finds the following description of the state of the art:

Why do you only study self-similar sets? The reason is that self-similar sets are perhaps the simplest and most basic structures in the theory of fractals. They give us much information on what would happen in the general case of fractals. Although there have been many studies on analysis on fractals, we are still near the beginning in the exploration of this field.

According to the above quotations fractal geometry and fractal analysis search for footpaths through the fractal wilderness. Our own contributions may be found in [Triδ] and [Triϵ]. It is one of the main aims of this book to continue these considerations. This will be done in Chapter 7. Some preparations and some descriptions are given in the present subsection and the following ones. They are based on wider classes of fractal measures, which need not to be self-similar, and related fractal elliptic operators. These considerations rely on building blocks of function spaces, especially on quarkonial decompositions as described in Section 1.6. In other words, we hope that the recent theory of function spaces as presented here opens a new track in the fractal wilderness, somehow away from the fashionable resorts of self-similarity and iterated function systems.

On the one hand, only in the last few decades has non-smooth or fractal geometry and analysis attracted attention on a larger scale. On the other hand it was present in mathematics for a long time, either as (counter-)examples to enhance the smooth analysis (such as the famous one-third Cantor set) or, surprisingly, as an effective tool to handle smooth problems. One of the most spectacular proofs of one of the most famous problems in mathematics is connected with twisted (*spiralling*, [Nash54]) or what one would call nowadays *fractal* embeddings of abstract structures in Euclidean spaces. J. Nash proved in [Nash54], [Nash56], that every n-dimensional Riemannian C^∞ manifold can be isometrically embedded in an Euclidean space $\mathbb{R}^N$ with $2N = n(n+1)(3n+11)$. We refer to [Aub98], p. 123, for details. This goal was reached in [Nash56] based on the preceding paper [Nash54]. The C^1-isometries treated in [Nash54] are related to fractal embeddings. We quote from [Nas98], p. 158:

He (Nash) *showed that you could fold the manifold like a silk handkerchief, without distorting it. Nobody would have expected Nash's theorem to be true. In fact, everyone would have expected it to be false... There has been some tendency in recent decades to move from harmonic to chaos. Nash says chaos is just around the corner.*

Although different in detail but related in spirit there is a more recent analogue of the technique used in [Nash54], resulting in so-called *snowflaked* bi-Lipschitzian embeddings of abstract homogeneous spaces (X, ϱ, μ), consisting of a set X, a quasi-metric ϱ and a measure μ (satisfying the doubling condition) into a high-dimensional Euclidean space $\mathbb{R}^n$ (Euclidean charts). This gives the possibility to develop first a theory of function spaces on sets in $\mathbb{R}^n$ and to transfer the outcome afterwards to homogeneous spaces (X, ϱ, μ). We refer to Section 1.17 for a description of the basic material, including references, and to Chapter 8 for a detailed study of function spaces on quasi-metric spaces.

1.12.2 Radon measures

We assume that the reader is familiar with basic measure and integration theory. Short descriptions of what is needed here may be found in [Mat95], pp. 7–13, [Fal85], pp. 1–6, and [Triδ], pp. 1–2. However to avoid misunderstandings we recall some notation and a few properties. We mostly assume that μ is a Radon measure in $\mathbb{R}^n$ with

$$\Gamma = \operatorname{supp} \mu \subset \{x : |x| < 1\} \quad \text{and} \quad 0 < \mu(\mathbb{R}^n) < \infty. \tag{1.405}$$

Let $0 < p \le \infty$. Then $L_p(\Gamma, \mu)$ is the usual complex quasi-Banach space, quasi-normed by

$$\|f \,|L_p(\Gamma, \mu)\| = \left(\int_{\mathbb{R}^n} |f(x)|^p \, \mu(\mathrm{d}x) \right)^{1/p} = \left(\int_{\Gamma} |f(\gamma)|^p \, \mu(\mathrm{d}\gamma) \right)^{1/p} \tag{1.406}$$

with the obvious modification if $p = \infty$. Recall that a Radon measure, say, with (1.405), has the following approximation properties:

$$\mu(V) = \sup\{\mu(K) : K \subset V;\ K \text{ compact}\} \tag{1.407}$$

for any open set V in $\mathbb{R}^n$, and

$$\mu(A) = \inf\{\mu(V) : A \subset V;\ V \text{ open}\} \tag{1.408}$$

for any μ-measurable set A in $\mathbb{R}^n$. Let T_μ,

$$T_\mu : \quad \varphi \mapsto \int_{\mathbb{R}^n} \varphi(x)\, \mu(\mathrm{d}x), \qquad \varphi \in S(\mathbb{R}^n), \tag{1.409}$$

be the tempered distribution generated by μ.

Proposition 1.123. *Let μ^1 and μ^2 be two Radon measures with* (1.405). *Then*

$$T_{\mu^1} = T_{\mu^2} \text{ in } S'(\mathbb{R}^n) \qquad \text{if, and only if,} \quad \mu^1 = \mu^2. \tag{1.410}$$

Proof. Since $S(\mathbb{R}^n)$ is dense in the Banach space $C_0(\mathbb{R}^n)$ of all complex-valued bounded continuous functions on $\mathbb{R}^n$ tending to zero if $|x| \to \infty$, one can interpret

T_μ given by (1.409) as a linear and continuous functional on $C_0(\mathbb{R}^n)$. Then it follows by the Riesz representation theorem as stated in [Mall95], Theorem 6.6, p. 97, that μ is uniquely determined. Hence we have (1.410). □

Remark 1.124. The above proposition can be extended immediately to all elements of the spaces $L_1(\Gamma,\mu)$ interpreting $f \in L_1(\Gamma,\mu)$ as the complex (or signed) Radon measure $f\mu$. This is also covered by [Mall95]. In particular one has for $1 \le p \le \infty$ the one-to-one relation between

$$f \in L_p(\Gamma,\mu) \qquad \text{and} \qquad f\mu \in S'(\mathbb{R}^n). \tag{1.411}$$

This justifies (as usual) the identification of μ and $f \in L_p(\Gamma,\mu)$ with the corresponding tempered distribution and to write

$$\mu \in S'(\mathbb{R}^n) \qquad \text{and} \qquad f \in S'(\mathbb{R}^n), \tag{1.412}$$

respectively.

1.12.3 The μ-property

Properties of Radon measures in $\mathbb{R}^n$ have been considered in great detail both in fractal geometry and in the theory of function spaces. As far as our own contributions are concerned we refer to [Triδ] and in particular to [Triϵ], Section 9. We continue these considerations in Chapter 7. But it seems to be reasonable to give a description of the general background now and to indicate a few assertions.

Let Q_{jm} with $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$ be the closed cubes in $\mathbb{R}^n$ centred at $2^{-j}m$ and with side-length 2^{-j+1} (mildly overlapping each other for fixed $j \in \mathbb{N}_0$).

Definition 1.125. *Let μ be a Radon measure in $\mathbb{R}^n$ with* (1.405)*. Let $0 < p \le \infty$, $0 < q \le \infty$ and $\lambda \in \mathbb{R}$. Then*

$$\mu^\lambda_{pq} = \left(\sum_{j=0}^{\infty} 2^{j\lambda q} \left(\sum_{m\in\mathbb{Z}^n} \mu(Q_{jm})^p \right)^{q/p} \right)^{1/q} \tag{1.413}$$

with the obvious modifications if p and/or q are infinite.

Remark 1.126. The behavior of quantities of type

$$\sum_{m\in\mathbb{Z}^n} \mu(Q_{jm})^p \qquad \text{if} \quad j \to \infty \tag{1.414}$$

has been studied since the late 1980s apparently somewhat parallel and independently of each other both in fractal geometry (multifractal formalism) and in the theory of function spaces (equivalent quasi-norms, traces of functions on sets $\Gamma = \operatorname{supp}\mu$). In the multifractal analysis one asks typically for the behavior of

$$\limsup_{j\to\infty} \frac{1}{j} \log \left(\sum_{m\in\mathbb{Z}^n} \mu(Q_{jm})^p \right) \tag{1.415}$$

where log is taken to base 2 and $0 < p < \infty$. This is essentially the dyadic version of the function $\tau(p)$ in [Heu98], p. 310, formula (1). But there are several modifications. The recent development of this theory may be found in [BMP92], [Ols95], [Ngai97], [NBH02]. Some aspects of multifractal analysis have been surveyed in [Fal97], Chapter 11. Modifications may be found in [Str93], [Str94]. In the theory of function spaces characterisations in terms of local dyadic means as described in Section 1.4, Theorem 1.10 and Corollary 1.12 have been known since the late 1980s. Some references may be found in Remark 1.11. As will be detailed later on, afterwards it is only a minor step to arrive at quantities of type (1.413). The first explicit formulation of equivalent quasi-norms based on μ^λ_{pq} was given in [Win95] and had been used afterwards by several authors. A discussion including some references may be found in [Triϵ], p. 125. In [Triϵ], Section 9, we considered traces of function spaces on fractal sets Γ of type

$$\mathrm{tr}_\mu \; : \; B^s_{pq}(\mathbb{R}^n) \hookrightarrow L_r(\Gamma, \mu) \tag{1.416}$$

for some r with $0 < r \le \infty$. Then the quantities μ^λ_{pq} (preferably with $p > 1$ and $\lambda \ge 0$) come in quite naturally. Such problems have a long history which may be found in [Triϵ], Section 9. We refer in addition especially to [AdH96]. In this connection there are always problems of type (1.414).

To provide a better understanding of what follows we complement the above considerations by the following simple observation. Let

$$\sigma^-_p = \min\left(0, n - \frac{n}{p}\right), \qquad 0 < p \le \infty. \tag{1.417}$$

Proposition 1.127. *Let μ be a Radon measure in $\mathbb{R}^n$ with* (1.405).

(i) *Then*

$$\mu \in B^0_{1,\infty}(\mathbb{R}^n) \qquad \textit{and} \qquad \|\mu \,|B^0_{1,\infty}(\mathbb{R}^n)\| \sim \mu(\mathbb{R}^n) \tag{1.418}$$

with equivalence constants which are independent of μ.

(ii) *Let $0 < p \le \infty$. Then*

$$\mu^\lambda_{pq} \sim \mu(\mathbb{R}^n) \quad \textit{if} \quad \begin{cases} \textit{either} & 0 < q \le \infty, \;\; \lambda < \sigma^-_p, \\ \textit{or} & q = \infty, \;\; \lambda = \sigma^-_p, \end{cases} \tag{1.419}$$

with equivalence constants which are independent of μ.

Proof. Step 1. We prove part (i). By (1.15) and well-known properties of the Fourier transform we have

$$(\varphi_k \widehat{\mu})^\vee(x) = c\, 2^{(k-1)n} \int_{\mathbb{R}^n} \varphi_1^\vee \left(2^{k-1}(x-y)\right) \mu(\mathrm{d}y), \qquad k \in \mathbb{N}. \tag{1.420}$$

Taking the L_1-norm (with respect to x), complemented by an obvious counterpart with $(\varphi_0\widehat{\mu})^\vee$ it follows by (1.17) that

$$\mu \in B^0_{1,\infty}(\mathbb{R}^n) \qquad \text{and} \qquad \|\mu \,|B^0_{1,\infty}(\mathbb{R}^n)\| \le c\,\mu(\mathbb{R}^n). \tag{1.421}$$

As for the converse we may assume that $\varphi = \varphi_0$ in (1.14) is non-negative and radially symmetric. Then $\varphi^\vee$ is real in $\mathbb{R}^n$ and positive in some ball centred at the origin. If μ has a support in a ball centred at the origin and of a sufficiently small radius then we get

$$(\varphi\widehat{\mu})^\vee(x) = c\int_{\mathbb{R}^n} \varphi^\vee(x-y)\,\mu(\mathrm{d}y) \ge c'\,\mu(\mathbb{R}^n) \quad \text{if} \quad |x| \le \varepsilon \tag{1.422}$$

for some $\varepsilon > 0$, where $c' > 0$ is independent of these measures. Together with a dilation argument one gets part (i).

Step 2. If $1 \le p \le \infty$ and, hence, $\sigma_p^- = 0$, then one gets (1.419) from (1.413). If $0 < p < 1$ then (1.419) follows from

$$\Big(\sum_{m\in\mathbb{Z}^n} \mu(Q_{jm})^p\Big)^{1/p} \le c\,2^{jn\frac{1-p}{p}} \sum_{m\in\mathbb{Z}^n} \mu(Q_{jm}) \sim 2^{jn\frac{1-p}{p}}\,\mu(\mathbb{R}^n). \tag{1.423}$$

$\square$

Remark 1.128. We refer also to [Triϵ], Section 9.26, p. 146, where we discussed further elementary properties of the quantities μ^λ_{pq} introduced in (1.413).

Definition 1.129. *Let* $0 < p \le \infty$ *(*$p < \infty$ *for the* F*-spaces),* $0 < q \le \infty$*,* $s \in \mathbb{R}$ *and*

$$s - \frac{n}{p} = \lambda - n. \tag{1.424}$$

(i) *Then* $B^s_{pq}(\mathbb{R}^n)$ *is said to have the* μ*-property if*

$$\mu \in B^s_{pq}(\mathbb{R}^n) \quad \Longleftrightarrow \quad \mu^\lambda_{pq} < \infty \tag{1.425}$$

for all Radon measures μ *with* (1.405).

(ii) *Then* $F^s_{pq}(\mathbb{R}^n)$ *is said to have the* μ*-property if*

$$\mu \in F^s_{pq}(\mathbb{R}^n) \quad \Longleftrightarrow \quad \mu^\lambda_{pp} < \infty. \tag{1.426}$$

Remark 1.130. Obviously, (1.425) means that μ is an element of $B^s_{pq}(\mathbb{R}^n)$ if, and only if, μ^λ_{pq} with (1.424) is finite. Similarly, (1.426).

Theorem 1.131. *Let* $0 < p \le \infty$ *(*$p < \infty$ *for the* F*-spaces), and* $0 < q \le \infty$*.*

(i) *Let* $s > 0$*. Then neither* $B^s_{pq}(\mathbb{R}^n)$ *nor* $F^s_{pq}(\mathbb{R}^n)$ *has the* μ*-property.*

(ii) *Let* $s < 0$. *Then both* $B^s_{pq}(\mathbb{R}^n)$ *and* $F^s_{pq}(\mathbb{R}^n)$ *have the* μ-*property. Furthermore with* λ *given by* (1.424),

$$\|\mu\,|B^s_{pq}(\mathbb{R}^n)\| \sim \mu^\lambda_{pq} \tag{1.427}$$

and

$$\|\mu\,|F^s_{pq}(\mathbb{R}^n)\| \sim \mu^\lambda_{pp} \tag{1.428}$$

(*equivalent quasi-norms where the equivalence constants are independent of* μ *with* (1.405)).

Remark 1.132. This is a simplified version of Theorem 1 in [Tri03b]. We return to this subject in Chapter 7 in greater detail, where we give a proof of this assertion and discuss what happens on the critical line $s = 0$. Quite obviously according to (1.418) the space $B^0_{1,\infty}(\mathbb{R}^n)$ fits in this scheme. Then we have $p = 1$, $q = \infty$, $\lambda = 0$. This is a special case of

$$\|\mu\,|B^s_{p\infty}(\mathbb{R}^n)\| \sim \mu(\mathbb{R}^n) \quad \text{if} \quad s = \min\left(0, n\left(\frac{1}{p} - 1\right)\right) \tag{1.429}$$

where $0 < p \le \infty$. Furthermore we get for $0 < p \le \infty$ ($p < \infty$ in the F-case), $0 < q \le \infty$, and A either B or F,

$$\|\mu\,|A^s_{pq}(\mathbb{R}^n)\| \sim \mu(\mathbb{R}^n) \quad \text{if} \quad s < \min\left(0, n\left(\frac{1}{p} - 1\right)\right). \tag{1.430}$$

Both (1.429) and (1.430) are in good agreement with (1.419) and (1.424). Further discussions are shifted to Section 1.14 below. The independence of the right-hand side of (1.428) on q is remarkable. It follows from the independence of the positive cones of the corresponding spaces on q. We give a formulation. First we recall that $f \in F^s_{pq}(\mathbb{R}^n)$ is called a positive distribution if

$$f(\varphi) \ge 0 \qquad \text{for any} \quad \varphi \in S(\mathbb{R}^n) \quad \text{with} \quad \varphi \ge 0. \tag{1.431}$$

Then it follows from the Radon-Riesz theorem that $f = \mu$ is a (not necessarily compactly supported) Radon measure. We refer to [Mall95], pp. 61/62 and 71, 75. The positive cone $\overset{+}{F}{}^s_{pq}(\mathbb{R}^n)$ is the collection of all positive $f \in F^s_{pq}(\mathbb{R}^n)$.

Proposition 1.133. *Let* $0 < p < \infty$ *and* $s < 0$. *Then*

$$\overset{+}{F}{}^s_{pq_1}(\mathbb{R}^n) = \overset{+}{F}{}^s_{pq_2}(\mathbb{R}^n) \quad \textit{for} \quad 0 < q_1 \le q_2 \le \infty. \tag{1.432}$$

Remark 1.134. As has been said the positive cone of $F^s_{pq}(\mathbb{R}^n)$ consists entirely of positive Radon measures. The history of the above assertion may be found in [AdH96], p. 126. A proof restricted to $1 < p < \infty$ and $1 < q_1 \le q_2 < \infty$ was given in [AdH96], Corollary 4.3.9, p. 103 (with a reference to an earlier paper by D.R. Adams, 1989). The full proof is due to [JPW90] and [Net89].

1.13 Fractal operators

1.13.1 The classical theory

Recall that this introductory Chapter 1 has several aims. First we wish to provide the background on which the following chapters are built. Secondly we describe some remarkable developments in the theory of function spaces and its applications in the 1990s in continuation of Chapter 1 in [Triγ] including some assertions considered later on in this book in greater detail. Beginning with the preceding Section 1.12 the remaining subsections deal with applications of function spaces to non-smooth objects, first fractal measures, and now fractal elliptic operators which leads in the following subsection to fractal characteristics of measures.

First we describe the classical background of boundary value problems for regular elliptic differential operators restricting us to the simplest and most prominent example, the Dirichlet Laplacian in bounded C^∞ domains in the plane $\mathbb{R}^2$.

Let Ω be a bounded C^∞ domain in the plane $\mathbb{R}^2$ with the boundary $\partial\Omega$. Let $L_2(\Omega)$ be the usual complex Hilbert space normed according to (1.294) with $p = 2$. Let

$$H^l(\Omega) = W_2^l(\Omega) \qquad \text{where} \quad l \in \mathbb{N}, \tag{1.433}$$

be the classical Sobolev spaces as briefly discussed in Section 1.11.10. In particular we have (1.393) and (1.394) with $p = 2$. Let

$$H_0^2(\Omega) = \{ f \in H^2(\Omega) \,:\, f|\partial\Omega = 0 \} \tag{1.434}$$

and

$$\mathring{H}^1(\Omega) = \{ f \in H^1(\Omega) \,:\, f|\partial\Omega = 0 \}, \tag{1.435}$$

where the latter space will be equipped with the scalar product

$$(f, g)_{\mathring{H}^1(\Omega)} = \sum_{m=1}^{2} \int_\Omega \frac{\partial f}{\partial x_m} \cdot \frac{\partial \overline{g}}{\partial x_m} \, \mathrm{d}x. \tag{1.436}$$

Here the trace $f|\partial\Omega$ of $f \in H^l(\Omega)$ with $l \in \mathbb{N}$ makes sense and has the usual meaning. Recall that $D(\Omega) = C_0^\infty(\Omega)$ is dense in $\mathring{H}^1(\Omega)$ (but not in $H_0^2(\Omega)$, which explains the different way of writing). Furthermore, according to Friedrichs' inequality there is a number $c > 0$ such that

$$\| f \,| L_2(\Omega) \| \le c \left(\sum_{m=1}^{2} \int_\Omega \left| \frac{\partial f}{\partial x_m}(x) \right|^2 \mathrm{d}x \right)^{1/2} \qquad \text{for all} \quad f \in \mathring{H}^1(\Omega). \tag{1.437}$$

In particular, (1.436) generates a norm. All this is well known and very classical and may be found in many books, for example in [Tri72], Theorem 28.3, p. 385 (or p.

357 in the English translation). In agreement with Definition 1.95 we complement (1.433) by

$$H^{-1}(\Omega) = B^{-1}_{2,2}(\Omega), \tag{1.438}$$

(Sobolev spaces of order -1). We collect some very classical assertions needed later on.

Proposition 1.135. *Let Ω be a bounded C^∞ domain in $\mathbb{R}^2$. The Dirichlet Laplacian*

$$-\Delta = -\sum_{m=1}^{2} \frac{\partial^2}{\partial x_m^2}, \qquad \mathrm{dom}(-\Delta) = H^2_0(\Omega), \tag{1.439}$$

is a positive-definite self-adjoint operator in $L_2(\Omega)$ with pure point spectrum.

Remark 1.136. In particular,

$$-\Delta : \quad H^2_0(\Omega) \hookrightarrow L_2(\Omega) \tag{1.440}$$

is an isomorphic map. This assertion can be extended to scales of H^s_p and B^s_{pq} spaces and to more general regular elliptic operators. In this version it may be found in [Triα], Section 5.7.1, Remark 1 on p. 402 (slightly corrected in the 1995 edition), complemented in [Triβ], Sections 4.3.3, 4.3.4. A systematic treatment has been given in [RuS96], Section 3.5.2, p. 130. In this context it is reasonable to use the notation $-\Delta$ for mappings between pairs of spaces within these scales. Let $(-\Delta)^{-1}$ be the corresponding inverse. Then we have as a special case of the just-mentioned literature the isomorphic map

$$(-\Delta)^{-1} : \quad H^{-1}(\Omega) \hookrightarrow \mathring{H}^1(\Omega). \tag{1.441}$$

To prepare notationally our later considerations we prefer now

$$B = (-\Delta)^{-1} = (-\Delta)^{-1} \circ \mu_L, \tag{1.442}$$

where μ_L is the Lebesgue measure. Then B is a positive, self-adjoint, compact operator in $L_2(\Omega)$. Let ϱ_k be its positive eigenvalues repeated according to multiplicity and ordered by magnitude, and let u_k be its related eigenfunctions,

$$Bu_k = \varrho_k\, u_k, \qquad \varrho_1 > \varrho_2 \geq \cdots > 0, \qquad \varrho_k \to 0 \tag{1.443}$$

if $k \to \infty$. We recall the following outstanding properties (again notationally adapted to our later needs). Let $C^\infty(\overline{\Omega})$ be the space of all C^∞ functions in Ω such that any derivative can be extended continuously to $\overline{\Omega}$.

Theorem 1.137. *Let Ω be a connected bounded C^∞ domain in $\mathbb{R}^2$ and let B be the above inverse* (1.442) *of the Dirichlet Laplacian.*

(i) (H. Weyl, 1912, [Weyl12a], [Weyl12b]). *There are two constants $0 < c_1 \leq c_2 < \infty$ such that*

$$c_1\, k^{-1} \leq \varrho_k \leq c_2\, k^{-1}, \qquad k \in \mathbb{N}. \tag{1.444}$$

(ii) (R. Courant, 1924, [CoH24], p. 398/399). *The largest eigenvalue* $\varrho = \varrho_1$ *is simple and*

$$u_1(x) = c\, u(x) \quad \text{with} \quad 0 \neq c \in \mathbb{C} \quad \text{and} \quad u(x) > 0 \text{ in } \Omega \tag{1.445}$$

(Nullstellenfreiheit).

(iii) (Smoothness) *All eigenfunctions* $u_k \in C^\infty(\overline{\Omega})$ *where* $k \in \mathbb{N}$.

Remark 1.138. All assertions, including the C^∞ property in (iii) are very classical. The latter is covered by the above references, for example by [Triα], also on the much larger scale of arbitrary (smooth) regular elliptic differential operators. As had been said, (1.444) is adapted to our later purposes. Otherwise even the original version of H. Weyl is sharper than (1.444) including the volumes of Ω (main term) and of $\partial\Omega$ (remainder term). Since that time the spectral theory of (regular and singular elliptic) differential operators and pseudodifferential operators is an outstanding topic of analysis during the whole last century. The state of the art and in particular the techniques used nowadays may be found in [SaV97]. Courant's strikingly short elegant proof of (1.445) on less than one page entitled

Charakterisierung der ersten Eigenfunktion durch ihre Nullstellenfreiheit

indicates what follows in a few lines. Based on quadratic forms Courant relies (in recent notation) on H^1-arguments. But he did not bother very much about the technical rigour of his proof. More recent versions may be found in [Tay96], pp. 315–316. We refer also to [Tai96] for generalisations and to [ReS78], Theorem XIII.43, for an abstract version. The assumption that Ω is connected is only needed in part (ii) of the theorem.

1.13.2 The fractal theory

Again let Ω be a bounded C^∞ domain in the plane $\mathbb{R}^2$ and let $(-\Delta)^{-1}$ be the inverse of the corresponding Dirichlet Laplacian as introduced in the preceding Section 1.13.1, now preferably considered as an isomorphic map according to (1.441). We wish to replace the Lebesgue measure μ_L in (1.442) by more general Radon measures μ in $\mathbb{R}^2$ now with

$$\operatorname{supp} \mu = \Gamma \subset \Omega, \qquad 0 < \mu(\mathbb{R}^2) < \infty, \quad \text{and} \quad |\Gamma| = 0, \tag{1.446}$$

where $|\Gamma|$ is the Lebesgue measure of Γ. This is the counterpart of (1.405) where we now assume that μ is singular (with respect to the Lebesgue measure). Furthermore since Γ is compact it has a positive distance to the boundary $\partial\Omega$ of Ω. We use the notation and justifications given in Section 1.12.2 to identify μ with the generated distribution and the Hilbert space $L_2(\Gamma,\mu)$ with the respective subset of $S'(\mathbb{R}^2)$. First we wish to clarify under which conditions B given by (1.442), now with μ in place of μ_L, makes sense and how it is defined. It comes out that this question can be reduced to the trace problem asking for a constant $c > 0$ such that

$$\|\varphi \,|L_2(\Gamma,\mu)\| \leq c\, \|\varphi\, |H^1(\Omega)\| \qquad \text{for all} \quad \varphi \in D(\Omega). \tag{1.447}$$

If one has an affirmative answer then one can define the linear and bounded trace operator tr_μ,

$$\mathrm{tr}_\mu : \quad \mathring{H}^1(\Omega) \hookrightarrow L_2(\Gamma, \mu), \tag{1.448}$$

by completion. Sometimes it is desirable to have a more explicit description of $\mathrm{tr}_\mu f \in L_2(\Gamma, \mu)$ if $f \in H^1(\mathbb{R}^2)$. This can be done by a refined version of the theory of Lebesgue points. We shall not need this in what follows. A description may be found in [Triϵ], pp. 260/261, [Tri01] and the references given there especially to [AdH96]. But we need the duality assertion

$$H^{-1}(\Omega) = \left(\mathring{H}^1(\Omega)\right)' \tag{1.449}$$

in the framework of the dual pairing $(D(\Omega), D'(\Omega))$. Although quite often used, sometimes even taken as a definition of $H^{-1}(\Omega)$, it is a little bit like mathematical folklore. On the other hand, (1.449) remains valid for arbitrary domains Ω in $\mathbb{R}^n$, where $H^1(\Omega)$ and $H^{-1}(\Omega)$ according to Definition 1.95 are the restrictions of

$$H^1(\mathbb{R}^n) = B^1_{2,2}(\mathbb{R}^n) \qquad \text{and} \qquad H^{-1}(\mathbb{R}^n) = B^{-1}_{2,2}(\mathbb{R}^n) \tag{1.450}$$

to Ω, respectively, and $\mathring{H}^1(\Omega)$ is the completion of $D(\Omega)$ in $H^1(\Omega)$.

Proposition 1.139. *Let Ω be an arbitrary domain in $\mathbb{R}^n$. Then*

$$\left(\mathring{H}^1(\Omega)\right)' = H^{-1}(\Omega) \qquad (\textit{equivalent norms}) \tag{1.451}$$

in the framework of the dual pairing $(D(\Omega), D'(\Omega))$.

Remark 1.140. We refer to [Triϵ], Proposition 20.3 and Remark 20.4, pp. 297/298.

We return to bounded C^∞ domains in the plane $\mathbb{R}^2$. We interpret $f \in L_2(\Gamma, \mu)$ according to Section 1.12.2, especially (1.411), (1.412) as a distribution, hence

$$(\mathrm{id}_\mu f)(\varphi) = \int_\Gamma f(\gamma)\, \varphi(\gamma)\, \mu(\mathrm{d}\gamma), \qquad \varphi \in S(\mathbb{R}^2), \tag{1.452}$$

or $\varphi \in D(\Omega)$, which is the same, and call id_μ the *identification operator*. As usual we identify $L_2(\Gamma, \mu)$ with its dual. With (1.448) it comes out that

$$\mathrm{id}_\mu = \mathrm{tr}'_\mu : \quad L_2(\Gamma, \mu) \hookrightarrow H^{-1}(\Omega), \tag{1.453}$$

is just the dual of tr_μ where one has to use (1.449). We refer for details to [Triϵ], Section 9.2, pp. 123/124.

Proposition 1.141. *Let Ω be a bounded C^∞ domain in $\mathbb{R}^2$ and let μ be a Radon measure with* (1.446) *such that the trace according to* (1.447), (1.448) *exists. Then*

$$\mathrm{id}^\mu = \mathrm{id}_\mu \circ \mathrm{tr}_\mu : \quad \mathring{H}^1(\Omega) \hookrightarrow H^{-1}(\Omega) \tag{1.454}$$

and

$$B = (-\Delta)^{-1} \circ \mathrm{id}^\mu : \quad \mathring{H}^1(\Omega) \hookrightarrow \mathring{H}^1(\Omega) \tag{1.455}$$

are linear and bounded operators.

Proof. This follows immediately from (1.448), (1.453) and (1.441). □

Remark 1.142. Consequently B in (1.455), abbreviated by

$$B = (-\Delta)^{-1} \circ \mu, \tag{1.456}$$

is the right substitute for (1.442) where μ_L is the Lebesgue measure. Hence our way to deal with the operator B in (1.456) depends on the question of whether the trace according to (1.447), (1.448) exists or not. To describe some assertions we rely on the notation of dyadic cubes (now squares) as introduced at the beginning of Section 1.5.1 and used in Definition 1.125. Hence let Q_{jm} with $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^2$ be the square in $\mathbb{R}^2$ centred at $2^{-j}m$ and with side length 2^{-j+1} (for fixed j some overlap is admitted). With μ given by (1.446) we put

$$\mu_j = \sup_{m \in \mathbb{Z}^2} \mu(Q_{jm}), \qquad j \in \mathbb{N}_0, \tag{1.457}$$

and for $f \in L_1(\Gamma, \mu)$,

$$f_{jm} = \int_{Q_{jm}} f(\gamma)\, \mu(\mathrm{d}\gamma), \qquad j \in \mathbb{N}_0, \quad m \in \mathbb{Z}^2. \tag{1.458}$$

Theorem 1.143. *Let Ω be a bounded C^∞ domain in $\mathbb{R}^2$ and let μ be a Radon measure with* (1.446).

(i) *Then* tr_μ *according to* (1.447), (1.448) *exists if, and only if,*

$$\sup \sum_{j=0}^\infty \sum_{m \in \mathbb{Z}^2} f_{jm}^2 < \infty, \tag{1.459}$$

where the supremum is taken over all

$$f \in L_2(\Gamma, \mu) \quad \text{with} \quad f \geq 0 \quad \text{and} \quad \| f \,| L_2(\Gamma, \mu)\| \leq 1. \tag{1.460}$$

(ii) *If* tr_μ *exists, then*

$$\sum_{j=0}^\infty \sum_{m \in \mathbb{Z}^2} \mu(Q_{jm})^2 < \infty. \tag{1.461}$$

Conversely, if

$$\sum_{j=0}^\infty \mu_j < \infty, \tag{1.462}$$

then tr_μ *exists and is compact.*

(iii) *If*

$$\sum_{j=0}^{\infty} \sqrt{\mu_j} < \infty \tag{1.463}$$

then B, *given by* (1.455), (*or likewise* (1.456)) *is compact and*

$$B: \quad \mathring{H}^1(\Omega) \hookrightarrow \mathring{H}^1(\Omega) \cap C(\overline{\Omega}). \tag{1.464}$$

Remark 1.144. The criterion (1.459) is a special case of [Triϵ], Theorem 9.3, p. 125. Inserting $f(\gamma) = c > 0$ (a constant) one gets (1.461) as a necessary condition. Furthermore for fixed $j \in \mathbb{N}_0$ and f given by (1.460) one gets by Hölder's inequality,

$$\sum_{m \in \mathbb{Z}^2} f_{jm}^2 \le c \, \|f \,|L_2(\Gamma, \mu)\|^2 \, \mu_j \le c \, \mu_j \, . \tag{1.465}$$

Hence, (1.462) ensures that both tr_μ and B given by (1.448) and (1.455) are linear and bounded operators. We return later on in Chapter 7 to this subject in a larger and more systematic context. Then the remaining assertions of the above theorem are special cases of a more comprehensive theory. This applies to the claimed compactness of tr_μ if one has (1.462) and to part (iii). We refer in particular to Remark 7.64. We only mention that (1.463) is stronger than (1.462) with the consequence that (1.455) can be strengthened by

$$B: \quad \mathring{H}^1(\Omega) \hookrightarrow \mathring{B}^1_{2,1}(\Omega) \hookrightarrow \mathring{H}^1(\Omega) \cap C(\overline{\Omega}), \tag{1.466}$$

where the latter is a sharp embedding theorem of $B^1_{2,1}(\Omega)$ into the space of all complex-valued continuous functions on $\overline{\Omega}$ denoted by $C(\overline{\Omega})$.

Based on Theorem 1.143 we can now describe the fractal counterpart of Theorem 1.137. We always assume that $\mathring{H}^1(\Omega)$ is normed according to (1.436). Furthermore if Γ is a compact subset of Ω then $\mathring{H}^1(\Omega \backslash \Gamma)$ denotes the completion of $D(\Omega \backslash \Gamma)$ in $H^1(\Omega)$, again normed according to (1.436).

Theorem 1.145. *Let* Ω *be a connected bounded* C^∞ *domain in* $\mathbb{R}^2$ *and let* μ *be a Radon measure satisfying* (1.446) *and* (1.463). *Then* B *according to* (1.455) (*or* (1.456)) *is a self-adjoint, compact, non-negative operator in* $\mathring{H}^1(\Omega)$ *with null-space*

$$N(B) = \mathring{H}^1(\Omega \backslash \Gamma). \tag{1.467}$$

Furthermore, B *is generated by the quadratic form*

$$(Bf, g)_{\mathring{H}^1(\Omega)} = \int_\Gamma f(\gamma) \, \overline{g(\gamma)} \, \mu(\mathrm{d}\gamma), \quad f \in \mathring{H}^1(\Omega), \quad g \in \mathring{H}^1(\Omega). \tag{1.468}$$

Let ϱ_k *be the positive eigenvalues of* B, *repeated according to multiplicity and ordered by decreasing magnitude and let* u_k *be the related eigenfunctions,*

$$Bu_k = \varrho_k \, u_k, \qquad k \in \mathbb{N}. \tag{1.469}$$

(i) *The largest eigenvalue is simple,*

$$\varrho = \varrho_1 > \varrho_2 \geq \cdots > 0, \qquad \varrho_k \to 0 \quad \textit{if} \quad k \to \infty, \tag{1.470}$$

and the related eigenfunctions u_1 *have no zeros in* Ω,

$$u_1(x) = c\, u(x) \quad \textit{with} \quad 0 \neq c \in \mathbb{C} \quad \textit{and} \quad u(x) > 0 \quad \textit{in} \quad \Omega. \tag{1.471}$$

(ii) *The eigenfunctions* u_k *are* (*classical*) *harmonic functions in* $\Omega \backslash \Gamma$,

$$\Delta u_k(x) = 0 \quad \textit{if} \quad x \in \Omega \backslash \Gamma \quad \textit{and} \quad u_k \in C(\overline{\Omega}). \tag{1.472}$$

Proof. (*Explanations*). One can prove the above theorem by combining corresponding arguments from [Triϵ], Section 19, with the above Theorem 1.143. First we refer to [Triϵ], Theorem 19.7, pp. 264–270, where we dealt with the operator B in bounded C^∞ domains in $\mathbb{R}^n$ in the more general context of some B^s_{pq}-spaces but under the severe additional assumption that Γ is a d-set. Here we always assume that $n = 2$. Then the measures considered in [Triϵ], Theorem 19.7, have in addition to (1.446) the property

$$\mu\left(B(\gamma, r)\right) \sim r^d \quad \text{where} \quad \gamma \in \Gamma \quad \text{and} \quad 0 < r < 1, \tag{1.473}$$

with $0 < d < 2$. Here $B(\gamma, r)$ denotes a circle centred at γ and of radius r, and $\sim$ means that the corresponding equivalence constants are independent of γ and r. Obviously such measures are special cases of (1.463). Now one can check which arguments from the proof of [Triϵ], Theorem 19.7, and the preceding explanations apply also to the above more general situation. Since tr_μ exists as a compact operator, the arguments from [Triϵ], Section 19.3, can be carried over. In particular one has (1.468) where the right-hand side is considered as a quadratic form in $\mathring{H}^1(\Omega)$. As a consequence, B is a self-adjoint, compact, non-negative operator in $\mathring{H}^1(\Omega)$ with null space

$$N(B) = \left\{ f \in \mathring{H}^1(\Omega) \;:\; \mathrm{tr}_\mu f = 0 \right\}. \tag{1.474}$$

Obviously, the right-hand side of (1.467) is a subspace of the space on the right-hand side of (1.474). The proof that these two spaces coincide follows from the same arguments as in the proof of [Triϵ], Proposition 19.5, pp. 260–263, which in turn is based on the deep Theorem 10.1.1 in [AdH96], p. 281, due to Yu.V. Netrusov. Afterwards one gets the parts (i) and (ii) of the above theorem in the same way as in [Triϵ], p. 266–270, now based on Theorem 1.143(iii). The additional assumption that Ω is connected is needed only to prove that the largest eigenvalue ϱ is simple and that the corresponding eigenfunctions have the property (1.471). We overlooked this point in [Triϵ], Theorem 19.7, which must be corrected in this way. But it is covered by Step 5 of the proof of this theorem on pages 269/270. □

Remark 1.146. Both Theorem 1.143 and Theorem 1.145 had been formulated first in [Tri04b] without proofs. We return to this subject in greater detail and in greater generality in Chapter 7. Comparing the above theorem with the classical assertions in Theorem 1.137 we observe that the Courant property (1.445) has the fractal counterpart (1.471). One might consider (1.472) as a substitute of the C^∞-smoothness of the classical eigenfunctions as stated in Theorem 1.137(iii). But the question arises whether $u_k \in C(\overline{\Omega})$ in (1.472) can be improved in dependence on the quality of μ. This is the case and will again be considered in detail later on. What about the Weyl property (1.444)? If μ satisfies (1.473) with $0 < d < 2$, then one has again (1.444). This follows from [Triϵ], Theorem 19.7, p. 265, with a reference to [Triδ], Theorem 30.2, p. 234. Whether this property remains valid for all measures considered in the above theorem is not clear. But the investigation of this phenomenon was one of the major topics of the relevant parts in [Triδ], [Triϵ], closely related with the music of drums having fractal membranes. We return to this subject both in this introductory chapter and in greater detail in Chapter 7. We refer in this context also to the survey [Tri02b].

1.14 Fractal characteristics of measures

Beginning with Section 1.12 the remaining subsections of this first introductory chapter deal with non-smooth structures from the point of view of function spaces. In particular in the Sections 1.12–1.16 we describe diverse aspects of fractal measures. Now we convert what has been done so far in the two preceding subsections into diverse fractal characteristics and compare them afterwards.

Again let Ω be a bounded C^∞ domain in the plane $\mathbb{R}^2$ and let μ be a (singular) Radon measure in $\mathbb{R}^2$ with

$$\operatorname{supp} \mu = \Gamma \subset \Omega, \quad 0 < \mu(\mathbb{R}^2) < \infty, \quad \text{and} \quad |\Gamma| = 0, \tag{1.475}$$

where $|\Gamma|$ is the Lebesgue measure of Γ. Then we can apply both the Definitions 1.125, 1.129 and the Theorems 1.131, 1.145, where again Q_{jm} are squares centred at $2^{-j}m$ with $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^2$ and of side-length 2^{-j+1}. The q-index in (1.413) does not play any role in what follows. We choose $q = \infty$ and put

$$\mu_p^\lambda = \mu_{p\infty}^\lambda = \begin{cases} \sup\limits_{j \in \mathbb{N}_0} 2^{j\lambda} \left(\sum\limits_{m \in \mathbb{Z}^2} \mu(Q_{jm})^p \right)^{1/p} & \text{if} \quad 0 < p < \infty, \\ \sup\limits_{j \in \mathbb{N}_0} 2^{j\lambda} \mu_j & \text{if} \ \ p = \infty, \end{cases} \tag{1.476}$$

with

$$\mu_j = \sup_{m \in \mathbb{Z}^2} \mu(Q_{jm}), \qquad j \in \mathbb{N}_0, \tag{1.477}$$

as in (1.457). Furthermore under the hypotheses of Theorem 1.145 we denote the function u in (1.471) as the *Courant function* writing now $u = u_\mu^\Omega$ to indicate that it depends on Ω and μ.

Definition 1.147. *Let Ω be a connected bounded C^∞ domain in the plane $\mathbb{R}^2$ and let μ be a Radon measure in $\mathbb{R}^2$ with* (1.475). *Let $0 \le t = 1/p < \infty$.*

(i) *Then*

$$\lambda_\mu(t) = \sup\{\lambda : \mu_p^\lambda < \infty\} \tag{1.478}$$

are multifractal characteristics of μ.

(ii) *Then*

$$s_\mu(t) = \sup\{s : \mu \in B^s_{p\infty}(\mathbb{R}^2)\} \tag{1.479}$$

are Besov characteristics of μ.

(iii) *Let, in addition,*

$$\sum_{j=0}^\infty \sqrt{\mu_j} < \infty \tag{1.480}$$

and let u^Ω_μ be the above Courant function. Then

$$\omega^\Omega_\mu(t) = \sup\big\{\omega : u^\Omega_\mu \in B^\omega_{p\infty}(\Omega)\big\} \tag{1.481}$$

are Courant characteristics of μ.

Remark 1.148. We discuss the above quantities. We always take log to base 2. It follows that

$$\lambda_\mu(t) = \begin{cases} -t \limsup\limits_{j\to\infty} \dfrac{1}{j} \log\Bigg(\displaystyle\sum_{m\in\mathbb{Z}^2} \mu(Q_{jm})^{1/t}\Bigg) & \text{if} \quad 0 < t < \infty, \\ -\limsup\limits_{j\to\infty} \dfrac{1}{j} \log \mu_j & \text{if} \quad t = 0, \end{cases} \tag{1.482}$$

where μ_j is given by (1.477). Using that μ is the indicated measure it follows easily from (1.423) and Hölder's inequality that these $\limsup_{j\to\infty}$ exist for all t with $0 \le t < \infty$. By the references in Remark 1.126 and (1.415) it follows that $\lambda_\mu(t)$ are typical quantities as considered nowadays in (multi)fractal geometry and analysis. According to Theorem 1.131 they are naturally related to some Besov spaces and hence to $s_\mu(t)$ in (1.479). We have

$$\lambda_\mu(1) = 0 \qquad \text{and} \qquad s_\mu(1) = 0. \tag{1.483}$$

The first assertion is obvious. The second one follows from (1.418) and the assumption that μ is a singular measure and hence a singular distribution, Figure 1.14. We collect what can be said about the above quantities and their interrelations. We say that a curve on $[0,\infty)$ is increasing if it is non-decreasing.

Theorem 1.149. *Let Ω be a connected bounded C^∞ domain in the plane $\mathbb{R}^2$ and let μ be a Radon measure in $\mathbb{R}^2$ with* (1.475).

(i) *Then $s = s_\mu(t)$ with $0 \le t < \infty$ according to* (1.479) *is an increasing concave function in the (t,s)-diagram in Figure 1.14 with*

$$2(t-1) \le s_\mu(t) \le 0 \qquad \text{if} \quad 0 \le t \le 1, \tag{1.484}$$

in particular $s_\mu(1) = 0$.

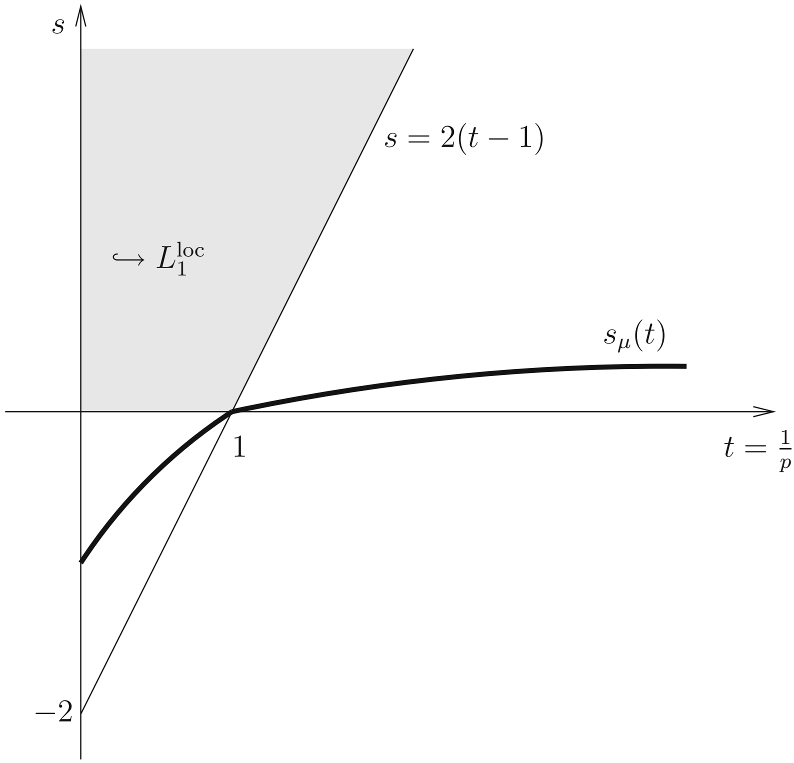

Figure 1.14

(ii) *Then*

$$s_\mu(t) \begin{cases} = \lambda_\mu(t) + 2(t-1) & \text{if } 0 \le t \le 1, \\ \ge \lambda_\mu(t) + 2(t-1) & \text{if } t > 1. \end{cases} \tag{1.485}$$

(iii) *Let, in addition,* (1.480) *be satisfied and let* u^Ω_μ *be the related Courant function. Then* $\mu^\Omega = u^\Omega_\mu \, \mu$ *is a Radon measure satisfying* (1.475) *and*

$$\omega^\Omega_\mu(t) = \begin{cases} s_{\mu^\Omega}(t) + 2 & \text{if } 0 \le t < \infty, \\ s_\mu(t) + 2 & \text{if } 0 \le t \le 1. \end{cases} \tag{1.486}$$

Remark 1.150. Parts (i) and (ii) are covered by [Tri03b]. Otherwise we followed partly [Tri04b]. Detailed proofs in an n-dimensional setting will be given in Chapter 7. Then 2 in (1.484), (1.485) must be replaced by n. We will discuss also in detail under which conditions the upper line in (1.485) can be extended to all t with $0 \le t < \infty$. But we comment on (1.486) now. According to Theorem 1.145 the Courant function $u = u^\Omega_\mu$ is on Γ positive and continuous. Hence μ^Ω is a Radon measure satisfying (1.475) and s_{μ^Ω} looks similar to $s_\mu(t)$. Let $\zeta \in D(\Omega)$

with $\zeta(x) = 1$ near Γ. Then it follows by (1.469), (1.456) that

$$-\varrho\,\Delta u = u\,\mu \tag{1.487}$$

and

$$\varrho\,(\mathrm{id} - \Delta)\zeta u = \zeta u\mu + \varrho\zeta u + v \quad \text{for some} \quad v \in D(\Omega). \tag{1.488}$$

This can be extended from Ω to $\mathbb{R}^2$. Since u is continuous in $\overline{\Omega}$ the right-hand side belongs to any space $B^s_{p\infty}(\mathbb{R}^2)$ with $1 \le p \le \infty$ and $s < s_{\mu^\Omega}(t)$ where again $t = 1/p$. Hence $\zeta u \in B^{s+2}_{p\infty}(\mathbb{R}^2)$ which proves the upper line in (1.486) if $0 \le t \le 1$. In particular $\zeta u \in B^\sigma_{1,\infty}(\mathbb{R}^2)$ for any $\sigma < 2$. If $1 < t = 1/p$ then one gets again by (1.488) that $\zeta u \in B^{s+2}_{p\infty}(\mathbb{R}^2)$ for any $s < \min(2, s_{\mu^\Omega}(t))$. Iterative application of this argument results finally in the upper line in (1.486) for all $t \ge 0$. If $0 \le t \le 1$ then it follows easily from (1.485) and (1.478) that $s_{\mu^\Omega}(t) = s_\mu(t)$. This proves the lower line of (1.486).

1.15 Isotropic measures

The aim of this subsection is twofold. First we introduce isotropic compactly supported Radon measures in $\mathbb{R}^n$ and collect some properties preparing our later considerations. Secondly we indicate very briefly the impact of this specification on some assertions of the two preceding subsections. Then we are back in the plane $\mathbb{R}^2$.

1.15.1 Some notation and basic assertions

We always assume that μ is a positive Radon measure in $\mathbb{R}^n$ with

$$\Gamma = \operatorname{supp}\mu \quad \text{compact} \qquad \text{and} \quad 0 < \mu(\mathbb{R}^n) < \infty. \tag{1.489}$$

According to Section 1.12.2 we identify μ with the generated distribution. (Obviously it is immaterial whether the compact set Γ is a subset of the unit ball as in (1.405) or not.) Again a ball in $\mathbb{R}^n$ centred at $x \in \mathbb{R}^n$ and of radius r is denoted by $B(x, r)$. A non-negative function h on the unit interval $[0, 1]$ is called strictly increasing if $h(t_1) > h(t_2)$ for $t_1 > t_2$.

Definition 1.151. *Let μ be a Radon measure in $\mathbb{R}^n$ according to* (1.489).

(i) *Then μ is called isotropic if there is a continuous strictly increasing function h on the interval* $[0, 1]$ *with $h(0) = 0$, $h(1) = 1$, and*

$$\mu\,(B(\gamma, r)) \sim h(r) \qquad \text{with} \quad \gamma \in \Gamma \quad \text{and} \quad 0 < r < 1 \tag{1.490}$$

(*where the equivalence constants are independent of γ and r*).

(ii) *The isotropic measure* μ *according to part* (i) *is called strongly isotropic if there is a number* $k \in \mathbb{N}$ *such that*

$$h(2^{-j-k}) \le \frac{1}{2}\, h(2^{-j}) \qquad \text{for all} \quad j \in \mathbb{N}_0. \tag{1.491}$$

(iii) *Then* μ *is called strongly diffuse if there is a number* λ *with* $0 < \lambda < 1$ *such that*

$$\mu(Q_1) \le \frac{1}{2}\, \mu(Q_0) \tag{1.492}$$

for any cube Q_0 *centred at some* $\gamma_0 \in \Gamma$ *and of side-length* r *with* $0 < r < 1$, *and any sub-cube* Q_1 *with* $Q_1 \subset Q_0$ *centred at some* $\gamma_1 \in \Gamma$ *and of side-length* λr.

(iv) *Then* μ *is called doubling if there is a number* $c \ge 1$ *such that*

$$\mu\left(B(\gamma, 2r)\right) \le c\, \mu\left(B(\gamma, r)\right) \tag{1.493}$$

for all $\gamma \in \Gamma$ *and all* $0 < r < 1$.

Remark 1.152. We add a discussion of the above definition. By $h(0) = 0$ in part (i) we exclude measures with atoms. Hence the isotropic measure μ in part (i) is *diffuse* according to [Bou56], §5.10, p. 61. This may justify the notation in part (iii) which is a slight modification of [Triϵ], p. 277. The assumption $h(1) = 1$ is convenient but immaterial. We collect a few simple properties of isotropic measures.

Proposition 1.153. *Let* μ *be an isotropic measure according to Definition* 1.151(i).

(i) *Then* μ *is doubling.*

(ii) *Furthermore,* μ *is strongly isotropic if, and only if, it is strongly diffuse.*

(iii) *Furthermore,*

$$|\Gamma| = 0 \quad \text{if, and only if,} \quad \lim_{r \to 0} \frac{r^n}{h(r)} = 0. \tag{1.494}$$

Proof. There is a number $N \in \mathbb{N}$ such that $\Gamma \cap B(\gamma, 2r)$ for any $\gamma \in \Gamma$ and any r with $0 < r < 1$ can be covered by at most N balls centred at Γ and of radius r. Hence μ is doubling. Now also part (ii) is more or less obvious. Let $|\Gamma| = 0$. Then for any $\varepsilon > 0$ there is an open neighborhood Γ_ε of Γ with $|\Gamma_\varepsilon| \le \epsilon$. This follows from (1.408). If $r > 0$ is small then Γ can be covered by $\sim h^{-1}(r)$ balls centred at Γ, of radius r, and of controlled overlapping. Hence $h^{-1}(r)\, r^n \le c\varepsilon$ for some $c > 0$ which is independent of ε and r. This proves the right-hand side of (1.494). But this argument works also in the other direction. □

Remark 1.154. Compact sets Γ in $\mathbb{R}^n$ for which there is an isotropic measure μ with (1.489) and (1.490) are called *h-sets*. The best known *h-sets* are *d-sets* with $h(r) = r^d$ where $0 < d < n$, hence

$$\mu\left(B(\gamma, r)\right) \sim r^d, \qquad \gamma \in \Gamma, \qquad 0 < r < 1, \tag{1.495}$$

(with a minor abuse of notation since h is a function and d is a number). These sets have been studied in great detail in fractal geometry. One may consult the corresponding books mentioned in Section 1.12.1 or [Triδ], pp. 5–7, where one finds further references. Perturbed d-sets, so-called (d, Ψ)-sets where

$$h(r) = r^d \, \Psi(r) \quad \text{typically with} \quad \Psi(r) = \left| \log \frac{r}{2} \right|^b$$

for some $b \in \mathbb{R}$ and related spaces of Besov type have been introduced in [EdT98] and considered in detail in [Mou99], [Mou01a], [Mou01b], [CaM04a], [CaM04b] and [HaM04]. In particular one has to prove that such isotropic measures exist. It came out that the more general question to characterise all isotropic measures μ in $\mathbb{R}^n$ and all functions h with (1.490) is a rather tricky problem. It was finally solved by M. Bricchi in the following way.

Theorem 1.155. *Let h be a continuous strictly increasing function on the interval $[0,1]$ with $h(0) = 0$ and $h(1) = 1$. Then there is a compact set Γ and a Radon measure μ in $\mathbb{R}^n$ with* (1.489) *and* (1.490) *if, and only if, there are two constants $0 < c_1 \le c_2 < \infty$ and a continuous increasing function h^* on $[0,1]$ with*

$$c_1 \, h^*(t) \le h(t) \le c_2 \, h^*(t) \qquad \textit{and} \quad 0 \le t \le 1 \tag{1.496}$$

and

$$h^*(2^{-j}) \le 2^{kn} \, h^*(2^{-j-k}) \qquad \textit{for all} \quad j \in \mathbb{N}_0 \quad \textit{and all} \quad k \in \mathbb{N}_0. \tag{1.497}$$

Remark 1.156. If one has (1.495) with $0 < d \le n$ in mind then (1.497) is reasonable. This applies also to the generalisations indicated in Remark 1.154. However the proof of the above theorem is not so obvious. The final assertion may be found in [Bri03], Theorem 2.7. We refer also to [Bri02b], [Bri04]. There one can also find lists of standard and non-standard examples of such measure-generating functions h.

1.15.2 Traces and fractal operators

One can expect that the assertions in the preceding Sections 1.12–1.14 about the μ-property of function spaces, traces on sets and fractal operators simplify considerably if one assumes in addition that the underlying measure μ is isotropic. We return to these questions in detail in Chapter 7. Here we restrict ourselves to an illustration. As in Section 1.13.2 we assume that Ω is a bounded C^∞ domain in the plane $\mathbb{R}^2$ and that the Radon measure μ satisfying (1.446), hence

$$\operatorname{supp} \mu = \Gamma \subset \Omega, \quad 0 < \mu(\mathbb{R}^2) < \infty, \quad |\Gamma| = 0, \tag{1.498}$$

is isotropic, this means that

$$\mu\left(B(\gamma, r)\right) \sim h(r) \qquad \text{with} \quad \gamma \in \Gamma \quad \text{and} \quad 0 < r < 1 \tag{1.499}$$

according to Definition 1.151. Otherwise $\mathring{H}^1(\Omega)$ has the same meaning as in (1.433), (1.435), normed via the scalar product (1.436). Furthermore the operator B has the same meaning as in (1.455), (1.456) under the assumption that the trace tr_μ according to (1.448) exists. Now the Theorems 1.143, 1.145 can be complemented as follows.

Theorem 1.157. *Let Ω be a bounded C^∞ domain in $\mathbb{R}^2$ and let μ be an isotropic Radon measure with* (1.498), (1.499).

(i) *The following three assertions are equivalent to each other:*

1. *The trace operator* tr_μ*,*
$$\mathrm{tr}_\mu : \quad \mathring{H}^1(\Omega) \hookrightarrow L_2(\Gamma,\mu) \tag{1.500}$$
exists (as a linear and bounded operator).
2. *The operator* tr_μ *in* (1.500) *is compact.*
3. *It holds*
$$\sum_{j=0}^{\infty} h(2^{-j}) < \infty. \tag{1.501}$$

(ii) *The measure μ is strongly isotropic according to Definition 1.151 if, and only if,* (1.501) *is strengthened by*
$$\sum_{j\ge J} h(2^{-j}) \sim h(2^{-J}) \qquad \textit{with} \quad J \in \mathbb{N}_0, \tag{1.502}$$
where the equivalence constants are independent of J.

(iii) *Let μ be strongly isotropic and let, in addition, Ω be connected. Then B,*
$$B = (-\Delta)^{-1} \circ \mathrm{id}^\mu = (-\Delta)^{-1} \circ \mu : \quad \mathring{H}^1(\Omega) \hookrightarrow \mathring{H}^1(\Omega) \tag{1.503}$$
is a self-adjoint, compact, non-negative operator with null space (1.467). *Again let ϱ_k be the positive eigenvalues of B, repeated according to multiplicity and ordered as in* (1.470), *and let u_k be the related eigenfunctions,*
$$Bu_k = \varrho_k \, u_k. \tag{1.504}$$
Then the largest eigenvalue $\varrho = \varrho_1$ is simple and the corresponding eigenfunctions u_1 have the Courant property (1.471). *The eigenfunctions u_k are harmonic in $\Omega\backslash\Gamma$, satisfying* (1.472). *In addition the eigenvalues ϱ_k have the Weyl property,*
$$\varrho_k \sim k^{-1} \qquad \textit{with} \quad k \in \mathbb{N}, \tag{1.505}$$
where the equivalence constants are independent of k.

Proof. Since μ is doubling according to Proposition 1.153 one gets for any $j \in \mathbb{N}_0$,

$$\sum_{m\in\mathbb{Z}^2} \mu(Q_{jm})^2 \sim h(2^{-j}) \sum_{m\in\mathbb{Z}^2} \mu(Q_{jm}) \sim h(2^{-j}). \tag{1.506}$$

Hence (1.461) and (1.462) coincide. Now part (i) follows from Theorem 1.143(ii). If one has (1.491) then one gets (1.502). Conversely since $h(2^{-j})$ is monotonically decreasing, (1.491) follows easily from (1.502). (We return later on in Section 7.1.4 to assertions of this type in greater detail and in a more general context). This proves (ii). By (1.491) we have also (1.463) with $\mu_j \sim h(2^{-j})$. Then all assertions in (iii) are covered by Theorem 1.145 with exception of (1.505). By Proposition 1.153(ii) the measure μ is strongly diffuse. Then (1.505) is a special case of [Triϵ], Theorem 19.17, p. 280. □

Remark 1.158. Part (iii) of the above theorem is the perfect fractal counterpart of the classical Theorem 1.137 including the Weyl property (1.505) under the assumption that the underlying measure μ is strongly isotropic. In particular the distinguished measures μ according to (1.495) with $0 < d < 2$ are strongly isotropic. This applies also to the slightly more general measures mentioned in Remark 1.154 and to many other measures which may be found in [Bri02b], [Bri04] quoted in Remark 1.156. On the other hand, the Courant property (1.471) seems to be more universal than the Weyl property since it applies also to measures which are not necessarily isotropic (even used in Definition 1.147 and Theorem 1.149 as a fractal characteristic). But it comes out that this applies also to the Weyl property (1.505) which we are going to discuss in the following subsection.

1.16 Weyl measures

In connection with the above Remark 1.158 it seems to be reasonable to have a closer look at measures for which the eigenvalues of the generated operators B have the Weylian behavior (1.505).

Definition 1.159. *Let μ be a Radon measure in the plane $\mathbb{R}^2$ such that*

$$\operatorname{supp} \mu = \Gamma \quad \textit{compact}, \qquad 0 < \mu(\mathbb{R}^2) < \infty, \quad |\Gamma| = 0. \tag{1.507}$$

Then μ is said to be a Weyl measure if for any bounded C^∞ domain in $\mathbb{R}^2$ with $\Gamma \subset \Omega$, the operator B,

$$B = (-\Delta)^{-1} \circ \mathrm{id}^\mu = (-\Delta)^{-1} \circ \mu : \quad \mathring{H}^1(\Omega) \hookrightarrow \mathring{H}^1(\Omega), \tag{1.508}$$

according to Proposition 1.141 *is compact and*

$$\varrho_k \sim k^{-1}, \qquad k \in \mathbb{N}, \tag{1.509}$$

where ϱ_k are the positive eigenvalues of B, repeated according to multiplicity and ordered by decreasing magnitude as in (1.470).

Remark 1.160. The operator B introduced by Proposition 1.141 can be generated by the quadratic form (1.468). Hence it is always non-negative and self-adjoint in $\mathring{H}^1(\Omega)$. Assumed to be compact, the positive eigenvalues can be ordered as in (1.470) and the question (1.509) makes sense. It is the fractal counterpart of Weyl's observation (1.444).

Theorem 1.161. *Any doubling and strongly diffuse Radon measure in the plane* $\mathbb{R}^2$ *according to Definition* 1.151 *with* (1.507) *is a Weyl measure.*

Remark 1.162. The (somewhat complicated) proof of this theorem may be found in [Triϵ], Theorem 19.17, pp. 280–289. It is one of the main assertions of the book [Triϵ]. Both assumptions, to be doubling and to be strongly diffuse, suggest that Weyl measures might have some isotropic properties. But this is not necessarily the case and one can construct Weyl measures which are neither doubling nor strongly diffuse.

Proposition 1.163. *Let* μ_k *with* $k = 1, \ldots, K$ *be isotropic measures in the plane* $\mathbb{R}^2$ *according to Definition* 1.151(i) *with*

$$\mu_k\left(B(\gamma, r)\right) \sim h_k(r); \quad 0 < r < 1, \quad \gamma \in \Gamma_j = \operatorname{supp} \mu_j \tag{1.510}$$

and

$$\sum_{j \geq J} h_k(2^{-j}) \sim h_k(2^{-J}) \qquad \text{with} \quad J \in \mathbb{N}_0, \tag{1.511}$$

where the equivalence constants are independent of J. *Let*

$$0 < h_1(r) \leq h_2(r) \leq \cdots \leq h_K(r) \qquad \text{if} \quad 0 < r < 1 \tag{1.512}$$

and

$$\frac{h_k(r)}{h_{k+1}(r)} \to 0 \quad \text{if} \quad r \to 0, \qquad k = 1, \ldots, K-1. \tag{1.513}$$

Then

$$\mu = \sum_{k=1}^{K} \mu_k \tag{1.514}$$

is a Weyl measure.

Remark 1.164. It follows by Theorem 1.157 that each μ_k is a Weyl measure. Afterwards one can use Corollary 2 in [Tri04d] and its proof, appropriately adapted. If the sets Γ_k are embedded into each other then one cannot expect in general that the measure in (1.514) with $K \geq 2$ is doubling or strongly isotropic. The whole matter is somewhat tricky. It is closely related to the music of fractal drums (or better drums with fractal membranes). We refer in this context to [Triδ], Sections 28.9–28.11, p. 230, and Section 30, and in particular to the extensive discussion in [Triϵ], Section 19.18, pp. 288–291. Further assertions may be found in [Tri04c], [Tri04d]. So far we have no example of a Radon measure with (1.498) such that B in (1.503) is compact, but not Weylian. There remains the challenging problem:

Characterise all Weyl measures in the plane.

1.17 Spaces on fractals and on quasi-metric spaces

1.17.1 Fractal characteristics, revisited

We now always assume that μ is a (singular) Radon measure in $\mathbb{R}^n$ with

$$\operatorname{supp}\mu = \Gamma \quad \text{compact}, \quad 0 < \mu(\mathbb{R}^n) < \infty \quad \text{and} \quad |\Gamma| = 0, \tag{1.515}$$

where $|\Gamma|$ is the Lebesgue measure of Γ. We dealt in Section 1.14 with fractal characteristics of such measures in the plane $\mathbb{R}^2$ under the additional assumption $\Gamma \subset \Omega$ where Ω is a bounded C^∞ domain. Only the third of the three characteristics of μ introduced in Definition 1.147 depends on this additional assumption. The multifractal characteristics $\lambda_\mu(t)$ and the Besov characteristics $s_\mu(t)$ in (1.478) and (1.479) are largely independent of Ω and their generalisation from $\mathbb{R}^2$ to $\mathbb{R}^n$ is rather obvious. Furthermore, the whole subject will be considered in detail in Chapter 7 based on [Tri03b]. But to make what follows independently readable we fix some notation and assertions, restricting ourselves to the bare minimum.

Again let Q_{jm} with $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$ be closed cubes in $\mathbb{R}^n$ centred at $2^{-j}m$ with side-length 2^{-j+1}. Then we put as in Definition 1.125 now with respect to (1.515)

$$\mu^\lambda_{pq} = \left(\sum_{j=0}^\infty 2^{j\lambda q} \left(\sum_{m\in\mathbb{Z}^n} \mu(Q_{jm})^p \right)^{q/p} \right)^{1/q} \tag{1.516}$$

where $0 < p \le \infty$, $0 < q \le \infty$ and $\lambda \in \mathbb{R}$ (with the obvious modifications if p and/or q are infinite). In particular,

$$\mu^\lambda_p = \mu^\lambda_{p\infty} = \begin{cases} \sup\limits_{j\in\mathbb{N}_0} 2^{j\lambda} \left(\sum\limits_{m\in\mathbb{Z}^n} \mu(Q_{jm})^p \right)^{1/p} & \text{if } 0 < p < \infty, \\ \sup\limits_{j\in\mathbb{N}_0, m\in\mathbb{Z}^n} 2^{j\lambda}\, \mu(Q_{jm}) & \text{if } p = \infty \end{cases} \tag{1.517}$$

is the obvious extension of (1.476) from $\mathbb{R}^2$ to $\mathbb{R}^n$.

Definition 1.165. *Let μ be a Radon measure in $\mathbb{R}^n$ with* (1.515)*. Let $0 \le t = 1/p < \infty$. Then*

$$\lambda_\mu(t) = \sup\left\{ \lambda : \ \mu^\lambda_p < \infty \right\} \tag{1.518}$$

are multifractal characteristics of μ and

$$s_\mu(t) = \sup\left\{ s : \ \mu \in B^s_{p\infty}(\mathbb{R}^n) \right\} \tag{1.519}$$

are Besov characteristics of μ.

Remark 1.166. This is the obvious extension of parts (i) and (ii) in Definition 1.147. It applies also to (1.482) making again clear that $\lambda_\mu(t)$ are typical quantities as considered nowadays in (multi)fractal geometry and analysis.

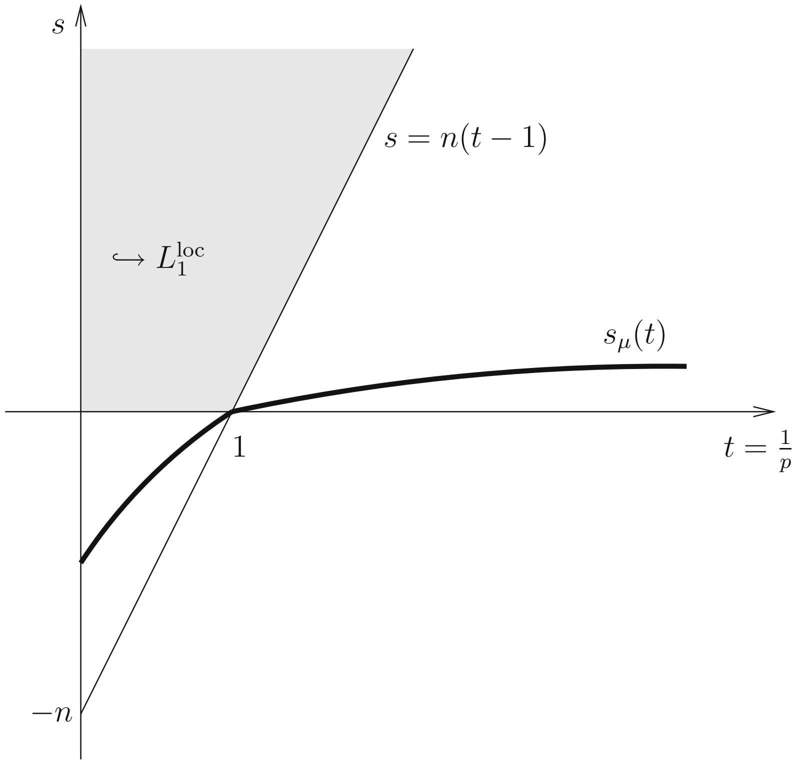

Figure 1.17.1

Theorem 1.167. *Let μ be a Radon measure in $\mathbb{R}^n$ with* (1.515). *Let $\lambda_\mu(t)$ and $s_\mu(t)$ be the characteristics according to Definition* 1.165. *Then $s = s_\mu(t)$ with $0 \le t < \infty$ is an increasing concave function in the (t,s)-diagram in Figure* 1.17.1 *with*

$$n(t-1) \le s_\mu(t) \le 0 \qquad \textit{if} \quad 0 \le t \le 1 \tag{1.520}$$

and

$$s_\mu(t) \begin{cases} = \lambda_\mu(t) + n(t-1) & \textit{if} \;\; 0 \le t \le 1, \\ \ge \lambda_\mu(t) + n(t-1) & \textit{if} \;\; t > 1. \end{cases} \tag{1.521}$$

Remark 1.168. This is the n-dimensional version of the parts (i) and (ii) of Theorem 1.149. We have $s_\mu(1) = \lambda_\mu(1) = 0$. We return to the subject in detail in Chapter 7. Furthermore, if $0 \le t = 1/p < 1$ and $s_\mu(t) < 0$ then

$$\mu \in B^{s_\mu(t)}_{pq}(\mathbb{R}^n) \qquad \text{if, and only if,} \quad \mu^{\lambda_\mu(t)}_{pq} < \infty. \tag{1.522}$$

This follows from Theorem 1.131.

1.17.2 Traces and trace spaces

Let μ be a Radon measure in $\mathbb{R}^n$ according to (1.515). Then $L_r(\Gamma,\mu)$ with $0 < r \le \infty$ has the usual meaning quasi-normed by

$$\|f\,|L_r(\Gamma,\mu)\| = \Big(\int_{\mathbb{R}^n} |f(x)|^r\,\mu(\mathrm{d}x)\Big)^{1/r} = \Big(\int_{\Gamma} |f(\gamma)|^r\,\mu(\mathrm{d}\gamma)\Big)^{1/r} \tag{1.523}$$

with the usual modification if $r=\infty$. According to Section 1.12.2 one can interpret $L_r(\Gamma,\mu)$ with $1 \le r \le \infty$ as a subset of $S'(\mathbb{R}^n)$ identifying $f \in L_r(\Gamma,\mu)$ with the complex finite measure $f\mu \in S'(\mathbb{R}^n)$, hence

$$(\mathrm{id}_\mu f)\,(\varphi) = \int_{\Gamma} f(\gamma)\,\varphi(\gamma)\,\mu(\mathrm{d}\gamma), \quad \varphi \in S(\mathbb{R}^n). \tag{1.524}$$

Again we call id_μ the *identification operator*. We investigate later on in Chapter 7 mapping properties of id_μ. At this moment we only remark that for $s<0$, $0<p\le\infty$ and $0<q\le\infty$

$$\mathrm{id}_\mu : \quad L_\infty(\Gamma,\mu) \hookrightarrow B^s_{pq}(\mathbb{R}^n) \tag{1.525}$$

exists (as a linear and bounded operator) if, and only if, $\mu \in B^s_{pq}(\mathbb{R}^n)$. This follows from Corollary 1.12 if one chooses there for the kernel k a non-negative function. Then it follows from Theorem 1.131 that id_μ according to (1.525) exists if, and only if,

$$\mu^\lambda_{pq} < \infty \qquad \text{with} \quad \lambda = s + n - \frac{n}{p}, \tag{1.526}$$

where μ^λ_{pq} is given by (1.516). By Theorem 1.167 and Figure 1.17.1 the case $1 < p \le \infty$ is of special interest. Then we have $s_\mu(t) \le 0$ with $t = 1/p$ for the Besov characteristics of μ. Hence id_μ in (1.525) exists if

$$0 \le t = 1/p < 1, \quad 0 < q \le \infty, \quad -\infty < s < s_\mu(t), \tag{1.527}$$

and it does not exist if

$$0 \le t = 1/p < 1, \quad 0 < q \le \infty, \quad s_\mu(t) < s < \infty. \tag{1.528}$$

Furthermore, id_μ exists for $0 \le t = 1/p < 1,\ 0 < q \le \infty$, if

$$s = s_\mu(t) < 0 \qquad \text{and} \quad \mu \in B^{s_\mu(t)}_{pq}(\mathbb{R}^n). \tag{1.529}$$

According to (1.522) this can be reformulated in terms of multifractal characteristics $\lambda_\mu(t)$. The reason for this somewhat specific consideration comes from its close connection with traces which we are going to define now.

Definition 1.169. *Let μ be a Radon measure in $\mathbb{R}^n$ with* (1.515). *Let*

$$1 < p < \infty, \qquad 0 < q < \infty \qquad \textit{and} \qquad s > 0. \tag{1.530}$$

Let for some $c > 0$,

$$\int_\Gamma |\varphi(\gamma)|\, \mu(\mathrm{d}\gamma) \le c\, \|\varphi \,|B^s_{pq}(\mathbb{R}^n)\| \quad \textit{for all} \quad \varphi \in S(\mathbb{R}^n). \tag{1.531}$$

Then the trace operator tr_μ*,*

$$\mathrm{tr}_\mu : \quad B^s_{pq}(\mathbb{R}^n) \hookrightarrow L_1(\Gamma, \mu), \tag{1.532}$$

is the completion of the pointwise trace $(\mathrm{tr}_\mu \varphi)(\gamma) = \varphi(\gamma)$ *with* $\varphi \in S(\mathbb{R}^n)$. *Furthermore, the image of* tr_μ *according to* (1.532) *is denoted by* $\mathrm{tr}_\mu B^s_{pq}(\mathbb{R}^n)$ *and quasi-normed by*

$$\|g \,|\mathrm{tr}_\mu B^s_{pq}(\mathbb{R}^n)\| = \inf \|f \,|B^s_{pq}(\mathbb{R}^n)\| \tag{1.533}$$

where the infimum is taken over all $f \in B^s_{pq}(\mathbb{R}^n)$ *with* $\mathrm{tr}_\mu f = g$.

Remark 1.170. We dealt in [Triϵ], Section 9, in detail with traces of spaces on sets Γ, preferably with $F^s_{pq}(\mathbb{R}^n)$ in place of $B^s_{pq}(\mathbb{R}^n)$. The above version is near to [Triϵ], Section 9.32, p. 151. We refer also to the above Section 1.13.2 where we used identification operators and trace operators in an L_2-setting in connection with the fractal Laplacian. To justify the above definition we first remark the well-known fact that $S(\mathbb{R}^n)$ is dense in $B^s_{pq}(\mathbb{R}^n)$ with (1.530). We refer to [Triβ], Theorem 2.3.3, p. 48. Then it makes sense to define tr_μ by standard arguments and to quasi-norm the trace space by (1.533). If one has (1.531) for some s, p, q, with (1.530) then one has also (1.531) for all spaces $B^{s+\varepsilon}_{pv}(\mathbb{R}^n)$ with $\varepsilon > 0$ and $0 < v \le \infty$. Under this assumption one can extend the above definition to $q = \infty$.

Remark 1.171. The restriction for p in (1.530) will be of some use for us in connection with duality arguments and for the description of the trace spaces $\mathrm{tr}_\mu B^s_{pq}(\mathbb{R}^n)$ in terms of quarkonial representations. Also the choice of $L_1(\Gamma, \mu)$ as the largest target space is very reasonable and the possible replacement of $L_1(\Gamma, \mu)$ in (1.532) by $L_p(\Gamma, \mu)$ might cause some extra trouble at least as long as the measures μ with (1.515) are not specified. If this is the case then several modifications of the above definition are not only reasonable but they shed additional light on our intentions and produce interesting assertions. We formulate the outcome. First we recall that a compact set Γ in $\mathbb{R}^n$ is called a d-set with $0 < d < n$ if there is a Radon measure μ with $\mathrm{supp}\, \mu = \Gamma$ and

$$\mu\left(B(\gamma, r)\right) \sim r^d, \qquad \gamma \in \Gamma, \quad 0 < r < 1, \tag{1.534}$$

where $B(\gamma, r)$ is a ball centred at γ and of radius r. For μ one can choose the restriction of the Hausdorff measure $\mathcal{H}^d$ in $\mathbb{R}^n$ on Γ. Secondly we modify (1.531) by asking whether there is a constant $c > 0$ such that

$$\left(\int_\Gamma |\varphi(\gamma)|^p \, \mu(\mathrm{d}\gamma) \right)^{1/p} \le c \, \| \varphi \, | B^s_{pq}(\mathbb{R}^n) \| \quad \text{for all} \quad \varphi \in S(\mathbb{R}^n) \tag{1.535}$$

admitting now

$$0 < p < \infty, \qquad 0 < q < \infty, \qquad s > 0. \tag{1.536}$$

Of course, the $L_p(\Gamma, \mu)$ with $p < 1$ are not very comfortable spaces and they cannot be interpreted as distributions. Nevertheless the question makes sense and one gets by completion tr_μ and the trace space $\mathrm{tr}_\mu B^s_{pq}(\mathbb{R}^n)$. One can replace $B^s_{pq}(\mathbb{R}^n)$ on the right-hand side of (1.535) by $F^s_{pq}(\mathbb{R}^n)$. Obviously, $\mathrm{tr}_\mu B^s_{pq}(\mathbb{R}^n)$ and $\mathrm{tr}_\mu F^s_{pq}(\mathbb{R}^n)$ are introduced similarly as in Definition 1.169.

Proposition 1.172. *Let Γ be a compact d-set in $\mathbb{R}^n$ with $0 < d < n$ and let μ be a corresponding measure with $\Gamma = \operatorname{supp} \mu$ and* (1.534). *Let tr_μ and the related trace spaces be defined as in the preceding Remark* 1.171. *Then*

$$\mathrm{tr}_\mu B^{\frac{n-d}{p}}_{pq}(\mathbb{R}^n) = L_p(\Gamma, \mu) \qquad \text{if} \quad 0 < p < \infty, \quad 0 < q \le \min(1, p), \tag{1.537}$$

and

$$\mathrm{tr}_\mu F^{\frac{n-d}{p}}_{pq}(\mathbb{R}^n) = L_p(\Gamma, \mu) \qquad \text{if} \quad 0 < p \le 1, \quad 0 < q \le \infty. \tag{1.538}$$

Proof. According to [Triϵ], Remark 9.19, any d-set with $d < n$ satisfies the so-called ball condition (or porosity condition). Then the above proposition follows immediately from [Triδ], Corollary 18.12, p. 142, where one finds also an explanation of how to incorporate $q = \infty$ in (1.538). $\square$

Remark 1.173. Assertions of this type have some history, in particular if $d = n-1$ and if Γ is replaced by $\mathbb{R}^{n-1}$. Comments and references may be found in [Triγ], Section 4.4.3, pp. 220–221.

After this useful and illuminating digression we return to traces as introduced in Definition 1.169. Recall that $\frac{1}{p} + \frac{1}{p'} = 1$ where $1 \le p \le \infty$. Furthermore, μ^λ_{pq} has been introduced in (1.516).

Theorem 1.174. *Let μ be a Radon measure in $\mathbb{R}^n$ with* (1.515). *Let*

$$1 < p < \infty, \qquad 1 \le q < \infty, \qquad s > 0. \tag{1.539}$$

Then the following three assertions are equivalent to each other:

(i) *The trace tr_μ according to* (1.532) *exists,*

(ii) $\mu \in B^{-s}_{p'q'}(\mathbb{R}^n)$,

(iii) $\mu^\lambda_{p'q'} < \infty$ *with* $\lambda = \frac{n}{p} - s$.

Proof. The trace operator tr_μ and the identification operator id_μ as introduced in (1.524) are dual to each other, $\mathrm{id}_\mu = (\mathrm{tr}_\mu)'$. We refer for details and explanations to [Triϵ], Section 9.2, pp. 122–124. In particular, tr_μ according to (1.532) exists if, and only if,

$$\mathrm{id}_\mu : \quad L_\infty(\Gamma,\mu) \hookrightarrow B^{-s}_{p'q'}(\mathbb{R}^n) \tag{1.540}$$

exists. Here we used in addition the duality assertion

$$\big(B^s_{pq}(\mathbb{R}^n)\big)' = B^{-s}_{p'q'}(\mathbb{R}^n), \tag{1.541}$$

[Triβ], Theorem 2.11.2, p. 178. However according to (1.525), (1.526) and the explanations given there, (1.540) is equivalent to (ii), which, in turn, is equivalent to (iii). □

Recall that $\frac{1}{q} + \frac{1}{q'} = 1$ if $1 \le q \le \infty$ and $q' = \infty$ if $0 < q < 1$.

Corollary 1.175. *Let μ be a Radon measure in $\mathbb{R}^n$ with* (1.515). *Let $0 < 1/p = t < 1$ and let $s_\mu(t)$ be the Besov characteristics of μ as introduced in Definition* 1.165. *Then*

$$\mathrm{tr}_\mu : \quad B^s_{pq}(\mathbb{R}^n) \hookrightarrow L_1(\Gamma,\mu) \tag{1.542}$$

exists according to Definition 1.169 (*complemented by Remark* 1.170 *as far as $q = \infty$ is concerned*) *if*

$$\begin{cases} \textit{either} & s > |s_\mu(1-t)|, \ 0 < q \le \infty, \\ \textit{or} & s = |s_\mu(1-t)| > 0, \ 0 < q < \infty, \ \mu \in B^{s_\mu(1-t)}_{p'q'}(\mathbb{R}^n). \end{cases} \tag{1.543}$$

Proof. By the definition of the Besov characteristics one has

$$\mu \in B^{-s}_{p'v}(\mathbb{R}^n) \quad \text{if} \quad -s < s_\mu(1-t) \quad \text{and} \quad 0 < v \le \infty. \tag{1.544}$$

Then the upper line in (1.543) follows from Theorem 1.174 and elementary embeddings (including $q = \infty$). Similarly one gets the lower line. □

Remark 1.176. The above assertions are rather satisfactory. If $0 < s < |s_\mu(1-t)|$ with $0 < 1/p = t < 1$ then it follows by Theorem 1.174 that there is no trace. Hence $s = |s_\mu(1-t)|$ is just the limiting case. By Theorem 1.174 the lower line in (1.543) can be reformulated as follows: *If*

$$\begin{aligned} &s = |s_\mu(1-t)| > 0, \quad 1 \le q < \infty \quad \textit{and} \\ &\mu^\lambda_{p'q'} < \infty \quad \textit{with} \quad \lambda = tn + s_\mu(1-t) \end{aligned} \tag{1.545}$$

then one has (1.542).

Remark 1.177. We compare the above corollary with Proposition 1.172 and assume that Γ is a d-set with $0 < d < n$. Then it follows from (1.534) and (1.516) that

$$\mu^\lambda_{pq} \sim \left(\sum_{j=0}^\infty 2^{j\lambda q} \left(2^{-jdp+jd}\right)^{q/p} \right)^{1/q} = \left(\sum_{j=0}^\infty 2^{jq(\lambda - d + \frac{d}{p})} \right)^{1/q}. \tag{1.546}$$

Hence $\lambda_\mu(t) = d(1-t)$ and $\lambda_\mu(1-t) = td$, and according to (1.521) for $1 \le p \le \infty$,

$$s_\mu(1-t) = -t(n-d) = -\frac{n-d}{p}. \tag{1.547}$$

This shows that Proposition 1.172 fits in the above scheme, now with $1 < p < \infty$, and $q \le 1$ (hence $q' = \infty$).

Definition 1.178. *Let μ be a Radon measure in $\mathbb{R}^n$ with* (1.515). *Let $0 < 1/p = t < 1$ and let $s_\mu(t)$ be the Besov characteristics of μ according to Definition* 1.165.

(i) *Let $0 < q \le \infty$ and $s > 0$. Then*

$$B^s_{pq}(\Gamma,\mu) = \mathrm{tr}_\mu B^{s+|s_\mu(1-t)|}_{pq}(\mathbb{R}^n) \tag{1.548}$$

according to Definition 1.169 (*complemented by Remark* 1.170 *as far as $q = \infty$ is concerned*).

(ii) *Let $0 < q < \infty$, $s_\mu(1-t) < 0$ and $\mu \in B^{s_\mu(1-t)}_{p'q'}(\mathbb{R}^n)$. Then*

$$B^0_{pq}(\Gamma,\mu) = \mathrm{tr}_\mu B^{|s_\mu(1-t)|}_{pq}(\mathbb{R}^n). \tag{1.549}$$

Remark 1.179. By Corollary 1.175 both (1.548) and (1.549) make sense as subspaces of $L_1(\Gamma,\mu)$. The spaces in (1.548) and (1.549) are quasi-normed as usual,

$$\|f\,|B^s_{pq}(\Gamma,\mu)\| = \inf \|g\,|B^{s+|s_\mu(1-t)|}_{pq}(\mathbb{R}^n)\|$$

where the infimum is taken over all $g \in B^{s+|s_\mu(1-t)|}_{pq}(\mathbb{R}^n)$ with $\mathrm{tr}_\mu g = f$. In particular, $B^s_{pq}(\Gamma,\mu)$ are quasi-Banach spaces (Banach spaces if $q \ge 1$). We justify the notation. Let Γ be a compact d-set in $\mathbb{R}^n$ with $0 < d < n$, hence

$$\mathrm{supp}\,\mu = \Gamma, \quad \mu\,(B(\gamma,r)) \sim r^d, \quad \gamma \in \Gamma, \quad 0 < r < 1, \tag{1.550}$$

according to (1.534) and the explanations given there. Then we get by (1.545)–(1.549) for $s > 0$,

$$B^s_{pq}(\Gamma,\mu) = \mathrm{tr}_\mu B^{s+\frac{n-d}{p}}_{pq}(\mathbb{R}^n), \qquad 1 < p < \infty, \quad 0 < q \le \infty, \tag{1.551}$$

and

$$B^0_{p,1}(\Gamma,\mu) = \mathrm{tr}_\mu B^{\frac{n-d}{p}}_{p,1}(\mathbb{R}^n) = L_p(\Gamma,\mu), \qquad 1 < p < \infty, \tag{1.552}$$

where the latter equality is a special case of (1.537). Furthermore, (1.551) is in good agreement with the well-known trace theorem from $\mathbb{R}^n$ onto d-dimensional hyper-planes,

$$\mathrm{tr} B^{s+\frac{n-d}{p}}_{pq}(\mathbb{R}^n) = B^s_{pq}(\mathbb{R}^d), \qquad n > d \in \mathbb{N}, \tag{1.553}$$

where $s > 0$, $1 \le p \le \infty$, $0 < q \le \infty$, [Triβ], Theorem 2.7.2, p. 132.

Remark 1.180. In [Triϵ], Section 9, we dealt in detail with traces of function spaces on compact sets in $\mathbb{R}^n$, preferably in terms of the spaces $F^s_{pq}(\mathbb{R}^n)$, complemented by a few corresponding assertions for the spaces $B^s_{pq}(\mathbb{R}^n)$. There one finds also detailed references to the whole subject. Trace spaces on compact sets $\Gamma = \operatorname{supp} \mu$ are introduced in [Triϵ], Section 9, by extending quarkonial decompositions as considered in the above Section 1.6 from $\mathbb{R}^n$ to Γ. Then we proved afterwards in [Triϵ], Theorem 9.33, p. 151, that these spaces coincide with the above spaces as introduced in Definition 1.178. Now we prefer the reverse way dealing with quarkonial decompositions in Section 1.17.3 below. Trace spaces on compact sets Γ in $\mathbb{R}^n$ attracted some attention, especially in case of d-sets, based on other means. The first corresponding systematic studies are due to A. Jonsson and H. Wallin. They summarised their results in [JoW84]. The corresponding spaces are defined with the help of first and higher differences $\Delta^M_h f(\gamma)$ and approximation procedures by polynomials. References to their subsequent work may be found in [Triϵ], Section 9.34(iii), p. 154. Our own considerations of trace spaces on compact d-sets in $\mathbb{R}^n$ began in [Triδ], especially in the Sections 18 and 20. They are based on (1.551) as a definition even in the larger context of $0 < p \le \infty$. We refer to [Triδ], Definition 20.2, p. 159. This approach has been generalised from d-sets to (d, Ψ)-sets and to h-sets, and to corresponding measures according to Definition 1.151 with the specification

$$h(t) = t^d \, \Psi(t), \qquad 0 < d < n, \tag{1.554}$$

and $\Psi(t)$ as in (1.267). However there is a somewhat curious point. In case of d-sets one has the satisfactory assertion (1.552). Nothing like this can be expected in the general situation covered by Definition 1.178 even if one has (1.549). We discussed this point in some detail in [Triϵ], Section 9.34(i), p. 153, and in Section 22.1, pp. 329–333, suggesting to take assertions of type (1.552) as a guide and to modify the spaces $B^s_{pq}(\mathbb{R}^n)$ in Definition 1.178 in case of (d, ψ)-sets and h-sets appropriately (*tailored spaces*). This works pretty well and results in spaces of generalised smoothness as briefly mentioned in Section 1.9.5, especially in (1.269) and (1.271). We do not go into detail and refer to the corresponding substantial work by S. Moura, [Mou99], [Mou01a], [Mou01b] in case of (d, Ψ)-sets and by M. Bricchi, [Bri01], [Bri04], [Bri02a] in case of (d, Ψ)-sets and h-sets. Furthermore, [EdT98], [EdT99] might be considered as forerunners as far as (d, Ψ)-sets are concerned. A corresponding brief description may also be found in [Triϵ], Section 22.

One point is of interest for us later on. It is well known that the trace according to (1.553) is a retraction. This means that there is a linear and bounded extension operator ext,

$$\mathrm{ext}: \quad B^s_{pq}(\mathbb{R}^d) \hookrightarrow B^{s+\frac{n-d}{p}}_{pq}(\mathbb{R}^n) \tag{1.555}$$

such that

$$\mathrm{tr} \circ \mathrm{ext} = \mathrm{id} \qquad (\text{identity in } B^s_{pq}(\mathbb{R}^d)). \tag{1.556}$$

This is also covered by [Triβ], Theorem 2.7.2, p. 132. There is the following counterpart for arbitrary d-sets:

Theorem 1.181. *Let* Γ *be a compact* d*-set in* $\mathbb{R}^n$ *with* $0 < d < n$ *and let* μ *be a corresponding Radon measure with* (1.550)*. Let* $0 < s < 1$, $1 < p < \infty$, $1 \le q \le \infty$, *and let* tr_μ *be the trace operator according to* (1.551)*. Then there is a common linear and bounded extension operator* ext_μ *with*

$$\mathrm{ext}_\mu \ : \quad B^s_{pq}(\Gamma,\mu) \hookrightarrow B^{s+\frac{n-d}{p}}_{pq}(\mathbb{R}^n) \tag{1.557}$$

and

$$\mathrm{tr}_\mu \circ \mathrm{ext}_\mu = \mathrm{id} \qquad (\textit{identity in } B^s_{pq}(\Gamma,\mu)). \tag{1.558}$$

Remark 1.182. This assertion is covered by [JoW84, Theorem 1, p. 103, Theorem 3, p. 155]. The extension operator constructed there is independent of s, p, q. The extension of this assertion to $s \ge 1$ is more complicated and depends on the so-called jet-definition of trace spaces. The situation improves substantially if the d-set satisfies the so-called *Markov property*. This applies to d-sets with $n-1 < d < n$. But otherwise it is a severe restriction. We refer to [JoW84] and the more recent papers [Wal91, Jon93, Jon96, FaJ01].

1.17.3 Quarkonial representations

In Section 1.6 we described quarkonial representations of the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ based on the approximate lattices (1.96)–(1.98) and the related β-quarks as introduced in Definition 1.36. Now we develop a corresponding theory for the trace spaces $B^s_{pq}(\Gamma,\mu)$ according to Definition 1.178. Again let μ be a Radon measure in $\mathbb{R}^n$ satisfying (1.515). First we adapt (1.96)–(1.98) to the compact support Γ of μ. Let

$$\Gamma_\varepsilon = \{x \in \mathbb{R}^n \ : \ \mathrm{dist}(x,\Gamma) < \varepsilon\}, \qquad \varepsilon > 0, \tag{1.559}$$

be the ε-neighborhood of Γ. Let $k \in \mathbb{N}_0$ and let

$$\left\{\gamma^{k,m}\right\}_{m=1}^{M_k} \subset \Gamma \quad \text{and} \quad \left\{\psi^{k,m}\right\}_{m=1}^{M_k} \tag{1.560}$$

be *approximate lattices* and *subordinated resolutions of unity* with the following properties: There are positive numbers c_1 and c_2 with

$$|\gamma^{k,m_1} - \gamma^{k,m_2}| \ge c_1\, 2^{-k}, \qquad m_1 \ne m_2, \quad k \in \mathbb{N}_0, \tag{1.561}$$

and

$$\Gamma_{\varepsilon_k} \subset \bigcup_{m=1}^{M_k} B\left(\gamma^{k,m}, 2^{-k-2}\right), \qquad k \in \mathbb{N}_0, \tag{1.562}$$

where $\varepsilon_k = c_2 2^{-k}$. Recall that $B(x,r)$ has the same meaning as in (1.95): a ball centred at $x \in \mathbb{R}^n$ and of radius $r > 0$. Furthermore, $\psi^{k,m}$ are non-negative C^∞ functions in $\mathbb{R}^n$ with

$$\operatorname{supp} \psi^{k,m} \subset B\left(\gamma^{k,m}, 2^{-k-1}\right), \quad k \in \mathbb{N}_0; \quad m = 1, \ldots, M_k, \tag{1.563}$$

$$|D^\alpha \psi^{k,m}(x)| \le c_\alpha\, 2^{k|\alpha|}, \qquad k \in \mathbb{N}_0; \quad m = 1, \ldots, M_k, \tag{1.564}$$

for all $\alpha \in \mathbb{N}_0^n$ and suitable constants c_α, and

$$\sum_{m=1}^{M_k} \psi^{k,m}(x) = 1, \qquad k \in \mathbb{N}_0, \quad x \in \Gamma_{\varepsilon_k}. \tag{1.565}$$

We always assume that the approximate lattices $\{\gamma^{k,m}\}$ and the subordinated resolutions of unity $\{\psi^{k,m}\}$ can be extended to $\mathbb{R}^n$ such that one gets approximate lattices $\{x^{k,m}\}$ and related resolutions of unity according to (1.96)–(1.101) with a sufficiently large K in (1.101). In other words, of interest are those approximate lattices $\{x^{k,m}\}$ in $\mathbb{R}^n$ and subordinated resolutions of unity $\{\psi^{k,m}\}$ according to (1.96)–(1.101) which are adapted to Γ in the way described above. Next we need the counterparts of β-quarks in (1.102) and of the sequence spaces in (1.108). Recall that

$$\gamma^\beta = \gamma_1^{\beta_1} \cdots \gamma_n^{\beta_n} \qquad \text{where} \quad \gamma \in \Gamma \subset \mathbb{R}^n \quad \text{and} \quad \beta \in \mathbb{N}_0^n. \tag{1.566}$$

Definition 1.183. *Let μ be a Radon measure in $\mathbb{R}^n$ according to* (1.515). *Let $0 < 1/p = t < 1$ and let $\lambda_\mu(t)$ be the multifractal characteristics of μ according to Definition* 1.165. *Let $\{\gamma^{k,m}\}_{m=1}^{M_k}$ and $\{\psi^{k,m}\}_{m=1}^{M_k}$ be as in* (1.560)–(1.565).

(i) *Let $s \ge 0$. Then*

$$(\beta\text{-qu})_{km}(\gamma) = 2^{-k(s-\lambda_\mu(1-t))}\, 2^{k|\beta|}\, (\gamma - \gamma^{k,m})^\beta\, \psi^{k,m}(\gamma) \tag{1.567}$$

with

$$\gamma \in \Gamma, \quad \beta \in \mathbb{N}_0^n, \qquad k \in \mathbb{N}_0 \quad \textit{and} \quad m = 1, \ldots, M_k\,, \tag{1.568}$$

is called an (s,p)-β-quark related to the ball in (1.563).

(ii) *Let $0 < q \le \infty$, $\varrho \in \mathbb{R}$ and*

$$\nu = \{\nu_{km}^\beta \in \mathbb{C} : \; k \in \mathbb{N}_0; \; \beta \in \mathbb{N}_0^n; \; m = 1, \ldots, M_k\}. \tag{1.569}$$

Then

$$b_{pq}^{\Gamma,\varrho} = \left\{\nu : \; \|\nu \,|b_{pq}^{\Gamma,\varrho}\| < \infty\right\} \tag{1.570}$$

with

$$\|\nu \,|b_{pq}^{\Gamma,\varrho}\| = \sup_{\beta \in \mathbb{N}_0^n} 2^{\varrho|\beta|} \left(\sum_{k=0}^\infty \left(\sum_{m=1}^{M_k} |\nu_{km}^\beta|^p \right)^{q/p} \right)^{1/q} \tag{1.571}$$

(*with the usual modification if $q = \infty$*).

Remark 1.184. Since $0 < t < 1$ it follows by (1.521) that

$$\lambda_\mu(1-t) = tn + s_\mu(1-t). \tag{1.572}$$

Hence replacing the multifractal characteristics in (1.567) by the Besov characteristics one gets

$$(\beta\text{-qu})_{km}(\gamma) = 2^{-k(s+|s_\mu(1-t)|-tn)+k|\beta|}\,(\gamma-\gamma^{k,m})^\beta\,\psi^{k,m}(\gamma). \tag{1.573}$$

These are the restrictions of the $(s+|s_\mu(1-t)|,p)$-β-quarks in (1.102) to Γ, in good agreement with the spaces in the right-hand sides of (1.548), (1.549). If Γ is a d-set with $0<d<n$ then we have (1.547) and (1.573) reduces to

$$(\beta\text{-qu})_{km}(\gamma) = 2^{-k(s-\frac{d}{p})+k|\beta|}\,(\gamma-\gamma^{k,m})^\beta\,\psi^{k,m}(\gamma) \tag{1.574}$$

again in good agreement with our previous considerations in [Triδ], 20.8–20.9, pp. 170–172. In [KnZ06] one finds corresponding constructions for the more general h-sets according to Definition 1.151 and Remark 1.154.

Theorem 1.185. *Let μ be a Radon measure in $\mathbb{R}^n$ according to* (1.515)*. Let $0 < 1/p = t < 1$ and $\varrho \ge 0$.*

(i) *Let $0<q\le\infty$ and $s>0$. Let $(\beta\text{-qu})_{km}$ be the (s,p)-β-quarks and let $b^{\Gamma,\varrho}_{pq}$ be the sequence spaces according to Definition* 1.183*. Then $B^s_{pq}(\Gamma,\mu)$ given by* (1.548) *is the collection of all $f\in L_1(\Gamma,\mu)$ which can be represented by*

$$f(\gamma) = \sum_{\beta\in\mathbb{N}^n_0}\sum_{k=0}^\infty\sum_{m=1}^{M_k}\nu^\beta_{km}\,(\beta\text{-qu})_{km}(\gamma), \qquad \|\nu\,|b^{\Gamma,\varrho}_{pq}\| < \infty, \tag{1.575}$$

$\gamma\in\Gamma$*. Furthermore,*

$$\|f\,|B^s_{pq}(\Gamma,\mu)\| \sim \inf \|\nu\,|b^{\Gamma,\varrho}_{pq}\| \tag{1.576}$$

where the infimum is taken over all admissible representations.

(ii) *Let $0<q<\infty$ and $\mu\in B^{s_\mu(1-t)}_{p'q'}(\mathbb{R}^n)$ with $s_\mu(1-t)<0$. Let $(\beta\text{-qu})_{km}$ be the $(0,p)$-β-quarks and $b^{\Gamma,\varrho}_{pq}$ be the sequence spaces according to Definition* 1.183*. Then $B^0_{pq}(\Gamma,\mu)$ given by* (1.549) *is the collection of all $f\in L_1(\Gamma,\mu)$ which can be represented by* (1.575)*. Furthermore one has* (1.576) *now with $s=0$.*

Remark 1.186. Recall that $\mu\in B^{s_\mu(1-t)}_{p'q'}(\mathbb{R}^n)$ with $s_\mu(1-t)<0$ at the beginning of part (ii) is equivalent to $\mu^{\lambda_\mu(1-t)}_{p'q'}<\infty$. This follows from Theorem 1.174 and (1.572). We add a few explanations. To avoid any misunderstanding we emphasize that ϱ with $\varrho\ge0$ and $(\beta\text{-qu})_{km}$ according to (1.567), originating from given $\{\gamma^{k,m}\}$ and given $\{\psi^{k,m}\}$ are fixed in (1.575), (1.576). In other words, for given ϱ and $(\beta\text{-qu})_{km}$ there are two positive constants c_1 and c_2, depending on ϱ and $(\beta\text{-qu})_{km}$ (and s, p, q) such that

$$c_1\,\|f\,|B^s_{pq}(\Gamma,\mu)\| \le \inf\|\nu\,|b^{\Gamma,\varrho}_{pq}\| \le c_2\,\|f\,|B^s_{pq}(\Gamma,\mu)\| \tag{1.577}$$

for all $f \in B^s_{pq}(\Gamma, \mu)$. Otherwise the material of this point, including (1.559)–(1.565), Definition 1.183 and Theorem 1.185 is a modified and notationally updated version of [Triϵ], Sections 9.24–9.33, pp. 144–152. There one finds also additional explanations including further references and a proof of Theorem 1.185. However we not only adapted the formulations to the multifractal characteristics and Besov characteristics as introduced in Definition 1.165 but reversed also the order of presentation: In [Triϵ], Sections 9.24–9.33, pp. 144–152, we took (1.575) as a definition of corresponding spaces and proved afterwards that they coincide with the trace spaces $B^s_{pq}(\Gamma, \mu)$. In particular, according to [Triϵ], Proposition 9.31, p. 150, the series (1.575) converges absolutely (and hence unconditionally) in $L_1(\Gamma, \mu)$. As far as further technicalities are concerned we refer to Remark 1.44. As there one can replace the summation over $k \in \mathbb{N}_0$ by a summation over $k - K \in \mathbb{N}_0$ where $K \in \mathbb{N}_0$ is a given number. This also explains the somewhat mysterious role of $\varrho \geq 0$, which can be chosen arbitrarily large at the expense of the constants c_1 and c_2 in (1.577).

1.17.4 Quasi-metric spaces

So far we dealt in Section 1.17 with Radon measures μ in $\mathbb{R}^n$ according to (1.515) with compact support Γ. Complemented by the natural metric

$$\varrho_n(x, y) = |x - y|, \qquad x \in \mathbb{R}^n, \quad y \in \mathbb{R}^n, \tag{1.578}$$

one gets the compact metric space (Γ, ϱ_n, μ). We described in Sections 1.17.1–1.17.3 the theory of function spaces on these specific metric spaces where Theorem 1.185 might be considered as the intrinsic description of corresponding Besov spaces $B^s_{pq}(\Gamma, \mu)$. It comes out that this way to study function spaces can be extended to huge classes of abstract quasi-metric spaces reducing them to the above specific spaces (Γ, ϱ_n, μ) by the method of *Euclidean charts*. Here we describe the general background and shift the specific studies to Chapter 8.

Let X be a set. A non-negative function $\varrho(x, y)$ defined on $X \times X$ is called a *quasi-metric* if it has the following properties:

$$\varrho(x, y) = 0 \qquad \text{if, and only if,} \quad x = y, \tag{1.579}$$

$$\varrho(x, y) = \varrho(y, x) \qquad \text{for all } \ x \in X \ \text{ and all } \ y \in X, \tag{1.580}$$

there is a real number A with $A \geq 1$ such that for all $x,\ y,\ z \in X$,

$$\varrho(x, y) \leq A[\varrho(x, z) + \varrho(z, y)]. \tag{1.581}$$

Of course, if $A = 1$ is admissible, then ϱ is a metric and (1.581) is the triangle inequality. As usual, a quasi-metric $\overline{\varrho}$ on X is said to be equivalent to ϱ, written as $\varrho \sim \overline{\varrho}$, if there are two positive numbers c_1 and c_2 such that

$$c_1\, \varrho(x, y) \leq \overline{\varrho}(x, y) \leq c_2\, \varrho(x, y), \qquad x \in X, \quad y \in X. \tag{1.582}$$

Theorem 1.187. *Let ϱ be a quasi-metric on a set X.*

(i) *There is a number ε_0 with $0 < \varepsilon_0 \le 1$ such that ϱ^ε for any ε with $0 < \varepsilon \le \varepsilon_0$ is equivalent to a metric.*

(ii) *Let $0 < \varepsilon \le \varepsilon_0$ and let $\overline{\varrho} \sim \varrho$ such that $\overline{\varrho}^\varepsilon$ is a metric according to part* (i). *Then there is a positive number c such that for all $x \in X$, $y \in X$, $z \in X$,*

$$|\overline{\varrho}(x,y) - \overline{\varrho}(x,z)| \le c\overline{\varrho}(y,z)^\varepsilon \left[\overline{\varrho}(x,y) + \overline{\varrho}(x,z)\right]^{1-\varepsilon}. \tag{1.583}$$

Remark 1.188. Part (i) is a remarkable observation. A proof may be found in [Hei01], Proposition 14.5, pp. 110–112. The assertion is even slightly stronger: If $\varrho \sim \overline{\varrho}$ such that $\overline{\varrho}^{\varepsilon_0}$ is a metric, then $\overline{\varrho}^\varepsilon$ is also a metric whenever $0 < \varepsilon \le \varepsilon_0$. This follows from the triangle inequality for metrics and the well-known observation

$$(a+b)^p \le a^p + b^p \qquad \text{where} \quad a \ge 0, \quad b \ge 0, \quad 0 < p \le 1. \tag{1.584}$$

Part (ii) follows from part (i) and the mean value theorem,

$$\begin{aligned} &|\overline{\varrho}^\varepsilon(x,y)^{1/\varepsilon} - \overline{\varrho}^\varepsilon(x,z)^{1/\varepsilon}| \\ &\le |\overline{\varrho}^\varepsilon(x,y) - \overline{\varrho}^\varepsilon(x,z)| \, \frac{1}{\varepsilon} \left(\overline{\varrho}^{\varepsilon(\frac{1}{\varepsilon}-1)}(x,y) + \overline{\varrho}^{\varepsilon(\frac{1}{\varepsilon}-1)}(x,z) \right) \\ &\le c\, \overline{\varrho}(y,z)^\varepsilon \left[\overline{\varrho}(x,y) + \overline{\varrho}(x,z)\right]^{1-\varepsilon}. \end{aligned} \tag{1.585}$$

Independently of part (i) the inequality (1.583) has been known for some time. It is a corner-stone of the analysis on quasi-metric spaces. We refer to [MaS79], Theorem 2, p. 259.

Let the set X and the quasi-metric ϱ be as above. Then the equivalent quasi-metric $\overline{\varrho}$ according to Theorem 1.187 generates a topology on X taking the balls

$$B(x,r) = \{y \in X : \overline{\varrho}(x,y) < r\} \tag{1.586}$$

as a basis of neighborhoods of $x \in X$, where $r > 0$.

Definition 1.189. *Let ϱ be a quasi-metric on the set X equipped with a topology as just indicated.*

(i) *Then (X, ϱ, μ) is called a space of homogeneous type if μ is a non-negative regular Borel measure on X such that there is a constant A with*

$$0 < \mu\,(B(x,2r)) \le A\,\mu\,(B(x,r)) < \infty \qquad \textit{for all} \quad x \in X, \quad r > 0, \tag{1.587}$$

(*doubling condition*).

(ii) *Let $d > 0$. Then (X, ϱ, μ) is called a d-space if it is a complete space of homogeneous type according to part* (i) *with*

$$\operatorname{diam} X = \sup\{\varrho(x,y) : x \in X, \ y \in X\} < \infty \tag{1.588}$$

and

$$\mu\,(B(x,r)) \sim r^d \quad \textit{for all} \quad x \in X \quad \textit{and} \quad 0 < r \le \operatorname{diam} X. \tag{1.589}$$

Remark 1.190. In case of part (ii) we have

$$\mu(\{x\}) = 0 \qquad \text{for every} \quad x \in X, \quad \text{and} \quad \mu(X) < \infty. \tag{1.590}$$

Furthermore, μ is a Radon measure. As far as the measure-theoretical notation is concerned we refer to [Mat95], pp. 7–13. Otherwise the above notation of d-spaces had been introduced in [TrY02] imitating the notation of d-sets in $\mathbb{R}^n$ according to (1.495) which are special d-spaces. In [TrY02], Sections 2.13 and 2.14, one finds also a discussion of this subject, especially of the necessity to switch from arbitrary quasi-metrics to equivalent quasi-metrics with (1.583). As for further background information we refer to [MaS79], [HaS94], [HaY02]. In particular, (1.589) is not so special as it seems at first glance: If (X, ϱ, μ) is a space of homogeneous type then one can find a quasi-metric ϱ' generating the same topology such that one has (1.589) with $d = 1$, based on ϱ'. But in general ϱ and ϱ' are not equivalent to each other. However the equivalence of the admitted quasi-metrics is indispensable in [TrY02], in what follows later on and also in connection with, say, d-sets in $\mathbb{R}^n$.

Remark 1.191. There has always been some interest in an analysis on spaces of homogeneous type according to Definition 1.189. We refer to [CoW71], [DJS85] and [Chr90]. But in the last decade, especially in the last few years there were some new developments to create a substantial intrinsic analysis on fractals and on (quasi-)metric spaces, at least partly closely connected with the problem how to measure smoothness on these structures. The underlying measures have often the property (1.589), but sometimes only the doubling condition (1.587) is required. We refer to the recent books [DaS97], [Hei01], [Kig01], [Sem01]. As far as function spaces on spaces of homogeneous type are concerned we give some more specific references later on. However our own corresponding approach is different from what has been done so far in literature. It will be presented in detail in Chapter 8. It is based on the theory of function spaces on d-sets in $\mathbb{R}^n$ on the one hand and the following remarkable observation on the other hand.

Theorem 1.192. *Let (X, ϱ, μ) be a space of homogeneous type according to Definition* 1.189(i)*. Let $0 < \varepsilon_0 \le 1$ be the same number as in Theorem* 1.187 *and let $0 < \varepsilon < \varepsilon_0$. Then there are an $n \in \mathbb{N}$ and a bi-Lipschitzian mapping*

$$H : \quad X \mapsto \mathbb{R}^n \tag{1.591}$$

from $(X, \varrho^\varepsilon, \mu)$ into $\mathbb{R}^n$. The dimension n (and also the Lipschitzian constants) can be chosen to depend only on ε and on the doubling constant A in (1.587)*.*

Remark 1.193. If ϱ is a metric then $(X, \varrho^\varepsilon, \mu)$ with $\varepsilon < 1$ is called the *snowflaked version* of (X, ϱ, μ). In case of quasi-metrics one has even to assume that $\varepsilon < \varepsilon_0$. Then (1.591) means that there is a bi-Lipschitzian map H of the snowflaked version $(X, \varrho^\varepsilon, \mu)$ onto its image in $\mathbb{R}^n$,

$$|H(x) - H(y)| \sim \varrho^\varepsilon(x, y), \qquad x \in X, \quad y \in X, \tag{1.592}$$

where the equivalence constants are independent of x and y. Usually this assertion is formulated and proved for metric spaces (X, ϱ) with a so-called doubling metric. We refer for a definition of a doubling metric to [Hei01], p. 81. However if there is a doubling measure then the corresponding metric is also doubling, [Hei01], p. 82. Hence one can apply the corresponding assertions in [Sem01], Theorem 2.2, p. 13, and [Hei01], Theorem 12.2 on p. 98, to $(X, \varrho^{\varepsilon_0}, \mu)$ resulting in the above theorem. The original proof of this remarkable assertion is due to P. Assouad, [Ass79], [Ass83]. More details may be found in [Hei01] and [Sem01] as specified above and also in [DaS97], 5.4, p. 21.

1.17.5 Spaces on quasi-metric spaces

So far we discussed in the Sections 1.17.2, 1.17.3 the spaces $B^s_{pq}(\Gamma, \mu)$ as introduced in Definition 1.178 where μ is a Radon measure in $\mathbb{R}^n$ with (1.515). They are trace spaces which can be characterised in terms of quarkonial representations. It is mainly an updated description of our corresponding previous considerations in [Triδ] and, especially, in [Triϵ]. If μ is an isotropic measure generating a d-set, a (d, Ψ)-set or an h-set then one has good reasons to switch from the spaces B^s_{pq} (on $\mathbb{R}^n$ or on Γ) to corresponding spaces of generalised smoothness. This results in so-called tailored spaces and we indicated this possibility in Remark 1.180 where one finds also corresponding references especially to the work of S. Moura and M. Bricchi. In case of d-sets there is the different well-known approach by A. Jonsson and H. Wallin, [JoW84], which is of interest for us especially in connection with the above Theorem 1.181. We mentioned a few more recent papers in Remark 1.182. There are some attempts to extend this theory to more general sets and measures. We refer in addition to the above papers to [Jon94] and [Byl94].

In the last decade, but especially in very recent times, a lot has been done to develop a substantial analysis on fractals (as subsets in $\mathbb{R}^n$ or intrinsically) or on (quasi-)metric spaces asking,

How to measure smoothness on (X, ϱ, μ)*?*

Here X is a set, ϱ is a quasi-metric and μ is a Borel measure. The d-spaces as introduced in Definition 1.189 are special cases. Quite often, but not always, something like (1.589) is required, occasionally weakened assuming only the doubling condition (1.587). Quite recently sometimes even the convenient doubling condition has been abandoned. This is now a fashionable subject. However it is not our aim to report on these developments. Our own method will be presented in detail in Chapter 8 and outlined in the following Section 1.17.6 reducing function spaces on (X, ϱ, μ) by Theorem 1.192 to corresponding function spaces on fractals in $\mathbb{R}^n$. Although this way of dealing with function spaces is (as far as we know) unrelated to other proposals it seems to be reasonable to mention at least a very few papers dealing with different approaches and to hint on the most elaborated method by indicating a few details. Diverse aspects how to handle smoothness (intrinsically) on fractals and on (quasi-)metric spaces may be found in the recent books [DaS97], [Hei01], [Kig01], [Sem01]. Based on [Kig01], R.S. Strichartz developed in [Str03] a theory of Hölder-Zygmund, Besov and Sobolev spaces of higher order on the Sierpinski gasket and other post critical finite self-similar fractals. In general, the spaces considered are different from trace spaces as introduced in Definition 1.178. For a long time it has been well known that classical Besov spaces and (fractional) Sobolev spaces on $\mathbb{R}^n$ are closely connected with Gauss-Weierstrass semi-groups, Cauchy-Poisson semi-groups and the underlying Bessel potentials and Riesz potentials. We refer to [Triα], 2.5.2, 2.5.3 for details and to [Triγ], 1.8.1 for a short description. There one finds also the necessary references. It is quite natural to ask for fractal counterparts. This works to some extent and under some restrictions (d-sets or Sierpinski gaskets). This will not be considered here. Details and further references may be found in [Zah02], [Zah04], [HuZ04], [Kum04]. As far as Sobolev spaces on metric spaces are concerned we refer for different approaches to [HaK00] and [LLW02]. The most recent paper in this connection is [YaY05] where one finds also many references.

We give now a brief description of some aspects of the most elaborated theory of function spaces on quasi-metric spaces. We restrict ourselves to the simplest case. We follow partly the presentation given in [TrY02]. Let $d > 0$ and let (X, ϱ, μ) be a d-space according to Definition 1.189. According to Theorem 1.187 we suppose that for some ε_0 with $0 < \varepsilon_0 \le 1$ and all ε with $0 < \varepsilon \le \varepsilon_0$,

$$|\varrho(x,y) - \varrho(y,z)| \le c\, \varrho^\varepsilon(y,z)\ [\varrho(x,y) + \varrho(x,z)]^{1-\varepsilon} \tag{1.593}$$

for $x \in X$, $y \in X$, $z \in X$ and a suitable constant $c > 0$. Let $1 \le p \le \infty$. Then the complex Banach space $L_p(X)$ has the usual meaning, normed by

$$\|f\,|L_p(X)\| = \left(\int_X |f(x)|^p\, \mu(\mathrm{d}x) \right)^{1/p} \tag{1.594}$$

with the usual modification if $p = \infty$. Let $\mathrm{Lip}^\varepsilon(X)$ be the Banach space of all complex continuous functions on X such that

$$\|f\,|\mathrm{Lip}^\varepsilon(X)\| = \sup_{x\in X} |f(x)| + \sup_{x\ne y} \frac{|f(x) - f(y)|}{\varrho^\varepsilon(x,y)} < \infty. \tag{1.595}$$

Then $\mathrm{Lip}^\varepsilon(X)$ is dense in $L_2(X)$. We refer to [DJS85] and [Han97]. Hence the dual pairing $(\mathrm{Lip}^\varepsilon(X), (\mathrm{Lip}^\varepsilon(X))')$ with the usual identification $(L_2(X))' = L_2(X)$ makes sense. According to Theorem 1.10 in Section 1.4 one can define the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ in terms of local means within the dual pairing $(S(\mathbb{R}^n), S'(\mathbb{R}^n))$. It is remarkable that there is a partial substitute under the above assumptions which goes back to [DJS85]. We rely here on the modification according to [Han97] adapted to our situation. There is a sequence of complex-valued functions

$$S_m(x,y) \qquad \text{with} \quad x \in X, \quad y \in X, \quad \text{and} \quad m \in \mathbb{N}_0, \tag{1.596}$$

and a number $C > 0$ with the following properties:

$$S_m(x,y) = 0 \text{ if } \varrho(x,y) \ge C2^{-m} \quad \text{and} \quad |S_m(x,y)| \le C2^{dm}, \tag{1.597}$$

$$\begin{cases} |S_m(x,y) - S_m(x',y)| \le C\, 2^{m(d+\varepsilon)} \varrho^\varepsilon(x,x'), \\ |S_m(x,y) - S_m(x,y')| \le C\, 2^{m(d+\varepsilon)} \varrho^\varepsilon(y,y'), \end{cases} \tag{1.598}$$

$$\begin{aligned} &|S_m(x,y) - S_m(x',y) - S_m(x,y') + S_m(x',y')| \\ &\le C\, 2^{m(d+2\varepsilon)}\, \varrho^\varepsilon(x,x')\, \varrho^\varepsilon(y,y') \end{aligned} \tag{1.599}$$

for all $x \in X$, $x' \in X$, $y \in X$, $y' \in X$; and

$$\int_X S_m(x,y)\, \mu(\mathrm{d}y) = \int_X S_m(x,y)\, \mu(\mathrm{d}x) = 1. \tag{1.600}$$

Let

$$E_0(x,y) = S_0(x,y) \qquad \text{and} \qquad E_m(x,y) = S_m(x,y) - S_{m-1}(x,y) \tag{1.601}$$

if $m \in \mathbb{N}$. Then we have

$$\int_X E_m(x,y)\, \mu(\mathrm{d}y) = \int_X E_m(x,y)\, \mu(\mathrm{d}x) = 0 \qquad \text{for} \quad m \in \mathbb{N}. \tag{1.602}$$

Now E_l given by

$$(E_l f)(x) = \int_X E_l(x,y)\, f(y)\, \mu(\mathrm{d}y) \qquad \text{with} \quad l \in \mathbb{N}_0, \tag{1.603}$$

are the local means we are looking for. One can compare the above kernels $S_m(x,y)$ and $E_m(x,y)$ with the kernels $2^{m\cdot n}\, k\,(2^m(y-x))$ in the local means $k(2^{-m},f)(x)$ in (1.41). There is an immediate counterpart of (1.597) with $d = n$. Furthermore if k_0 is normalised by $k_0^\vee(0) = 1$ and if S_m is generated by $2^{mn}k_0(2^m\cdot)$, one has (1.600) and (1.602), where the latter corresponds now to (1.42) with $|\alpha| = 0$ and hence $s < 1$. In contrast to Section 1.4, now only the limited Lip^ε-smoothness is available, which results in (1.598) and (1.599). But it is not a surprise that one can reduce the required smoothness for the kernels in Section 1.4 in dependence on s. We have done this in a slightly different but nearby context of atoms in Section 1.5.2, which is also well reflected by atomic decompositions of the spaces $B^s_{pq}(X)$ that we are going to introduce now. In other words, E_l in (1.603) is a promising replacement of the local means in Section 1.4 as long as the smoothness s of the respective spaces B^s_{pq} and F^s_{pq} is restricted at least by $|s| \le \varepsilon$.

Definition 1.194. *Let $d > 0$ and let (X, ϱ, μ) be a d-space according to Definition* 1.189 *equipped with the quasi-metric ϱ satisfying* (1.593) *where $0 < \varepsilon = \varepsilon_0 \le 1$ is the same number as in Theorem* 1.187. *Let*

$$1 \le p \le \infty, \quad 1 \le q \le \infty, \quad \textit{and} \quad -\varepsilon < s < \varepsilon. \tag{1.604}$$

Let $E_l f$ be the local means as introduced in (1.603). *Then $B^s_{pq}(X)$ is the collection of all $f \in (\mathrm{Lip}^\varepsilon(X))'$ such that*

$$\|f \,|B^s_{pq}(X)\| = \left(\sum_{j=0}^{\infty} 2^{jsq} \, \|E_j f \,|L_p(X)\|^q \right)^{1/q} < \infty \tag{1.605}$$

with the usual modification if $q = \infty$.

Remark 1.195. This definition might be considered as the counterpart of Theorem 1.10. Since $\mathrm{Lip}^\varepsilon(X)$ is dense in $L_2(X)$,

$$\mathrm{Lip}^\varepsilon(X) \hookrightarrow L_2(X) = (L_2(X))' \hookrightarrow (\mathrm{Lip}^\varepsilon(X))' \tag{1.606}$$

justifies the definition. Furthermore, $B^s_{pq}(X)$ are Banach spaces which are independent of the chosen local means (equivalent norms). These are inhomogeneous Besov spaces on d-spaces. Similarly one can introduce inhomogeneous spaces $F^s_{pq}(X)$ with

$$1 < p < \infty, \quad 1 \le q \le \infty, \quad \text{and} \quad -\varepsilon < s < \varepsilon, \tag{1.607}$$

imitating Theorem 1.10(ii). Furthermore there are several extensions of (homogeneous and non-homogeneous) spaces of B-type and F-type on more general spaces of homogeneous type according to Definition 1.189(i) and to some values with $p \le 1$ and $q \le 1$. However the admitted smoothness is limited by $|s| < \varepsilon$. Spaces of this type have now a rather substantial history covering in particular the above special case. The analysis on spaces of homogeneous type started with [CoW71]. As far as the above function spaces are concerned we refer here to the surveys [HaS94], [HaY02], the contributions [Han97], [HLY99] and the recent papers [HLY01], [Yang02], [TrY02], [Yang03], [HaY03], [Yang04], [YaL04], [DHY05], [HaY04], reflecting at least partly some recent developments. One may also consult Section 9.3.2, especially in connection with the last two references.

Of special interest for us (and a strong reason to incorporate the above material) is the following observation. Recall that a compact set Γ in $\mathbb{R}^n$ is called a *d-set* with $0 < d < n$ if there is a Radon measure μ with $\operatorname{supp} \mu = \Gamma$ and

$$\mu(B(\gamma, r)) \sim r^d, \qquad \gamma \in \Gamma, \quad 0 < r < 1. \tag{1.608}$$

Then we have in particular (1.551) for the trace spaces $B^s_{pq}(\Gamma, \mu)$. Complemented by the natural metric ϱ_n in (1.578), one gets the *d-space* (X, ϱ, μ) with $X = \Gamma$ and $\varrho = \varrho_n$ according to Definition 1.189(ii). Of course one may now choose $\varepsilon = 1$ in (1.593) and Definition 1.194. Again the resulting spaces are denoted by $B^s_{pq}(X)$.

Theorem 1.196. *Let* Γ *be a compact* d*-set in* $\mathbb{R}^n$ *with* $0 < d < n$ *and let*

$$0 < s < 1, \qquad 1 < p < \infty, \qquad 1 \le q \le \infty. \tag{1.609}$$

Then

$$B^s_{pq}(\Gamma, \mu) = B^s_{pq}(X). \tag{1.610}$$

Remark 1.197. This assertion is due to D. Yang, [Yang03], Theorem 3.1. His proof uses atomic representations for the spaces $B^s_{pq}(X)$ as established in [HLY99] and [HaY02] on the one hand and quarkonial representations for the spaces $B^s_{pq}(\Gamma, \mu)$ according to Theorem 1.185 on the other hand. This result answers a question posed in [Triϵ], 9.34(x), pp. 159–160.

1.17.6 Function spaces on *d*-spaces: an approach via Euclidean charts

Let (X, ϱ, μ) be a d-space according to Definition 1.189 with $d > 0$. We may assume that the quasi-metric ϱ satisfies (1.593) for all ε with $0 < \varepsilon \le \varepsilon_0$ where $0 < \varepsilon_0 \le 1$ is the same number as in Theorem 1.187. Since (X, ϱ, μ) is assumed to be complete it is also compact. We apply Theorem 1.192 to the snowflaked version $(X, \varrho^\varepsilon, \mu)$ of (X, ϱ, μ) now with $0 < \varepsilon < \varepsilon_0$. Hence there is a bi-Lipschitzian map H, the *snowflaked transform*,

$$H : \quad (X, \varrho^\varepsilon, \mu) \quad \text{onto} \quad \big(\Gamma, \varrho_n, \mathcal{H}^{d_\varepsilon}|\Gamma\big) \tag{1.611}$$

where Γ is a compact d_ε-set with $d_\varepsilon = d/\varepsilon$ in some $\mathbb{R}^n$ according to Remark 1.179 (or the explanations given in front of Theorem 1.196), where again

$$\varrho_n(x, y) = |x - y|, \qquad x \in \mathbb{R}^n, \quad y \in \mathbb{R}^n, \tag{1.612}$$

is the usual Euclidean distance in $\mathbb{R}^n$ and the measure μ in (1.608) can be fixed (up to equivalences) as the restriction of the Hausdorff measure $\mathcal{H}^{d_\varepsilon}$ to Γ. We call Γ with ϱ_n and $\mathcal{H}^{d_\varepsilon}|\Gamma$ an *Euclidean* ε*-chart* of (X, ϱ, μ). In particular,

$$HX = \Gamma, \quad \varrho_n\left(H(x), H(y)\right) \sim \varrho^\varepsilon(x, y), \quad x \in X, \quad y \in X, \tag{1.613}$$

and one may fix (again up to equivalences)

$$\mu = (\mathcal{H}^{d_\varepsilon}|\Gamma) \circ H \tag{1.614}$$

(image measure). Now it is the basic idea to transfer the spaces $B^s_{pq}(\Gamma, \mu)$ according to (1.551) and in particular their quarkonial characterisations in Theorem 1.185 to $(X, \varrho^\varepsilon, \mu)$, hence

$$B^s_{pq}(X, \varrho, \mu; H) = B^{s/\varepsilon}_{pq}\Big(\Gamma, \varrho_n, \mathcal{H}^{d/\varepsilon}|\Gamma\Big) \circ H, \tag{1.615}$$

where

$$1 < p < \infty, \qquad 0 < q \le \infty, \qquad s > 0. \tag{1.616}$$

These spaces can be characterised in terms of quarkonial representations now on X. Taking the intersection of these spaces one can impress a C^∞-structure on X. This gives the possibility to introduce by duality distributional spaces B^s_{pq} on X with $s < 0$. This will be done in detail in Chapter 8. Obviously one gets in this way many scales of spaces $B^s_{pq}(X, \varrho, \mu; H)$ which are different from each other in general. But there will be a counterpart of Theorem 1.196, saying that for $1 < p < \infty$, $1 \le q \le \infty$,

$$B^s_{pq}(X, \varrho, \mu; H) = B^s_{pq}(X) \qquad \text{if} \quad 0 < s < \varepsilon, \tag{1.617}$$

where $B^s_{pq}(X)$ are the spaces according to Definition 1.194.

Remark 1.198. The details of the outlined procedure will be given in Chapter 8. But we are more interested in the method than in generality. In particular the restriction to d-spaces is the simplest case but by no means necessary for this method: The trace spaces as introduced in Definition 1.169 and their quarkonial representations according to Theorem 1.185 apply to arbitrary Radon measures μ according to (1.515) and the corresponding metric spaces (Γ, ϱ_n, μ). For the snowflaked transform as described by Theorem 1.192 one needs that the measure μ is doubling according to (1.587) but not more. Hence these two ingredients apply to much more general situations. But this will not be done. First candidates would be spaces according to Definition 1.189(ii) where (1.589) is replaced by arbitrary isotropic measures as introduced in Definition 1.151 or with the specification (1.554). Also the assumption (1.588) and that (X, ϱ, μ) is compact might not be necessary. One could deal with complete but not necessarily compact spaces (X, ϱ, μ) and begin with a resolution of unity on X in terms of Lip^ε-functions according to (1.595). This is possible and the compact pieces can be treated as indicated. Afterwards one has to glue these pieces together appropriately. But then one is very near to the usual way as (n-dimensional) manifolds (C^∞ or Riemannian) are represented by an atlas consisting of local charts. In our case one has Euclidean charts and for compact spaces (X, ϱ, μ) one chart is sufficient. On the other hand the idea of local charts has also been used quite extensively in connection with function spaces on Riemannian manifolds and on Lie groups. We refer to [Triγ], [Triϵ]. Hence following these suggestions one might well arrive at

atlases of Euclidean charts and snowflaked bi-Lipschitzian maps.

On the other hand, the theory of function spaces on $\mathbb{R}^n$, on domains in $\mathbb{R}^n$ but also on spaces of homogeneous type as described in Section 1.17.5 applies equally to spaces of type B^s_{pq} and of type F^s_{pq}. But this is not the case for the trace spaces according to Definition 1.178 and for spaces of type (1.615). They strongly prefer B^s_{pq} spaces, so far.

1.18 Fractal characteristics of distributions

For Radon measures μ with (1.515) we introduced in Definition 1.165 the multifractal characteristics $\lambda_\mu(t)$ and the Besov characteristics $s_\mu(t)$ and collected in Theorem 1.167 some properties. It is quite natural to ask for corresponding notation and assertions for arbitrary distributions,

$$f \in S'(\mathbb{R}^n) \quad \text{with} \quad \operatorname{supp} f \quad \text{compact} \quad \text{and} \quad \operatorname{sing\,supp} f \neq \emptyset. \tag{1.618}$$

Recall that the *singular support* of $f \in S'(\mathbb{R}^n)$ is given by

$$\operatorname{sing\,supp} f = \{x \in \mathbb{R}^n \;:\; f|B(x,r) \notin C^\infty \text{ for any } r > 0\}, \tag{1.619}$$

where again $B(x,r)$ is the open ball centred at $x \in \mathbb{R}^n$ and of radius r, and $f|\Omega \in D'(\Omega)$ is the restriction of f to a domain Ω in $\mathbb{R}^n$. We refer to [Hor83], Definition 2.2.3, p. 42. We return to this subject in Chapter 7. Then we introduce the multifractal characteristics $\lambda^f(t)$ in modification and generalisation of $\lambda_\mu(t)$ in (1.518) in case of $f = \mu$. We have $\lambda^\mu(t) = \lambda_\mu(t)$ at least if $0 \le t \le 1$. On the other hand the Besov characteristics

$$s_f(t) = \sup\left\{s \;:\; f \in B^s_{p\infty}(\mathbb{R}^n)\right\}, \qquad 0 \le t = 1/p < \infty, \tag{1.620}$$

are the direct generalisations of (1.519). Since f is not C^∞ in $\mathbb{R}^n$ it follows that $s_f(t) < \infty$ for any $0 \le t < \infty$. Otherwise some of the properties in Theorem 1.167 can be carried over. As there we say that a real function $s(t)$ is increasing for $t \ge 0$ if it is non-decreasing. Any concave increasing real function $s(t)$, where $t \ge 0$, is absolutely continuous. Furthermore, $s'(t)$ exists a.e., it is non-negative and decreasing, [Heu90], p. 113. In particular $s'(\infty) = \lim_{t\to\infty} s'(t)$ exists. Recall that $|\Gamma|$, $\mathcal{H}^d(\Gamma)$ and $\dim_H \Gamma$ are the Lebesgue measure, the Hausdorff measure and the Hausdorff dimension of the set Γ in $\mathbb{R}^n$. Details about Hausdorff measures and Hausdorff dimensions may be found in [Fal85, Section 1.2], [Mat95, Section 4] and [Triδ, Section 4].

Theorem 1.199.

(i) *Let $\lambda^f(t)$ and $s_f(t)$ be the multifractal characteristics and the Besov characteristics of f as in* (1.618), *according to Chapter* 7 *below and* (1.620). *Then $s_f(t)$ with $0 \le t < \infty$ is an increasing concave function in the (t,s)-diagram in Figure* 1.17.1 *of slope smaller than or equal to n. Furthermore,*

$$s_f(t) = \lambda^f(t) + n(t-1), \qquad 0 \le t < \infty. \tag{1.621}$$

(ii) *For any real increasing concave function $s(t)$, where $0 \le t < \infty$, of slope smaller than or equal to n there is an f with* (1.618) *such that*

$$s_f(t) = s(t) \quad \textit{for} \quad 0 \le t < \infty, \qquad \textit{and} \quad |\Gamma| = 0, \tag{1.622}$$

where $\Gamma = \operatorname{sing\,supp} f$. If in addition $s'(\infty) < n$, then

$$\mathcal{H}^d(\Gamma) = 0 \quad \Longleftrightarrow \quad d \ge n - s'(\infty) \tag{1.623}$$

and, in particular,

$$\dim_H \Gamma = n - s'(\infty). \tag{1.624}$$

Remark 1.200. We return to this subject in Chapter 7 where we give also a definition of $\lambda^f(t)$ and discuss how it is related to Definition 1.165. Otherwise part (i) coincides with corresponding assertions in [Tri03c]. In case of $s'(\infty) = n$, and hence $s'(t) = n$ for all $0 \le t < \infty$, one can choose the lifted δ-distribution $f = I_\sigma \delta$ according to (1.5), (1.6), with $\sigma = -s(1)$. Then $\Gamma = \{0\}$ and (1.624) with $\dim_H \Gamma = 0$ remains valid also in this case. In connection with part (ii) we refer also to [Jaf00], Proposition 1, p. 58, and [Jaf01], Proposition 1, saying that for a given function $s(t)$ with the above properties there is an element f belonging locally to a homogeneous Besov space with $s(t) = s_f(t)$. The arguments there are based on wavelet expansions.

1.19 A black sheep of calculus becomes the king of function spaces

Before starting in earnest we look back at what has been said so far.

The black sheep. Beginners in calculus are usually surprised when they are told that the function

$$f(x) = e^{-1/x} \text{ if } x > 0 \qquad \text{and} \qquad f(x) = 0 \text{ if } x \le 0, \tag{1.625}$$

on the real line $\mathbb{R}$ is C^∞ but nevertheless cannot be expanded at the origin in a Taylor series. The function seems to promise more than it delivers. Even the more handsome version, now considered on $\mathbb{R}^n$,

$$g(x) = \begin{cases} e^{-\frac{1}{n-|x|^2}} & \text{if } |x| < \sqrt{n}, \\ 0 & \text{if } |x| \ge \sqrt{n}, \end{cases} \tag{1.626}$$

is, in the same way, discredited as a *black sheep of calculus*, a bad guy.

The rehabilitation. A remarkable rehabilitation came with the rise of the theory of distributions in the late 1940s due to an enthusiastic marketing and fiery lectures by Laurent Schwartz on this subject, [Gar97], p. 79. At that time it was a surprise even for experts (as it might be nowadays for students after their bad experiences with calculus) that g from (1.626) and its direct descendants play a decisive role in the foundation of the theory of distributions. One might think of C^∞ resolutions of unity of type (1.105), (1.106) and of the density of $D(\Omega) = C_0^\infty(\Omega)$ in $L_p(\mathbb{R}^n)$ with $1 \le p < \infty$ for domains Ω in $\mathbb{R}^n$. We refer for more historical details to [Gar97], Chapter 12.

The king. However the role of the function g from (1.626) goes far beyond what we just described (at least in our understanding). Beginning with [Triδ] we presented in [Triϵ] a new approach to spaces of type B^s_{pq} and F^s_{pq} on diverse sets such as $\mathbb{R}^n$, domains in $\mathbb{R}^n$, boundaries of these domains, manifolds and fractals in $\mathbb{R}^n$. This undertaking will be continued in this book and extended in several directions, especially to some function spaces on abstract quasi-metric spaces. Basically all these considerations have one point in common: The elements of these spaces are represented as sums over fixed *elementary building blocks* multiplied with complex numbers belonging to some sequence spaces. These elementary building blocks, which we called *β-quarks* can be derived from g (and nearby descendants) subject to a few well-established procedures. To give an impression of what is meant we assume that the underlying structure is $\mathbb{R}^n$. Then we have the classical (inhomogeneous) wavelet *procedures*,

$$x \mapsto 2^j x \quad \text{where } x \in \mathbb{R}^n,\ j \in \mathbb{N}_0, \qquad (\textbf{dilations}), \tag{1.627}$$

and

$$x \mapsto x - m \quad \text{where } x \in \mathbb{R}^n,\ m \in \mathbb{Z}^n, \qquad (\textbf{translations}), \tag{1.628}$$

complemented now by the multiplication of functions h with monomials $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$,

$$h(x) \mapsto x^\beta h(x), \quad \text{where } x \in \mathbb{R}^n,\ \beta \in \mathbb{N}_0^n, \qquad (\textbf{multiplications}). \tag{1.629}$$

Then one arrives at the β-quarks in (1.107) where one might think that ψ is an immaterial modification of

$$g(x) \cdot \Big(\sum_{m \in \mathbb{Z}^n} g(x-m) \Big)^{-1}, \qquad x \in \mathbb{R}^n.$$

Now by Theorem 1.39 all spaces $B^s_{pq}(\mathbb{R}^n)$ with $s > \sigma_p$ and $F^s_{pq}(\mathbb{R}^n)$ with $s > \sigma_{pq}$ can be represented by these elementary building blocks where the quality of the elements f in (1.110) and in (1.112) is measured in terms of corresponding sequences spaces as introduced in (1.108) and (1.109). For arbitrary smoothness $s \in \mathbb{R}$ one needs in addition derivatives of the building blocks caused by oscillations (or cancellations). It is sufficient to work with powers $(-\Delta)^L$ of the Laplacian $-\Delta$ in $\mathbb{R}^n$, hence

$$h(x) \mapsto (-\Delta)^L h(x), \quad \text{where } x \in \mathbb{R}^n,\ L \in \mathbb{N}, \qquad (\textbf{differentiations}). \tag{1.630}$$

Complementing dilations, translations and multiplications by differentiations according to (1.627)–(1.630), altogether, one gets elementary building blocks now for all spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ originating from g in (1.626). We refer to [Triϵ], Section 3. This remarkable success may justify the coronation of the black sheep of calculus g from (1.626) as the *king of function spaces*.

The proper empire. The above function g (and the indicated nearby descendants) subject to the procedures (1.627)–(1.630) generate elementary building blocks for all isotropic inhomogeneous spaces B^s_{pq} and F^s_{pq} in $\mathbb{R}^n$, where the corresponding complex coefficients belong to suitable sequence spaces. One may ask whether king g can extend his empire to other function spaces. This works on a surprisingly large scale. Looking first for isotropic inhomogeneous spaces in the context of $\mathbb{R}^n$ one may try to replace the whole $\mathbb{R}^n$ by (bounded smooth) domains Ω, manifolds in $\mathbb{R}^n$ (maybe boundaries of Ω) or bizarre (compact) fractal sets Γ in $\mathbb{R}^n$. Some adaptations are necessary. The function g itself is untouchable. But what about the procedures (1.627)–(1.630)? Dilations (1.627) and multiplications (1.629) are harmless. This applies also to differentiations (1.630) if the underlying structure is a (smooth bounded) domain Ω in $\mathbb{R}^n$. The situation is different in the case of (fractal compact) sets Γ in $\mathbb{R}^n$. But this difficulty can be avoided if one deals with spaces of positive smoothness, say, $B^s_{pq}(\Gamma)$ with $1 < p < \infty$ and $s > 0$. (Then differentiations are simply not needed.) But the translations (1.628) must be adapted to the underlying structures (domains Ω in $\mathbb{R}^n$ or fractal sets Γ in $\mathbb{R}^n$). This must be done on each level j of dilations resulting in the procedure

$$2^{-j}\mathbb{Z}^n \mapsto \{x^{j,m}\} \qquad (\textbf{distortions}), \tag{1.631}$$

where $\{x^{j,m}\}$ are approximate lattices which may be optimally adapted to domains Ω or to fractal sets Γ according to (1.96)–(1.98) and (1.560)–(1.562), correspondingly. Then the above function g subject to the procedures (1.631) (as the combined adapted replacement of dilations and translations according to (1.627) and (1.628)) and (1.629) (possibly complemented by (1.630) in case of domains) generate elementary building blocks for spaces of type $B^s_{pq}(\Omega)$, $F^s_{pq}(\Omega)$ on (smooth bounded) domains Ω in $\mathbb{R}^n$ and for spaces of type $B^s_{pq}(\Gamma)$, $s > 0$, on fractal sets Γ in $\mathbb{R}^n$. We refer to [Triϵ], Sections 5-7, as far as spaces on domains and manifolds are concerned, and to [Triϵ], Section 9, and the above Section 1.17, especially Section 1.17.3, for corresponding spaces on fractal sets Γ in $\mathbb{R}^n$. But this g-**philosophy**, which means the generation of elementary building blocks by application of some of the above procedures to the function g (and nearby descendants), can be extended to other isotropic inhomogeneous spaces in $\mathbb{R}^n$. As far as weighted spaces of type

$$B^s_{pq}(\mathbb{R}^n, w(\cdot)) \quad \text{with the weights } w(x) = (1 + |x|^2)^{\alpha/2}, \text{ where } \alpha \in \mathbb{R},$$

are concerned we refer to [Triϵ], Section 8. More challenging is the extension of the g-philosophy to diverse types of isotropic spaces of generalised smoothness $B^{\boldsymbol{\sigma}}_{pq}(\mathbb{R}^n)$ and $F^{\boldsymbol{\sigma}}_{pq}(\mathbb{R}^n)$, where the generalised smoothness $\boldsymbol{\sigma} = (\sigma_j)^\infty_{j=0}$ replaces the above scalar smoothness s. We refer to [Mou01b] and to [Bri02a], where the corresponding quarkonial expansions in the latter paper are based on [Bri04] and [FaL01], [FaL04].

The expanding empire. Empires, especially when deeply rooted in a philosophy, have the tendency to expand (and to collapse). The proper empire as described above is based on the g-philosophy in $\mathbb{R}^n$ and on diverse subsets. The procedures (1.627)–(1.631) are isotropic and intrinsically connected with $\mathbb{R}^n$. But the

g-philosophy can be exported to other structures and spaces, either within $\mathbb{R}^n$ or on abstract sets. The anisotropic spaces $B^{s,a}_{pq}(\mathbb{R}^n)$ and $F^{s,a}_{pq}(\mathbb{R}^n)$ are the first candidates in $\mathbb{R}^n$, where a stands for an anisotropy,

$$a = (a_1, \ldots, a_n), \quad 0 < a_1 \le \cdots \le a_n, \quad \sum_{j=1}^n a_j = n.$$

Then g, now subject to the anisotropic counterparts of the procedures (1.627)–(1.629), generates elementary building blocks for the above anisotropic spaces at least if s is so large that the anisotropic counterpart of (1.630) is not needed (which causes apparently some trouble). We refer to [Far00]. A corresponding theory for spaces with dominating mixed smoothness has been developed quite recently in [Vyb03], [Vyb06]. But maybe the boldest attempt to conquer other worlds by means of the g-philosophy has been indicated briefly in Section 1.17.6 and will be subject to detailed considerations in Chapter 8: According to (1.615) the **snowflaked transforms** H as described in (1.611) (and which might be considered as a further procedure complementing (1.631) and (1.629)) gives the possibility to transfer spaces of type B^s_{pq} on d-sets Γ in $\mathbb{R}^n$, which are based on the g-philosophy, to abstract d-spaces.

Resilient spaces. There is no doubt that further spaces can be incorporated in the above empire based on the g-philosophy. One might think of spaces obtained by interpolation or extrapolation of spaces which already can be represented by the same elementary building blocks (and different sequence spaces for the corresponding coefficients). On the other hand, there are some distinguished resilient spaces which can hardly be incorporated in the above scheme. As unfortunate examples we might cite such outstanding spaces as $L_1(\mathbb{R}^n)$, $L_\infty(\mathbb{R}^n)$ or $C(\mathbb{R}^n)$. But maybe they are not so resilient as they wish to be.

The good guy. The brother of the bad guy of calculus $g(x)$ in (1.626) is the good guy

$$G(x) = e^{-|x|^2/2}, \qquad x \in \mathbb{R}^n. \tag{1.632}$$

This function plays a crucial role not only in several branches of mathematics, especially in stochastics, but it also managed to be depicted on the former German banknote *Zehn Deutsche Mark*, which is dedicated to Gauß. At least by the opinion of German bankers it represents (one of) the greatest achievement(s) of Gauß. Incidentally, the curve ($n = 1$) on this banknote suggests that $G(x)$ has a compact support. Too good to be true and the bad guy $g(x)$ in (1.626) would oppose the idea violently (at least after becoming king just because of his compactness). Nevertheless having bestowed it with so much honor it is unavoidable to look for a proper place for $G(x)$ in the realm of function spaces as considered in this book. It comes out that $G(x)$ can be taken as the basic function of *Gausslets* we are dealing with in Section 3.3.1. In addition $G(x)$ plays a decisive role in Gabor analysis or time-frequency analysis which will be mentioned briefly at the beginning of Section 3.3.1.

Chapter 2

Atoms and Pointwise Multipliers

2.1 Notation, definitions and basic assertions

2.1.1 An introductory remark

This book consists of two parts. The first part coincides with Chapter 1, which is a historically-oriented survey in continuation of Chapter 1 in [Triγ] concentrating mainly on recent developments in the last 10–15 years. The other chapters might be considered as part 2 dealing now in detail with diverse topics of the recent theory of function spaces. Both parts are essentially self-contained. In other words, we repeat some notation and definitions if they are directly needed for the understanding of what follows. This causes a mild, but at the end rather limited, overlap. On the other hand we wish to benefit from Chapter 1 (and this was just one of the criteria for the topics selected): explanations, historical comments, references and assertions quoted on which our arguments rely will not be repeated. We just hint where one can find them.

Chapter 1, including the fairy-tale in Section 1.19, gives the impression that the recent theory of function spaces rests to a large extent on diverse building blocks. So it might be not a surprise that we deal first in the following chapters with several types of atoms, wavelet bases and wavelet frames.

2.1.2 Basic notation

First we fix some basic notation. Let $\mathbb{N}$ be the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{R}^n$ be Euclidean n-space, where $n \in \mathbb{N}$. Put $\mathbb{R} = \mathbb{R}^1$, whereas $\mathbb{C}$ is the complex plane. Let $S(\mathbb{R}^n)$ be the Schwartz space of all complex-valued, rapidly decreasing, infinitely differentiable functions on $\mathbb{R}^n$. By $S'(\mathbb{R}^n)$ we denote its topological dual, the space of all tempered distributions on $\mathbb{R}^n$. Furthermore, $L_p(\mathbb{R}^n)$ with $0 < p \le \infty$, is the standard quasi-Banach space with respect to Lebesgue measure, quasi-normed by

$$\|f\,|L_p(\mathbb{R}^n)\| = \left(\int_{\mathbb{R}^n} |f(x)|^p \,\mathrm{d}x \right)^{1/p} \tag{2.1}$$

with the obvious modification if $p = \infty$.

As usual, $\mathbb{Z}$ is the collection of all integers; and $\mathbb{Z}^n$ where $n \in \mathbb{N}$, denotes the lattice of all points $m = (m_1, \ldots, m_n) \in \mathbb{R}^n$ with $m_j \in \mathbb{Z}$. Let $\mathbb{N}_0^n$, where $n \in \mathbb{N}$, be the set of all multi-indices

$$\alpha = (\alpha_1, \ldots, \alpha_n) \quad \text{with} \quad \alpha_j \in \mathbb{N}_0 \quad \text{and} \quad |\alpha| = \sum_{j=1}^n \alpha_j. \tag{2.2}$$

If $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}_0^n$ then we put

$$x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n} \qquad \text{(monomials)}. \tag{2.3}$$

Let $Q_{\nu m}$ be the closed cube in $\mathbb{R}^n$ with sides parallel to the axes of coordinates, centred at $2^{-\nu}m$ with side length $2^{-\nu+1}$ where $m \in \mathbb{Z}^n$ and $\nu \in \mathbb{N}_0$. If Q is a cube in $\mathbb{R}^n$ and $r > 0$ then rQ is the cube in $\mathbb{R}^n$ concentric with Q and with side length r times the side length of Q. We denote by $\chi_{\nu m}^{(p)}$ the p-normalised characteristic function of $Q_{\nu m}$, which means

$$\chi_{\nu m}^{(p)}(x) = 2^{(\nu-1)n/p} \text{ if } x \in Q_{\nu m} \quad \text{and} \quad \chi_{\nu m}^{(p)}(x) = 0 \text{ if } x \notin Q_{\nu m}. \tag{2.4}$$

Of course,

$$\|\chi_{\nu m}^{(p)}\,|L_p(\mathbb{R}^n)\| = 1, \quad \text{where} \quad 0 < p \le \infty. \tag{2.5}$$

If $a \in \mathbb{R}$, then we put $a_+ = \max(a, 0)$. Throughout this book we use the abbreviations

$$\sigma_p = n\left(\frac{1}{p} - 1\right)_+ \qquad \text{and} \quad \sigma_{pq} = n\left(\frac{1}{\min(p,q)} - 1\right)_+ \tag{2.6}$$

where $0 < p \le \infty$, $0 < q \le \infty$.

As usual $T\colon A \hookrightarrow B$ means that T is a linear and bounded operator from the quasi-Banach space A into the quasi-Banach space B, also written as $T \in L(A, B)$. If $T = \mathrm{id}$ is a linear and bounded embedding then we shorten $\mathrm{id}\colon A \hookrightarrow B$ also by $A \hookrightarrow B$.

2.1.3 Spaces on Euclidean n-space

We are interested in spaces of type B^s_{pq} and F^s_{pq} on diverse structures. Our starting point is always $\mathbb{R}^n$. First we recall some notation.

If $\varphi \in S(\mathbb{R}^n)$ then

$$\widehat{\varphi}(\xi) = (F\varphi)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi}\, \varphi(x)\, \mathrm{d}x, \quad \xi \in \mathbb{R}^n, \tag{2.7}$$

denotes the Fourier transform of φ. As usual, $F^{-1}\varphi$ or $\varphi^\vee$, stands for the inverse Fourier transform, given by the right-hand side of (2.7) with i in place of $-i$. Here $x\xi$ denotes the scalar product in $\mathbb{R}^n$. Both F and F^{-1} are extended to $S'(\mathbb{R}^n)$ in the standard way. Let $\varphi_0 \in S(\mathbb{R}^n)$ with

$$\varphi_0(x) = 1 \text{ if } |x| \le 1 \quad \text{and} \quad \varphi_0(y) = 0 \text{ if } |y| \ge 3/2, \tag{2.8}$$

and let

$$\varphi_k(x) = \varphi_0(2^{-k}x) - \varphi_0(2^{-k+1}x), \qquad x \in \mathbb{R}^n, \quad k \in \mathbb{N}. \tag{2.9}$$

Then, since

$$1 = \sum_{j=0}^{\infty} \varphi_j(x) \qquad \text{for all } x \in \mathbb{R}^n, \tag{2.10}$$

the φ_j form a dyadic resolution of unity in $\mathbb{R}^n$. Recall that $(\varphi_j \widehat{f})^\vee$ is an entire analytic function on $\mathbb{R}^n$ for any $f \in S'(\mathbb{R}^n)$. In particular, $(\varphi_j \widehat{f})^\vee(x)$ makes sense pointwise.

Definition 2.1. *Let $\varphi = \{\varphi_j\}_{j=0}^{\infty}$ be the dyadic resolution of unity according to* (2.8)–(2.10).

(i) *Let*

$$0 < p \le \infty, \quad 0 < q \le \infty, \quad s \in \mathbb{R}, \tag{2.11}$$

and

$$\|f \,|B^s_{pq}(\mathbb{R}^n)\|_\varphi = \left(\sum_{j=0}^{\infty} 2^{jsq} \|(\varphi_j \widehat{f})^\vee \,|L_p(\mathbb{R}^n)\|^q \right)^{1/q} \tag{2.12}$$

(*with the usual modification if $q = \infty$*). *Then*

$$B^s_{pq}(\mathbb{R}^n) = \left\{ f \in S'(\mathbb{R}^n)\colon \ \|f \,|B^s_{pq}(\mathbb{R}^n)\|_\varphi < \infty \right\}. \tag{2.13}$$

(ii) *Let*

$$0 < p < \infty, \quad 0 < q \le \infty, \quad s \in \mathbb{R}, \tag{2.14}$$

and

$$\|f \,|F^s_{pq}(\mathbb{R}^n)\|_\varphi = \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} |(\varphi_j \widehat{f})^\vee(\cdot)|^q \right)^{1/q} \,|L_p(\mathbb{R}^n) \right\| \tag{2.15}$$

(*with the usual modification if $q = \infty$*). *Then*

$$F^s_{pq}(\mathbb{R}^n) = \left\{ f \in S'(\mathbb{R}^n)\colon \ \|f \,|F^s_{pq}(\mathbb{R}^n)\|_\varphi < \infty \right\}. \tag{2.16}$$

Remark 2.2. Recall that this book consists of two parts which we try to keep independent of each other as far as basic definitions and basic assertions are concerned. Part 1 coincides with Chapter 1, whereas part 2 covers the other chapters. But on the other hand we rely on Chapter 1 when it comes to explanations, further assertions and in particular to references to the literature. In this sense we restrict ourselves here to the above definition, which coincides with Definition 1.2, and to the formulation of the very basic assertion of the theory of these spaces in the next theorem and refer otherwise to Section 1.3 for the description of some fundamental properties.

Theorem 2.3. *Let* $\varphi = \{\varphi_j\}_{j=0}^\infty$ *be the dyadic resolution of unity according to* (2.8)–(2.10).

(i) *Let* p, q, s *be given by* (2.11). *Then* $B^s_{pq}(\mathbb{R}^n)$ *according to* (2.12), (2.13) *is a quasi-Banach space* (*Banach space if* $p \geq 1$, $q \geq 1$) *and it is independent of* φ (*equivalent quasi-norms*).

(ii) *Let* p, q, s *be given by* (2.14). *Then* $F^s_{pq}(\mathbb{R}^n)$ *according to* (2.15), (2.16) *is a quasi-Banach space* (*Banach space if* $p \geq 1$, $q \geq 1$) *and it is independent of* φ (*equivalent quasi-norms*).

Remark 2.4. Further details and results, explanations and, in particular, (historical) references may be found in the above Section 1.3. In Section 1.2 we listed some classical concrete spaces. As usual in this theory we do not distinguish between equivalent quasi-norms in a given space. This may justify our omission of the subscript φ in (2.12) and (2.15) writing simply

$$\|f\,|B^s_{pq}(\mathbb{R}^n)\| \qquad \text{and} \qquad \|f\,|F^s_{pq}(\mathbb{R}^n)\|. \tag{2.17}$$

2.1.4 Smooth atoms

Atoms are one of the basic building blocks of the recent theory of function spaces. We recall what we need in the sequel, restricting us to the bare minimum. We use the notation introduced in Section 2.1.2.

Definition 2.5. *Let* $s \in \mathbb{R}$, $0 < p \leq \infty$, $K \in \mathbb{N}_0$ *and* $C \geq 1$. *A continuous function* $a\colon \mathbb{R}^n \mapsto \mathbb{C}$ *for which there exist all* (*classical*) *derivatives* $D^\alpha a$ *if* $|\alpha| \leq K$ *is called an* $(s,p)_K$*-atom if*

$$\operatorname{supp} a \subset C\,Q_{\nu m} \qquad \text{for some } \nu \in \mathbb{N}_0,\ m \in \mathbb{Z}^n, \tag{2.18}$$

and

$$|D^\alpha a(x)| \leq 2^{-\nu(s-n/p)+|\alpha|\nu} \qquad \text{for} \quad |\alpha| \leq K. \tag{2.19}$$

Remark 2.6. This is a special case of Definition 1.15 with $L = 0$ in (1.60). Again we indicate the location and size of an $(s,p)_K$-atom with (2.18), (2.19) by writing $a_{\nu m}$ in place of a. Otherwise one may consult Section 1.5.1 for explanations, results and references, especially Theorem 1.19 and Remark 1.20.

2.2 Non-smooth atomic decompositions

We abbreviate

$$B^s_p(\mathbb{R}^n) = B^s_{pp}(\mathbb{R}^n) \qquad \text{where} \quad 0 < p \le \infty, \quad s \in \mathbb{R}. \tag{2.20}$$

In particular,

$$\mathcal{C}^s(\mathbb{R}^n) = B^s_\infty(\mathbb{R}^n), \qquad s > 0, \tag{2.21}$$

are the classical Hölder-Zygmund spaces according to (1.10), (1.12). Furthermore,

$$B^s_p(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n) \iff \begin{cases} s > n/p, & 1 < p \le \infty, \\ s \ge n/p, & 0 < p \le 1. \end{cases} \tag{2.22}$$

One can replace $C(\mathbb{R}^n)$ by $L_\infty(\mathbb{R}^n)$ in (2.22). This follows from [Triβ], Section 2.7.1, and (as far as the delicate limiting embeddings are concerned) from the above Theorem 1.73 including the explanations and references given there. Let σ_p be as in (2.6). If

$$\sigma_p < s < 1/p \qquad \text{where} \quad 0 < p < \infty, \tag{2.23}$$

then the characteristic functions of cubes are elements of $B^s_p(\mathbb{R}^n)$. Details and references may be found in Theorem 1.58 and Remark 1.59.

Definition 2.7. *Let $C \ge 1$,*

$$0 < p \le \infty, \qquad \sigma_p < s < \sigma < \infty. \tag{2.24}$$

Then $a_{\nu m} \in B^\sigma_p(\mathbb{R}^n)$ is called an $(s,p)^\sigma$-atom (more precisely $(s,p)^\sigma$-C-atom) if

$$\operatorname{supp} a_{\nu m} \subset C\, Q_{\nu m} \qquad \textit{where} \quad \nu \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n, \tag{2.25}$$

and

$$\| a_{\nu m} \,| B^\sigma_p(\mathbb{R}^n)\| \le 2^{\nu(\sigma - s)}. \tag{2.26}$$

Remark 2.8. This coincides essentially with Definition 1.25 indicating now the localisation of $a_{\nu m}$ as in Remark 2.6. Concerning notation we refer to Section 2.1.2. Quite obviously, (2.26) applies to a fixed quasi-norm in $B^\sigma_p(\mathbb{R}^n)$. By (2.22), if $s < n/p$ then there are unbounded $(s,p)^\sigma$-atoms. In case of (2.23) normalised characteristic functions of cubes are atoms.

In what follows we always assume that the quasi-norm in $B^\sigma_p(\mathbb{R}^n)$ is fixed in such a way that immaterial multiplicative constants can be avoided. Recall that σ_p is given by (2.6).

Proposition 2.9. *Let $C \ge 1$, $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$. Let $0 < p \le \infty$ and $\sigma_p < s < \sigma$.*

(i) *Let $\sigma < K \in \mathbb{N}$. Then any $(s,p)_K$-atom $a_{\nu m}$ according to Definition* 2.5, *Remark* 2.6, *is an $(s,p)^\sigma$-atom as introduced in Definition* 2.7.

(ii) *Let $a_{\nu m}$ be an $(s,p)^\sigma$-atom. Then*

$$\| a_{\nu m} \,| B^s_p(\mathbb{R}^n)\| \le 1 \qquad \textit{and} \qquad \| a_{\nu m} \,| L_p(\mathbb{R}^n)\| \le 2^{-\nu s}. \tag{2.27}$$

Proof. Step 1. We prove (i). First we recall the following homogeneity property: Let $0 < p \le \infty$, $s > \sigma_p$ and

$$g \in B^s_p(\mathbb{R}^n) \qquad \text{with} \qquad \operatorname{supp} g \subset \{y\colon |y| \le 1\}. \tag{2.28}$$

Then

$$\|g \,|B^s_p(\mathbb{R}^n)\| \sim 2^{-\nu(s-n/p)} \, \|g(2^\nu \cdot) \,|B^s_p(\mathbb{R}^n)\|, \quad \nu \in \mathbb{N}_0, \tag{2.29}$$

where the equivalence constants are independent of g and ν. This follows from [Triϵ], Section 5.16, p. 66 and $B^s_p = F^s_{pp}$ (based on [Triϵ], Sections 5.3–5.5). We apply this homogeneity property to $g(x) = a_\nu(2^{-\nu}x)$ where $a_\nu = a_{\nu m}$ with $m = 0$ is an $(s,p)_K$-atom located at the origin. Using

$$\|a_\nu(2^{-\nu} \cdot) \,|B^\sigma_p(\mathbb{R}^n)\| \lesssim 2^{-\nu(s-n/p)}, \qquad \nu \in \mathbb{N}_0, \tag{2.30}$$

as a consequence of (2.18), (2.19) (and Theorem 1.19) one gets

$$\|a_\nu \,|B^\sigma_p(\mathbb{R}^n)\| \sim 2^{\nu(\sigma-n/p)} \, \|a_\nu(2^{-\nu} \cdot) \,|B^\sigma_p(\mathbb{R}^n)\| \lesssim 2^{\nu(\sigma-s)}. \tag{2.31}$$

Hence, a_ν and also $a_{\nu m}$ with $m \in \mathbb{Z}^n$, are $(s,p)^\sigma$-atoms.

Step 2. We prove (ii) relying again on (2.28), (2.29). It is sufficient to prove (2.27) for $a_\nu = a_{\nu m}$ with $m = 0$. Then it follows that

$$\begin{aligned} \|a_\nu \,|B^s_p(\mathbb{R}^n)\| &\sim 2^{\nu(s-n/p)} \, \|a_\nu(2^{-\nu} \cdot) \,|B^s_p(\mathbb{R}^n)\| \\ &\lesssim 2^{\nu(s-n/p)} \, \|a_\nu(2^{-\nu} \cdot) \,|B^\sigma_p(\mathbb{R}^n)\| \\ &\lesssim 2^{-\nu(\sigma-s)} \, \|a_\nu \,|B^\sigma_p(\mathbb{R}^n)\| \\ &\lesssim 1. \end{aligned} \tag{2.32}$$

This proves the first assertion in (2.27). The second assertion follows by the same arguments with L_p in place of B^s_p. □

Remark 2.10. The proposition coincides essentially with Proposition 1.27 as far as the above $(s,p)_K$-atoms are concerned. Otherwise Proposition 1.27 covers also the atoms introduced in Definition 1.21 with $L = 0$. Having (2.28), (2.29) in mind one could incorporate these atoms in the above considerations. But this will not be needed here. Otherwise one may consult Section 1.5.2 for further information and references. Next we wish to prove Theorem 1.29. For this purpose we need the following special case of Definition 1.17.

Definition 2.11. *Let* $0 < p \le \infty$,

$$\lambda = \{\lambda_{\nu m} \in \mathbb{C}\colon \; \nu \in \mathbb{N}_0, \, m \in \mathbb{Z}^n\}, \tag{2.33}$$

and

$$\|\lambda \,|b_p\| = \left(\sum_{\nu=0}^\infty \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}|^p \right)^{1/p} \tag{2.34}$$

(*with the* sup-*norm in case of* $p = \infty$). *Then*

$$b_p = \{\lambda\colon \; \|\lambda \,|b_p\| < \infty\}. \tag{2.35}$$

Remark 2.12. We are interested in the series

$$\sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\, a_{\nu m}, \qquad \lambda \in b_p, \tag{2.36}$$

where $a_{\nu m}$ are $(s,p)^\sigma$-atoms according to Definition 2.7. If $p<\infty$ then we claim that (2.36) and also

$$\sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}| \cdot |a_{\nu m}(\cdot)| \tag{2.37}$$

converge in some space $L_r(\mathbb{R}^n)$ with $1<r<\infty$. If $p>1$ and $r=p$ then it follows by (2.25), (2.27) that

$$\begin{aligned}\bigg(\int_{\mathbb{R}^n} \Big(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}| \cdot |a_{\nu m}(x)| \Big)^p \,\mathrm{d}x \bigg)^{1/p} &\lesssim 2^{-\nu s} \Big(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|^p \Big)^{1/p} \\ &\lesssim 2^{-\nu s}\, \|\lambda \,|b_p\|. \end{aligned} \tag{2.38}$$

(This applies also to $p=\infty$.) Hence both (2.36) and (2.37) converge in $L_r(\mathbb{R}^n)$ with $r=p$. If $p\le 1$ then we use the embedding

$$B^s_p(\mathbb{R}^n) \hookrightarrow B^\varkappa_r(\mathbb{R}^n) \qquad \text{with} \quad s-n/p = \varkappa - n/r, \quad p\le r\le\infty, \tag{2.39}$$

[Triβ], Section 2.7.1, p. 129. Since $s>\sigma_p$ one has $\varkappa>0$ for some r with $1<r<\infty$ and consequently (2.27) with $\varkappa$ and r in place of s and p, correspondingly. Hence (2.36) and (2.37) converge in $L_r(\mathbb{R}^n)$ and we shall say that (2.36) *converges absolutely in* $L_r(\mathbb{R}^n)$. In particular, (2.36) converges unconditionally in $L_r(\mathbb{R}^n)$ and in $S'(\mathbb{R}^n)$ to some $f\in L_r(\mathbb{R}^n)$. One can replace $L_r(\mathbb{R}^n)$ also by $L_{\bar p}(\mathbb{R}^n)$ with $\bar p = \max(p,1)$. If $p=\infty$ then (2.36) converges absolutely in the weighted space $L_\infty(\mathbb{R}^n, w)$, normed by $\|wf\,|L_\infty(\mathbb{R}^n)\|$ with $w(x)=(1+|x|^2)^{\delta/2}$ where $\delta<0$. In particular, (2.36) always converges unconditionally in $S'(\mathbb{R}^n)$.

Theorem 2.13. *Let $C\ge 1$, $0<p\le\infty$ and $\sigma_p<s<\sigma$, where σ_p is given by* (2.6)*. Then $B^s_p(\mathbb{R}^n)$ is the collection of all $f\in S'(\mathbb{R}^n)$ which can be represented by* (2.36)*,*

$$f = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\, a_{\nu m} \qquad \textit{with} \quad \lambda\in b_p\,, \tag{2.40}$$

where $a_{\nu m}$ are $(s,p)^\sigma$-C-atoms according to Definition 2.7*. Furthermore,*

$$\|f\,|B^s_p(\mathbb{R}^n)\| \sim \inf \|\lambda\,|b_p\| \tag{2.41}$$

is an equivalent quasi-norm where the infimum is taken over all admissible representations (2.40)*.*

Proof. Step 1. According to Remark 2.12 the right-hand side of (2.40) converges absolutely in some space $L_r(\mathbb{R}^n)$ with $r > 1$ (with the indicated modification if $p = \infty$). Hence it is a regular distribution. Furthermore by Theorem 1.19 and Proposition 2.9 it remains to prove that there is a constant $c > 0$ such that

$$\|f \,|B^s_p(\mathbb{R}^n)\| \le c\,\|\lambda \,|b_p\| \tag{2.42}$$

for all representations (2.40). If $p \le 1$ then (2.42) is a consequence of

$$\|f \,|B^s_p(\mathbb{R}^n)\|^p \le \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|^p \cdot \|a_{\nu m} \,|B^s_p(\mathbb{R}^n)\|^p \tag{2.43}$$

and (2.27). Here we used that $B^s_p(\mathbb{R}^n)$ with $p \le 1$ is a p-Banach space. This follows from the corresponding assertion for $L_p(\mathbb{R}^n)$ and (2.12) with $q = p$.

Step 2. Hence it remains to prove (2.42) for $p > 1$. To avoid confusion we fix (and modify) our notation denoting elements

$$\text{of } \mathbb{N}_0 \text{ by } j,\, k \quad \text{and} \quad \text{of } \mathbb{Z}^n \text{ by } m,\, w, \tag{2.44}$$

whereas a,b,d refer to atoms and λ,η,ν to complex numbers or sequences of complex numbers. Then f in (2.40) is now written as

$$f = \sum_{k=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{k,m}\, a_{k,m} \qquad \text{with} \quad \lambda \in b_p \tag{2.45}$$

separating temporarily different entries by an extra comma. Then $a_{k,m}$ are $(s,p)^\sigma$-atoms ($C \ge 1$ is now fixed and not indicated) located in balls $B_{k,m}$ of radius $\sim 2^{-k}$ centred at $2^{-k}m$ (again simplifying notation). We expand $a_{k,m}(2^{-k}\cdot)$ optimally in $B^\sigma_p(\mathbb{R}^n)$ by smooth $(\sigma,p)_K$-atoms $b^{k,m}_{j,w}$ with $\sigma < K$ according to Theorem 1.19 based on Definition 2.5,

$$a_{k,m}(2^{-k}x) = \sum_{j=0}^{\infty} \sum_{w\in\mathbb{Z}^n} \eta^{k,m}_{j,w}\, b^{k,m}_{j,w}(x), \qquad x \in \mathbb{R}^n, \tag{2.46}$$

with

$$\operatorname{supp} b^{k,m}_{j,w} \subset B_{j,w}, \qquad |D^\alpha b^{k,m}_{j,w}(x)| \le 2^{-j(\sigma-n/p)+j|\alpha|}, \tag{2.47}$$

where $|\alpha| \le K$, and by (2.32),

$$\Big(\sum_{j,w} |\eta^{k,m}_{j,w}|^p\Big)^{1/p} \sim \|a_{k,m}(2^{-k}\cdot) \,|B^\sigma_p(\mathbb{R}^n)\| \lesssim 2^{-k(s-n/p)}. \tag{2.48}$$

Hence,

$$a_{k,m}(x) = \sum_{j=0}^{\infty} \sum_{w\in\mathbb{Z}^n} \eta^{k,m}_{j,w}\, b^{k,m}_{j,w}(2^k x), \qquad x \in \mathbb{R}^n, \tag{2.49}$$

where the functions $b^{k,m}_{j,w}(2^k\cdot)$ have supports in balls of radius $\sim 2^{-k-j}$. By (2.47) we have for $|\alpha| \le K$,

$$
\begin{aligned}
|D^\alpha b^{k,m}_{j,w}(2^k x)| &= 2^{k|\alpha|}\left|\left(D^\alpha b^{k,m}_{j,w}\right)(2^k x)\right| \\
&\le 2^{(j+k)|\alpha|}\,2^{-j(\sigma-n/p)} \\
&= 2^{(j+k)|\alpha|}\,2^{-(j+k)(s-n/p)}\,2^{-(j+k)(\sigma-s)}\,2^{k(\sigma-n/p)}.
\end{aligned} \tag{2.50}
$$

Replacing $j+k$ by j one obtains that

$$a_{k,m}(x) = 2^{k(\sigma-n/p)} \sum_{j\ge k}\sum_{w\in\mathbb{Z}^n} \eta^{k,m}_{j-k,w}\,2^{-j(\sigma-s)}\,d^{k,m}_{j,w}(x), \tag{2.51}$$

where $d^{k,m}_{j,w}$ are smooth $(s,p)_K$-atoms with supports in balls of radius $\sim 2^{-j}$ and in a ball centred at $2^{-k}m$ of radius $\sim 2^{-k}$. The last assertion follows from the multiplication of $a_{k,m}$ in (2.51) with a suitable cut-off function, $j \ge k$, and the properties of corresponding smooth atoms according to Definition 2.5. Let (j,w,k) with $k \le j$ be the set of all $m \in \mathbb{Z}^n$ producing non-vanishing atoms $d^{k,m}_{j,w}$ in (2.51). The cardinal number of (j,w,k) can be estimated from above by some $N \in \mathbb{N}$ which is independent of j, w, k. We insert (2.51) in (2.45) and collect for fixed $j \in \mathbb{N}_0$, $w \in \mathbb{Z}^n$ all non-vanishing terms. Then

$$d_{j,w}(x) = \frac{\sum_{k\le j} 2^{k(\sigma-n/p)} \sum_{m\in(j,w,k)} \eta^{k,m}_{j-k,w}\cdot\lambda_{k,m}\cdot d^{k,m}_{j,w}(x)}{\sum_{k\le j} 2^{k(\sigma-n/p)} \sum_{m\in(j,w,k)} |\eta^{k,m}_{j-k,w}|\cdot|\lambda_{k,m}|} \tag{2.52}$$

are correctly normalised smooth $(s,p)_K$-atoms according to Definition 2.5. Furthermore,

$$f = \sum_{j=0}^\infty \sum_{w\in\mathbb{Z}^n} \nu_{j,w}\,d_{j,w} \tag{2.53}$$

with

$$\nu_{j,w} = 2^{-j(\sigma-s)} \sum_{k\le j} 2^{k(\sigma-n/p)} \sum_{m\in(j,w,k)} |\eta^{k,m}_{j-k,w}|\cdot|\lambda_{k,m}|. \tag{2.54}$$

With $0<\varepsilon<\sigma-s$ one gets for $p<\infty$,

$$
\begin{aligned}
|\nu_{j,w}|^p &\lesssim \sum_{k\le j}\sum_{m\in(j,w,k)} 2^{-(j-k)(\sigma-s-\varepsilon)p}\cdot 2^{k(s-n/p)p}\cdot|\eta^{k,m}_{j-k,w}|^p\cdot|\lambda_{k,m}|^p \\
&\le \sum_{k\le j}\sum_{m\in\mathbb{Z}^n} 2^{k(s-n/p)p}\cdot|\eta^{k,m}_{j-k,w}|^p\cdot|\lambda_{k,m}|^p
\end{aligned} \tag{2.55}
$$

and hence

$$
\begin{aligned}
\sum_{j=0}^\infty\sum_{w\in\mathbb{Z}^n}|\nu_{j,w}|^p &\lesssim \sum_{k=0}^\infty\sum_{m\in\mathbb{Z}^n}|\lambda_{k,m}|^p \sum_{j\ge k}\sum_{w\in\mathbb{Z}^n} 2^{k(s-n/p)p}\cdot|\eta^{k,m}_{j-k,w}|^p \\
&\lesssim \sum_{k=0}^\infty\sum_{m\in\mathbb{Z}^n}|\lambda_{k,m}|^p.
\end{aligned} \tag{2.56}
$$

We used (2.48). If $p = \infty$ then one has to modify the last estimates appropriately. Hence, (2.53) is an expansion by smooth atoms and (2.42) follows from Theorem 1.19 and (2.56). $\square$

Remark 2.14. We followed essentially [Tri03d]. Otherwise one may consult Section 1.5 where one finds further atomic decomposition theorems and the necessary references. If $p \leq 1$ then the extension of Theorem 1.19 for the spaces $B^s_p(\mathbb{R}^n)$ from smooth to non-smooth atoms is rather simple and based on (2.43). In this case one can even admit in Definition 2.7 that $\sigma = s$. One might ask whether the above theorem can be extended to other spaces. First candidates are $B^s_{pq}(\mathbb{R}^n)$ with $s > \sigma_p$ and $F^s_{pq}(\mathbb{R}^n)$ with $s > \sigma_{pq}$ according to (2.6). In case of B-spaces one might use real interpolation, whereas the F-spaces enjoy the crucial homogeneity property (2.28), (2.29), [Triϵ], Section 5.16, p. 66. But nothing has been done so far. Our own interest in non-smooth atoms is connected with function spaces on d-spaces as outlined in Section 1.17.6 and considered in detail in Chapter 8. The proof of (1.617) is based on the above theorem. But non-smooth atoms can also be used for other purposes, in particular for pointwise multipliers which we are going to discuss next.

2.3 Pointwise multipliers and self-similar spaces

2.3.1 Definitions and preliminaries

Recall that σ_p is given by (2.6).

Definition 2.15.

(i) *Let $A(\mathbb{R}^n)$ be either $B^s_{pq}(\mathbb{R}^n)$ or $F^s_{pq}(\mathbb{R}^n)$ according to Definition* 2.1 *with $0 < p \leq \infty$ ($p < \infty$ in the F-case), $0 < q \leq \infty$ and $s > \sigma_p$. A locally integrable function m on $\mathbb{R}^n$ is called a pointwise multiplier for $A(\mathbb{R}^n)$ if*

$$f \mapsto mf \quad \text{generates a bounded map in } A(\mathbb{R}^n). \tag{2.57}$$

The collection of all pointwise multipliers for $A(\mathbb{R}^n)$ is denoted by $M(A(\mathbb{R}^n))$.

(ii) *A quasi-Banach space $A(\mathbb{R}^n)$ on $\mathbb{R}^n$ with*

$$S(\mathbb{R}^n) \subset A(\mathbb{R}^n) \subset S'(\mathbb{R}^n)$$

is said to be a multiplication algebra if $m_1 m_2 \in A(\mathbb{R}^n)$ for all $m_1 \in A(\mathbb{R}^n)$, $m_2 \in A(\mathbb{R}^n)$ and if there is a number $c > 0$ with

$$\|m_1 m_2 \,|A(\mathbb{R}^n)\| \leq c\, \|m_1 \,|A(\mathbb{R}^n)\| \cdot \|m_2 \,|A(\mathbb{R}^n)\| \tag{2.58}$$

for all $m_1 \in A(\mathbb{R}^n)$, $m_2 \in A(\mathbb{R}^n)$.

Remark 2.16. Since $s > \sigma_p$, the spaces from part (i) are embedded in some $L_r(\mathbb{R}^n)$ with $1 < r \le \infty$. Hence mf in (2.57) makes sense as a product of functions. Otherwise we refer for a more careful discussion of this somewhat delicate point to [RuS96], Section 4.2.

Proposition 2.17. *Naturally quasi-normed, $M(A(\mathbb{R}^n))$ from Definition 2.15*(i) *becomes a quasi-Banach space and a multiplication algebra. Furthermore,*

$$M(A(\mathbb{R}^n)) \hookrightarrow L_\infty(\mathbb{R}^n). \tag{2.59}$$

Remark 2.18. As for the use of $\hookrightarrow$ we refer to the end of Section 2.1.2. The above assertions are well-known. A short proof of (2.59) may be found in [Sic99a], Lemma 3, p. 213. However we do not deal here systematically with pointwise multipliers. On the contrary, we concentrate on two specific points. First we want to apply the non-smooth atomic decomposition Theorem 2.13 and secondly we wish to show that some notions from fractal geometry are useful also in this context. The study of pointwise multipliers is one of the key problems of the theory of function spaces. It attracted a lot of attention beginning with [Str67]. As far as classical Besov spaces and (fractional) Sobolev spaces are concerned we refer to [Maz85] and, in particular, to [MaSh85]. Pointwise multipliers in general spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ according to Definition 2.1 have been studied in great detail in [RuS96], Chapter 4, where one finds many references and historical comments, and in the recent papers [Sic99b], and especially, [Sic99a]. Our own contributions may be found in [Triβ], Section 2.8, and [Triγ], Section 4.2, including detailed references. Here we follow [Tri03d].

2.3.2 Uniform and self-similar spaces

Let ψ be a non-negative C^∞ function in $\mathbb{R}^n$ with

$$\operatorname{supp}\psi \subset \{y\colon\ |y| \le \sqrt{n}\} \tag{2.60}$$

and

$$\sum_{l\in\mathbb{Z}^n} \psi(x-l) = 1, \qquad x\in\mathbb{R}^n. \tag{2.61}$$

These are basic functions for building blocks. We refer to Remark 1.38 or to the descendants of the king's function in the fairy-tale in Section 1.19.

Definition 2.19. *Let $A(\mathbb{R}^n)$ be either $B^s_{pq}(\mathbb{R}^n)$ or $F^s_{pq}(\mathbb{R}^n)$ according to Definition* 2.1 *with $0<p\le\infty$ ($p<\infty$ in the F-case), $0<q\le\infty$ and $s\in\mathbb{R}$. Then $A_{\mathrm{unif}}(\mathbb{R}^n)$ is the collection of all $f\in S'(\mathbb{R}^n)$ such that*

$$\|f\,|A_{\mathrm{unif}}(\mathbb{R}^n)\|_\psi = \sup_{l\in\mathbb{Z}^n} \|\psi(\cdot-l)f\,|A(\mathbb{R}^n)\| < \infty \tag{2.62}$$

and $A_{\mathrm{selfs}}(\mathbb{R}^n)$ is the collection of all $f\in S'(\mathbb{R}^n)$ such that

$$\|f\,|A_{\mathrm{selfs}}(\mathbb{R}^n)\|_\psi = \sup_{j\in\mathbb{N}_0,l\in\mathbb{Z}^n} \|\psi(\cdot-l)\,f(2^{-j}\cdot)\,|A(\mathbb{R}^n)\| < \infty. \tag{2.63}$$

Remark 2.20. It follows by elementary pointwise multiplier assertions that $A_{\mathrm{unif}}(\mathbb{R}^n)$ is independent of ψ and that the corresponding quasi-norms in (2.62) are equivalent for admitted choices of ψ. Similarly for $A_{\mathrm{selfs}}(\mathbb{R}^n)$. This justifies our omission of ψ on the left-hand sides of (2.62) and (2.63) and to write simply

$$\|f\,|A_{\mathrm{unif}}(\mathbb{R}^n)\| \qquad \text{and} \qquad \|f\,|A_{\mathrm{selfs}}(\mathbb{R}^n)\|. \tag{2.64}$$

They are quasi-Banach spaces (Banach spaces if $p \geq 1$, $q \geq 1$). By embeddings of type (2.39) with $r = \infty$ according to [Triβ], Section 2.7.1, p. 129, and (2.21) one obtains

$$A^s_{pq,\mathrm{selfs}}(\mathbb{R}^n) \hookrightarrow A^s_{pq,\mathrm{unif}}(\mathbb{R}^n) \hookrightarrow \mathcal{C}^{s-n/p}_{\mathrm{unif}}(\mathbb{R}^n) = \mathcal{C}^{s-n/p}(\mathbb{R}^n) \tag{2.65}$$

where $A = B$ or $A = F$. The last equality follows from [Triγ], Section 2.4.7, p. 124. Spaces of type $A_{\mathrm{unif}}(\mathbb{R}^n)$ are very natural in connection with pointwise multiplications. We refer to [RuS96], Section 4.3.1, p. 150. We complemented these constructions in [Tri03d] by the spaces $A_{\mathrm{selfs}}(\mathbb{R}^n)$. Of interest for us are the spaces $A^s_{pq,\mathrm{selfs}}(\mathbb{R}^n)$ with $s > \sigma_p$. They consist of functions. In [Tri03d], Proposition 1, p. 462, we described explicitly a quasi-norm in $B^s_{p,\mathrm{selfs}}(\mathbb{R}^n)$ with $0 < p \leq \infty$, $s > \sigma_p$ in terms of differences of functions, which will be recalled briefly in Section 2.3.4, Remark 2.27. We used the abbreviation (2.20).

We prepare for the next theorem. Let

$$0 < p \leq \infty, \quad 0 < q \leq \infty, \quad s > \sigma_p\,, \quad (p < \infty \text{ in the } F\text{-case}), \tag{2.66}$$

with σ_p as in (2.6). Then $A(\mathbb{R}^n)$ with $A = B^s_{pq}$ or $A = F^s_{pq}$ is a multiplication algebra according to Definition 2.15, which means

$$A(\mathbb{R}^n) \cdot A(\mathbb{R}^n) \hookrightarrow A(\mathbb{R}^n), \quad \text{if, and only if,} \quad A(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n). \tag{2.67}$$

This can be specified (always under the assumption (2.66)) by

$$B^s_{pq}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n) \text{ if, and only if, } \begin{cases} \text{either} & s > n/p, \\ \text{or} & s = n/p,\ 0 < q \leq 1, \end{cases} \tag{2.68}$$

and

$$F^s_{pq}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n) \text{ if, and only if, } \begin{cases} \text{either} & s > n/p, \\ \text{or} & s = n/p,\ 0 < p \leq 1. \end{cases} \tag{2.69}$$

We refer to [RuS96], Section 4.6.4, p. 221/222, and [SiT95], Remark 3.3.2, p. 114.

Theorem 2.21. *Let $A(\mathbb{R}^n)$ be either $B^s_{pq}(\mathbb{R}^n)$ or $F^s_{pq}(\mathbb{R}^n)$ with p,q,s according to* (2.66)*. Let $A_{\mathrm{unif}}(\mathbb{R}^n)$ and $A_{\mathrm{selfs}}(\mathbb{R}^n)$ be the corresponding spaces introduced in Definition* 2.19*.*

(i) *Then* $A_{\mathrm{selfs}}(\mathbb{R}^n)$ *is a multiplication algebra according to Definition* 2.15(ii) *and*

$$A_{\mathrm{selfs}}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n). \tag{2.70}$$

(ii) *Then*

$$A_{\mathrm{unif}}(\mathbb{R}^n) = A_{\mathrm{selfs}}(\mathbb{R}^n) \tag{2.71}$$

if, and only if, $A(\mathbb{R}^n)$ *is a multiplication algebra.*

Proof. Step 1. We do not indicate $\mathbb{R}^n$ in the proof, hence A stands for $A(\mathbb{R}^n)$ etc. First we prove (2.70). Let $\varphi \in S(\mathbb{R}^n)$ with

$$\varphi(x) = 1 \text{ if } |x| \le 1 \quad \text{and} \quad \varphi(y) = 0 \text{ if } |y| \ge 3/2$$

and let (in modification of (2.9))

$$\varphi_k(x) = \varphi(2^{-k}x) - \varphi(2^{-k+1}x), \qquad x \in \mathbb{R}^n, \quad k \in \mathbb{Z}. \tag{2.72}$$

Since $s > \sigma_p$ it follows by [Triγ], Section 2.3.3, p. 98, that

$$\|f\,|A\| \sim \|f\,|\dot{A}\| + \|f\,|L_p\|, \qquad f \in A, \tag{2.73}$$

where $\|f\,|\dot{A}\|$ are the corresponding homogeneous quasi-norms

$$\|f\,|\dot{B}^s_{pq}\| = \left(\sum_{j=-\infty}^{\infty} 2^{jsq} \|(\varphi_j \widehat{f})^\vee\,|L_p\|^q \right)^{1/q} \tag{2.74}$$

and

$$\|f\,|\dot{F}^s_{pq}\| = \left\| \left(\sum_{j=-\infty}^{\infty} 2^{jsq}\,|(\varphi_j \widehat{f})^\vee(\cdot)|^q \right)^{1/q} |L_p \right\|. \tag{2.75}$$

As mentioned at the end of Section 2.3.3 in [Triγ], p. 100, it follows that

$$\|f(\lambda\cdot)\,|A\| \sim \lambda^{s-n/p}\,\|f\,|\dot{A}\| + \lambda^{-n/p}\,\|f\,|L_p\|, \quad f \in A, \quad \lambda > 0, \tag{2.76}$$

where the equivalence constants are independent of f and λ. We apply (2.76) to $f \in A_{\mathrm{selfs}}$. Then we get

$$\begin{aligned} &\|\psi(\cdot - l)\,f(2^{-j}\cdot)\,|A\| \\ &\sim 2^{-j(s-n/p)}\,\|\psi(2^j\cdot - l)\,f\,|\dot{A}\| + 2^{jn/p}\,\|\psi(2^j\cdot - l) f\,|L_p\| \end{aligned} \tag{2.77}$$

uniformly for all $j \in \mathbb{N}_0$ and $l \in \mathbb{Z}^n$ and, as a consequence,

$$2^{jn} \int_{\mathbb{R}^n} |\psi(2^j y - l)|^p\,|f(y)|^p\,\mathrm{d}y \lesssim \|f\,|A_{\mathrm{selfs}}\|^p. \tag{2.78}$$

(In (2.78) we may assume that $p < \infty$ since (2.70) with $p = \infty$ follows immediately from (2.65).) Hence the right-hand side of (2.78) is a uniform bound for $|f(\cdot)|^p$ at its Lebesgue points. This proves (2.70).

Step 2. We prove that A_{selfs} is a multiplication algebra. Let $f \in A_{\mathrm{selfs}}$ and $g \in A_{\mathrm{selfs}}$. Then it follows by [RuS96], Section 4.6.4, Theorem 2 on p. 222, (2.61), (2.70) and elementary multiplier assertions that

$$\begin{aligned} &\sup_{j,l} \|\psi(\cdot - l)\, f(2^{-j}\cdot)\, g(2^{-j}\cdot)\, |A\| \\ &\lesssim \sup_{j,l} \left(\|f\,|L_\infty\| \cdot \|\psi(\cdot - l)\, g(2^{-j}\cdot)\,|A\| + \|g\,|L_\infty\| \cdot \|\psi(\cdot - l)\, f(2^{-j}\cdot)\,|A\| \right). \end{aligned} \tag{2.79}$$

One gets by (2.70) that A_{selfs} is a multiplication algebra.

Step 3. We prove (ii). Let A be a multiplication algebra. Then we have

$$A \hookrightarrow L_\infty \quad \text{and} \quad \|\psi(2^j \cdot - l)\,|A\| \lesssim 2^{j(s-n/p)}, \quad j \in \mathbb{N}_0, \quad l \in \mathbb{Z}^n. \tag{2.80}$$

The first assertion follows from (2.67), the second one from (2.28), (2.29), real interpolation for the B-spaces and elementary embedding for the F-spaces. Then one obtains by (2.77), (2.73) that

$$\begin{aligned} \|\psi(\cdot - l)\, f(2^{-j}\cdot)\,|A\| &\lesssim 2^{-j(s-n/p)} \|\psi(2^j \cdot - l)\,|A\| \cdot \|f\,|A_{\mathrm{unif}}\| + \|f\,|L_\infty\| \\ &\lesssim \|f\,|A_{\mathrm{unif}}\|. \end{aligned} \tag{2.81}$$

This proves (2.71). Conversely, assuming (2.71) one gets (2.78) with A_{unif} in place of A_{selfs} and by $A \hookrightarrow A_{\mathrm{unif}}$, with A in place of A_{selfs}. Hence, $A \hookrightarrow L_\infty$, and (2.67) proves that A is a multiplication algebra. □

2.3.3 Pointwise multipliers

Recall that σ_p and σ_{pq} are given by (2.6) and that we introduced the pointwise multiplier spaces $M(A(\mathbb{R}^n))$ according to Definition 2.15 for all spaces $A(\mathbb{R}^n)$ with $A = B^s_{pq}$ and $A = F^s_{pq}$ restricted by (2.66). Let $\mathcal{C}^s = B^s_\infty = B^s_{\infty\infty}$ for $s > 0$.

Proposition 2.22.

(i) *Let $A_{\mathrm{unif}}(\mathbb{R}^n)$ and $A_{\mathrm{selfs}}(\mathbb{R}^n)$ be the spaces according to Definition* 2.19 *for the spaces $A(\mathbb{R}^n)$ where $A = B^s_{pq}$ or $A = F^s_{pq}$ with* (2.66)*. Then*

$$M(A(\mathbb{R}^n)) \hookrightarrow L_\infty(\mathbb{R}^n) \cap A_{\mathrm{unif}}(\mathbb{R}^n) \tag{2.82}$$

and

$$A_{\mathrm{selfs}}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n) \cap A_{\mathrm{unif}}(\mathbb{R}^n). \tag{2.83}$$

(ii) *Let $A(\mathbb{R}^n) = F^s_{pq}(\mathbb{R}^n)$ with*

$$0 < p < \infty, \quad 0 < q \le \infty, \quad s > \sigma_{pq}.$$

Then

$$M(A(\mathbb{R}^n)) \hookrightarrow A_{\mathrm{selfs}}(\mathbb{R}^n). \tag{2.84}$$

(iii) *Let either $A(\mathbb{R}^n) = F^s_{pq}(\mathbb{R}^n)$ be a multiplication algebra according to* (2.66), (2.67), (2.69), *or $A(\mathbb{R}^n) = \mathcal{C}^s(\mathbb{R}^n)$ with $s > 0$. Then*

$$M\left(A(\mathbb{R}^n)\right) = A_{\mathrm{unif}}(\mathbb{R}^n) = A_{\mathrm{selfs}}(\mathbb{R}^n). \tag{2.85}$$

Proof. Step 1. We prove (i). The inclusion (2.83) is an immediate consequence of (2.65), (2.70). Let $m \in M(A)$ (where we again do not indicate $\mathbb{R}^n$) and let ψ be as in Definition 2.19. Then it follows by elementary properties that

$$\|\psi(\cdot - l)\, m\,|A\| \lesssim \|m\,|M(A)\|, \qquad l \in \mathbb{Z}^n, \tag{2.86}$$

where the equivalence constants are independent of l and m. Now (2.59) and (2.86) prove (2.82).

Step 2. We prove (ii). The homogeneity property (2.28), (2.29), remains valid for F^s_{pq} with the indicated values of p,q,s in place of B^s_p and for $\mathcal{C}^s = B^s_\infty$. We refer to [Tri$\epsilon$], Section 5.16, p. 66 (based on [Triϵ], Sections 5.3–5.5). Let $m \in M(A)$ and ψ be as in Definition 2.19. Then

$$\begin{aligned} \|\psi\, m(2^{-j}\cdot)\,|A\| &\sim 2^{-j(s-n/p)}\, \|\psi(2^j\cdot)\, m\,|A\| \\ &\lesssim \|m\,|M(A)\|\, 2^{-j(s-n/p)}\, \|\psi(2^j\cdot)\,|A\| \\ &\lesssim \|m\,|M(A)\|. \end{aligned} \tag{2.87}$$

Then (2.84) follows from (2.63).

Step 3. The second equality in (2.85) is covered by Theorem 2.21(ii). As for the first equality we have so far (2.82). Hence it remains to prove that any $m \in A_{\mathrm{unif}}$ belongs also to $M(A)$. For the above spaces A one has the localisation principle,

$$\|f\,|A\| \sim \left(\sum_{l \in \mathbb{Z}^n} \|\psi(\cdot - l) f\,|A\|^p \right)^{1/p} \tag{2.88}$$

with the usual modification if $p = \infty$, hence $A = \mathcal{C}^s$. We refer to [Triγ], Section 2.4.7, p. 124. One may replace ψ by ψ^2. Let $m \in A_{\mathrm{unif}}$. Since A is a multiplication algebra one gets

$$\begin{aligned} \|mf\,|A\|^p &\sim \sum_{l \in \mathbb{Z}^n} \|\psi^2(\cdot - l) mf\,|A\|^p \\ &\lesssim \sum_{l \in \mathbb{Z}^n} \|\psi(\cdot - l) f\,|A\|^p\, \sup_k \|\psi(\cdot - k)\, m\,|A\|^p \\ &\lesssim \|m\,|A_{\mathrm{unif}}\|^p \cdot \|f\,|A\|^p \end{aligned} \tag{2.89}$$

(with obvious modification if $p = \infty$). This proves the first equality in (2.85). $\square$

The next theorem is the main assertion of Section 2.3: an application of the non-smooth atomic decomposition Theorem 2.13 for the spaces

$$A(\mathbb{R}^n) = B^s_p(\mathbb{R}^n) = B^s_{pp}(\mathbb{R}^n) \quad \text{with} \quad 0 < p \le \infty, \quad s > \sigma_p, \tag{2.90}$$

where again σ_p is given by (2.6). If $p = \infty$ then $\mathcal{C}^s = B^s_\infty$ with $s > 0$ are the Hölder-Zygmund spaces. First we complement Definition 2.19 restricted to the above spaces.

Definition 2.23. *Let* $0 < p \le \infty$ *and* $s > \sigma_p$. *Then*

$$B^{s+}_{p,\mathrm{selfs}}(\mathbb{R}^n) = \bigcup_{\sigma > s} B^\sigma_{p,\mathrm{selfs}}(\mathbb{R}^n). \tag{2.91}$$

Remark 2.24. According to Proposition 1 in [Tri03d], p. 462, one can describe $B^\sigma_{p,\mathrm{selfs}}(\mathbb{R}^n)$ and hence also $B^{s+}_{p,\mathrm{selfs}}(\mathbb{R}^n)$ explicitly in terms of differences of functions. We will return to this point in Section 2.3.4 below, Remark 2.27. By Theorem 2.21 the product of two functions belonging to $B^{s+}_{p,\mathrm{selfs}}(\mathbb{R}^n)$ is again an element of $B^{s+}_{p,\mathrm{selfs}}(\mathbb{R}^n)$. Furthermore,

$$B^{s+}_{p,\mathrm{selfs}}(\mathbb{R}^n) \subset B^s_{p,\mathrm{selfs}}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n), \tag{2.92}$$

where we preferred $\subset$ in the first inclusion since we reserved $\hookrightarrow$ for embeddings between quasi-Banach spaces.

Theorem 2.25. *Let* $0 < p \le \infty$ *and* $s > \sigma_p$, *where* σ_p *is given by* (2.6).

(i) *Then*

$$B^{s+}_{p,\mathrm{selfs}}(\mathbb{R}^n) \subset M\left(B^s_p(\mathbb{R}^n)\right) \hookrightarrow B^s_{p,\mathrm{selfs}}(\mathbb{R}^n). \tag{2.93}$$

(ii) *In addition, let* $0 < p \le 1$. *Then*

$$M\left(B^s_p(\mathbb{R}^n)\right) = B^s_{p,\mathrm{selfs}}(\mathbb{R}^n). \tag{2.94}$$

Proof. Step 1. We prove part (i). The right-hand side of (2.93) is covered by Proposition 2.22(ii) with $p = q$, complemented by (2.85) if $p = \infty$. We prove the left-hand side of (2.93) and assume $m \in B^\sigma_{p,\mathrm{selfs}}$ with $\sigma > s$ (again we do not indicate $\mathbb{R}^n$). Let

$$f = \sum_{j=0}^\infty \sum_{l \in \mathbb{Z}^n} \lambda_{jl}\, a_{jl}, \qquad \|\lambda \,|b_p\| \sim \|f \,|B^s_p\|, \tag{2.95}$$

be an optimal smooth atomic decomposition according to Theorem 1.19 and Definition 2.11, where a_{jl} are $(s,p)_K$-atoms with $K > \sigma$ as introduced in Definition 2.5 and Remark 2.6. We wish to prove that $m\, a_{jl}$ in

$$mf = \sum_{j,l} \lambda_{jl}\,(m a_{jl}), \tag{2.96}$$

after normalisation, are $(s,p)^\sigma$-atoms according to Definition 2.7. The support condition in (2.25) is obvious. If $l=0$ then we put $a_j = a_{jl} = a_{j,0}$. We may assume that

$$\operatorname{supp} a_j(2^{-j}\cdot) \subset \{y:\ |y| \le \sqrt{n}/2\}, \qquad j \in \mathbb{N}_0, \tag{2.97}$$

and that $\psi(y) > 0$ if $|y| < \sqrt{n}$ for the function ψ in (2.60). Then it follows from the pointwise multiplier assertion according to [Triγ], Corollary 4.2.2, p. 205, that

$$\|a_j(2^{-j}\cdot)\,\psi^{-1}\,|M(B^\sigma_p)\| \lesssim 2^{-j(s-n/p)}, \qquad j \in \mathbb{N}_0, \tag{2.98}$$

uniformly for all atoms. Using this observation and the homogeneity assertion (2.29) with σ in place of s one gets

$$\begin{aligned} \|m\,a_j\,|B^\sigma_p\| &\sim 2^{j(\sigma-n/p)}\,\|m(2^{-j}\cdot)\,a_j(2^{-j}\cdot)\,|B^\sigma_p\| \\ &\lesssim 2^{j(\sigma-s)}\,\|\psi\,m(2^{-j}\cdot)\,|B^\sigma_p\|. \end{aligned} \tag{2.99}$$

In case of a_{jl} with $l \in \mathbb{Z}^n$ one has (2.99) with a_{jl} and $\psi(\cdot - l)$ in place of a_j and ψ. Hence by (2.63)

$$\|m\,a_{jl}\,|B^\sigma_p\| \lesssim 2^{j(\sigma-s)}\,\|m\,|B^\sigma_{p,\mathrm{selfs}}\|, \quad j \in \mathbb{N}_0, \quad l \in \mathbb{Z}^n, \tag{2.100}$$

and it follows by Definition 2.7 that $m\,a_{jl}$ are $(s,p)^\sigma$-atoms multiplied with the indicated factor. By Theorem 2.13 and (2.95) one obtains that

$$\|mf\,|B^s_p\| \lesssim \|m\,|B^\sigma_{p,\mathrm{selfs}}\| \cdot \|f\,|B^s_p\|, \qquad f \in B^s_p. \tag{2.101}$$

This proves the left-hand side of (2.93).

Step 2. We prove (ii). Let $m \in B^s_{p,\mathrm{selfs}}$ and $p \le 1$. Then it follows from (2.100) with $\sigma = s$ that

$$\|ma_{jl}\,|B^s_p\| \lesssim \|m\,|B^s_{p,\mathrm{selfs}}\|, \quad j \in \mathbb{N}_0, \quad l \in \mathbb{Z}^n. \tag{2.102}$$

Using (2.95) and the counterpart of (2.43) applied to (2.96) one obtains that

$$\|mf\,|B^s_p\|^p \lesssim \|m\,|B^s_{p,\mathrm{selfs}}\|^p \cdot \|f\,|B^s_p\|^p. \tag{2.103}$$

Hence $m \in M(B^s_p)$. Then (2.94) follows from (2.103) and the right-hand side of (2.93). □

Remark 2.26. Both Proposition 2.22 and Theorem 2.25 make clear that typical procedures for building blocks of function spaces and of fractal analysis such as

$$\text{dilations } x \mapsto 2^j x \qquad \text{and} \qquad \text{translations } x \mapsto x + l \tag{2.104}$$

where $x \in \mathbb{R}^n$, $j \in \mathbb{N}_0$, $l \in \mathbb{Z}^n$, are also intimately related to pointwise multipliers. Some assertions of type (2.94) and also of explicit descriptions of the spaces $B^s_{p,\mathrm{selfs}}(\mathbb{R}^n)$ which will be mentioned in Section 2.3.4, Remark 2.27 below, are known, although the formulations given are different. In the case of $p = 1$ and $0 < s \notin \mathbb{N}$ we refer to [MaSh85], Section 3.4.2, p. 140 (a formulation of this result may also be found in [Sic99b], Theorem 2.9, pp. 299/300). As for $p < 1$ (and with some restrictions for $p = 1$) we refer to [Sic99a], Section 3.4.1, pp. 234–236, and Section 3.4.2, pp. 236–237. There are characterisations of type (2.94) with $\sigma_p < s < 1$ in the framework of the more general case of $M\left(F^s_{pq}(\mathbb{R}^n)\right)$.

2.3.4 Comments and complements

Pointwise multipliers are not a central subject of this book. The corresponding above results will not be needed later on. It was our main point to apply the non-smooth atomic representation Theorem 2.13 to get Theorem 2.25. Otherwise we refer to the literature mentioned in Remarks 2.18 and 2.26. We add now a few comments without proofs which are directly related to the above considerations. All spaces are defined on $\mathbb{R}^n$ which will not be indicated in what follows.

Remark 2.27. *Let* $0 < p \le \infty$ *and* $\sigma_p < s < N \in \mathbb{N}$, *where* σ_p *is given by* (2.6). *Then* $B^s_{p,\mathrm{selfs}}$ *according to Definition* 2.19 *with the abbreviation* (2.20) *is the collection of all* $f \in L_\infty$ *such that*

$$\|f\,|L_\infty\| + \sup_{l\in\mathbb{Z}^n, j\in\mathbb{N}_0} 2^{-j(s-n/p)} \left(\int\limits_{|x|+|h|\lesssim 2^{-j}} |h|^{-sp} \left|\Delta^N_h f(x+l2^{-j})\right|^p \frac{\mathrm{d}x\mathrm{d}h}{|h|^n} \right)^{1/p} < \infty \tag{2.105}$$

(*equivalent quasi-norms*). We refer to [Tri03d], Proposition 1, p. 462, and the above embedding (2.70). This makes the spaces in Theorem 2.25 more transparent.

Remark 2.28. One gets by real interpolation the following complement of (2.93): *Let* $0 < p \le \infty$, $0 < q \le \infty$ *and* $s > \sigma_p$. *Then*

$$B^{s+}_{p,\mathrm{selfs}} \subset M\left(B^s_{pq}\right). \tag{2.106}$$

But as follows from the next remark an extension of the right-hand side of (2.93) with B^s_{pq} in place of B^s_p is not possible in general.

Remark 2.29. For the spaces A covered by Definition 2.15 one may ask whether

$$M(A) = A_{\mathrm{selfs}} \tag{2.107}$$

with A_{selfs} (and A_{unif}, needed below) as in Definition 2.19. So far we have affirmative answers in each of the following cases:

$$A = F^s_{pq} \text{ is a multiplication algebra,} \tag{2.108}$$

$$A = \mathcal{C}^s \text{ with } s > 0, \tag{2.109}$$

$$A = B^s_p \text{ with } 0 < p \le 1,\ s > n(\tfrac{1}{p} - 1). \tag{2.110}$$

We refer to Proposition 2.22(ii) and Theorem 2.25(ii). According to Theorem 2.21(ii) in case of multiplication algebras the question (2.107) is equivalent to

$$M(A) = A_{\mathrm{unif}} = A_{\mathrm{selfs}}. \tag{2.111}$$

This applies to (2.108), (2.109), but not to (2.110) if $s < n/p$. According to [SiS99] the assertion (2.111) can be extended to the multiplication algebras

$$A = B^s_{pq} \qquad \text{with} \quad 1 \le p \le q \le \infty, \quad s > n/p. \tag{2.112}$$

On the other hand, (2.111) is no longer valid in case of the multiplication algebras

$$A = B^s_{pq} \qquad \text{with} \quad 1 \le q < p \le \infty, \quad s > n/p. \tag{2.113}$$

We refer to [Bor88], p. 162.

Remark 2.30. If the above space A is a multiplication algebra then it follows from Theorem 2.21(ii) that (2.107), (2.111) is the same as

$$M(A) = A_{\mathrm{unif}}. \tag{2.114}$$

This can be applied in particular to the spaces A in (2.108) and (2.109). In this version the corresponding pointwise multiplier assertions are known. We refer to [RuS96], Section 4.9, in particular Theorem 1 on p. 247, where one finds detailed considerations and further references. Let $\mathring{A} = \mathring{A}(\mathbb{R}^n)$ be the completion of $S(\mathbb{R}^n)$ in the spaces A covered by Definition 2.15, hence B^s_{pq} and F^s_{pq} with (2.66). One has $A = \mathring{A}$ if, and only if, $\max(p,q) < \infty$. [It is well known that $S(\mathbb{R}^n)$ is not dense in $B^s_{pq}(\mathbb{R}^n)$ if either $p = \infty$ or $q = \infty$. The corresponding assertion for $F^s_{p\infty}(\mathbb{R}^n)$ may be found in [Triϵ], p. 46.] Now one may extend question (2.114) to these spaces. We have done this in [Tri03d], Remark 10, pp. 480–481, in some detail for all spaces B^s_{pq} and F^s_{pq} with $\max(p,q) = \infty$. There one finds also further references. We mention only two examples in obvious notation,

$$M\left(\mathring{\mathcal{C}}^s\right) = \mathring{\mathcal{C}}^s \subsetneqq \mathcal{C}^s = M\left(\mathcal{C}^s\right) = \mathcal{C}^s, \qquad s > 0, \tag{2.115}$$

and

$$M\left(\mathring{B}^s_{\infty q}\right) = M\left(B^s_{\infty q}\right), \qquad 0 < q < \infty, \quad s > 0. \tag{2.116}$$

Remark 2.31. We restricted Definition 2.15 to $s > \sigma_p$. This is convenient and one gets quite easily (2.59). But this restriction is not necessary. One can extend this definition to all spaces A as introduced in Definition 2.1. One has to say what is meant by (2.57). But here we again refer to [RuS96], Section 4.2. One advantage of such an extension comes from the duality assertion

$$M\left(\mathring{A}\right) \hookrightarrow M\left((\mathring{A})'\right) \tag{2.117}$$

and the question of equality. As for dual spaces one may consult [Triβ], Section 2.11, or [RuS96], Section 2.1.5. In particular, if $1 < p < \infty$, $1 < q < \infty$, $s \in \mathbb{R}$, and $A = B$ or $A = F$, then $A = \mathring{A}$, and

$$\left(A^s_{pq}\right)' = A^{-s}_{p'q'}, \qquad M\left(A^s_{pq}\right) = M\left(A^{-s}_{p'q'}\right), \tag{2.118}$$

where $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$. Interpolation makes clear that spaces of smoothness $s = 0$ are of special interest. We refer for corresponding results to [KoS02] and [Tri03d].

Remark 2.32. Let χ_Ω be the characteristic function of the domain Ω in $\mathbb{R}^n$. A special, but nevertheless outstanding, problem is the question under which circumstances χ_Ω is a pointwise multiplier in some spaces according to Definition 2.1 (adopting the more general point of view indicated in Remark 2.31). As for the now older contributions we refer to [Triβ], Section 2.8.7, pp. 158–165; [Fra86a]; [FrJ90], §13; and the references given there. The state-of-the-art in the middle of the 1990s may be found in [RuS96], Section 4.6.3, pp. 207–221, 258. More recent results (and again additional references) are given in [Sic99a], Section 4; [Sic99b], Section 4; [Tri02a], Section 5; [Tri03d], Sections 2.4 and 4.3. The quality of the boundary $\Gamma = \partial\Omega$ is crucial for the question in which spaces B^s_{pq} or F^s_{pq} the characteristic function χ_Ω is a pointwise multiplier. In [Tri03d] we dealt with this problem in some detail assuming that Γ is a d-set or an h-set according to (1.495) and Definition 1.151, correspondingly. Furthermore another fundamental notion of fractal analysis comes in quite naturally. This is the ball condition or porosity condition which we mentioned briefly in connection with Proposition 1.172 and its proof. A compact set Γ in $\mathbb{R}^n$ is said to be *porous* if there is a number η with $0 < \eta < 1$ such that one finds, for any ball $B(x,r)$, centred at $x \in \mathbb{R}^n$ and of radius $0 < r < 1$, a ball $B(y,\eta r)$ with

$$B(y,\eta r) \subset B(x,r) \qquad \text{and} \qquad B(y,\eta r) \cap \Gamma = \emptyset. \tag{2.119}$$

We refer for details to [Triϵ], Sections 9.16-9.19, pp. 138–141, where this property was called the *ball condition*. The following assertion may be found in [Tri02a], Proposition 5.7(iii), p. 511:
Let Ω be a bounded domain in $\mathbb{R}^n$ such that $\Gamma = \partial\Omega$ is porous. Then

$$\chi_\Omega \in M\left(F^0_{pq}\right) \quad \textit{for all } 1 < p < \infty,\ 1 \le q \le \infty. \tag{2.120}$$

This applies in particular to bounded domains Ω where $\Gamma = \partial\Omega$ is a d-set with $n - 1 \le d < n$.

Chapter 3

Wavelets

This chapter deals with diverse types of wavelets in the (isotropic inhomogeneous unweighted) spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$. Some assertions will be extended later on in Section 4.2 to (isotropic) spaces in domains and in Chapters 5 and 6 to anisotropic spaces and weighted spaces in $\mathbb{R}^n$. In Section 3.1 we return to wavelet isomorphisms and wavelet bases as described in Section 1.7. In particular we give a proof of Theorem 1.64 and its counterpart in terms of Meyer wavelets and discuss some applications. Section 3.2 deals with wavelet frames as outlined so far in Section 1.8. As an outgrowth we develop a local smoothness theory. In Section 3.3 we complement these diverse types of wavelets by Gausslets and an application of the quarkonial representation Theorem 1.39.

3.1 Wavelet isomorphisms and wavelet bases

3.1.1 Definitions

We fix our notation and recall some definitions and assertions from Section 1.7 which are indispensable for what follows. Again we rely on [Tri04a].

With exception of Section 3.1.5 we always assume in Section 3.1 that the real compactly supported scaling function $\psi_F \in C^k(\mathbb{R})$ and the real compactly supported associated wavelet $\psi_M \in C^k(\mathbb{R})$ on the real line $\mathbb{R}$ have the same meaning as in Theorem 1.61(ii), where $k \in \mathbb{N}$. In particular, according to Proposition 1.51,

$$\psi^j_m(x) = \begin{cases} \psi_F(x-m) & \text{if } j=0,\ m \in \mathbb{Z}, \\ 2^{\frac{j-1}{2}}\psi_M(2^{j-1}x-m) & \text{if } j \in \mathbb{N},\ m \in \mathbb{Z}, \end{cases} \tag{3.1}$$

is an orthonormal basis in $L_2(\mathbb{R})$. These are the well-known one-dimensional (inhomogeneous) Daubechies wavelets which will be denoted as *k-wavelets*. Further details, explanations and references may be found in Section 1.7. In particular we always assume that (3.1) comes out of a multiresolution analysis.

We need the n-dimensional counterpart of these k-wavelets. We follow the construction described in Remark 1.52, Proposition 1.53 now assuming from the very beginning that ψ_F and ψ_M are the above functions resulting in the k-wavelets (3.1). Let $n \in \mathbb{N}$,

$$G = (G_1, \dots, G_n) \in \{F, M\}^{n*}, \tag{3.2}$$

where G_r is either F or M and where $*$ indicates that at least one of the components of G must be an M. Hence the cardinal number of $\{F, M\}^{n*}$ is $2^n - 1$. For $x \in \mathbb{R}^n$ we put

$$\Psi_m^G(x) = \prod_{r=1}^n \psi_{G_r}(x_r - m_r), \quad G \in \{F, M\}^{n*}, \quad m \in \mathbb{Z}^n. \tag{3.3}$$

Let $G^j = \{F, M\}^{n*}$ if $j \in \mathbb{N}$. We complement (3.3) by $G^0 = \{(F)^n\} = \{(F, \dots, F)\}$ and

$$\Psi_m^G(x) = \prod_{r=1}^n \psi_F(x_r - m_r), \quad G \in G^0, \quad m \in \mathbb{Z}^n. \tag{3.4}$$

Then it follows by Proposition 1.53 that

$$\Psi_m^{j,G}(x) = \begin{cases} \Psi_m^G(x) & \text{if } j = 0,\ G \in G^0,\ m \in \mathbb{Z}^n, \\ 2^{\frac{j-1}{2}n}\Psi_m^G(2^{j-1}x) & \text{if } j \in \mathbb{N},\ G \in G^j,\ m \in \mathbb{Z}^n, \end{cases} \tag{3.5}$$

is an orthonormal basis in $L_2(\mathbb{R}^n)$. The functions $\Psi_m^{j,G}$ (and the whole system) are denoted as *k-wavelets* where we reserve in Section 3.1 this notation exclusively for the above construction originating from the one-dimensional real Daubechies wavelets $\psi_F \in C^k(\mathbb{R})$ and $\psi_M \in C^k(\mathbb{R})$.

Our first aim is the proof of Theorem 1.64. For this purpose we recall the definition of the corresponding sequence spaces. Again let Q_{jm} with $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$ be the dyadic cubes in $\mathbb{R}^n$ as introduced in Section 2.1.2 and let χ_{jm} be the characteristic function of Q_{jm}.

Definition 3.1. *Let $0 < p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$,*

$$\lambda = \left\{\lambda_m^{j,G} \in \mathbb{C} :\ j \in \mathbb{N}_0,\ G \in G^j,\ m \in \mathbb{Z}^n\right\}, \tag{3.6}$$

$$\|\lambda \,|b_{pq}^s\| = \left(\sum_{j=0}^\infty 2^{j(s-n/p)q} \sum_{G \in G^j} \left(\sum_{m \in \mathbb{Z}^n} |\lambda_m^{j,G}|^p\right)^{q/p}\right)^{1/q} \tag{3.7}$$

and

$$\|\lambda \,|f_{pq}^s\| = \left\| \left(\sum_{j,G,m} 2^{jsq} \left|\lambda_m^{j,G}\chi_{jm}(\cdot)\right|^q \right)^{1/q} |L_p(\mathbb{R}^n) \right\| \tag{3.8}$$

(*with the usual modification if* $p = \infty$ *and/or* $q = \infty$). *Then*

$$b^s_{pq} = \big\{ \lambda : \ \|\lambda \,| b^s_{pq}\| < \infty \big\} \tag{3.9}$$

and

$$f^s_{pq} = \big\{ \lambda : \ \|\lambda \,| f^s_{pq}\| < \infty \big\}. \tag{3.10}$$

Remark 3.2. This definition is the adapted counterpart of Definition 1.17. It formalises (1.156)–(1.159). In (3.8) the index set (j, G, m) is the same as in (3.7). Obviously, b^s_{pq} and f^s_{pq} are quasi-Banach spaces.

3.1.2 Some preparations

The proof of Theorem 1.64 will be based on the observation that the k-wavelets according to (3.5) can be used simultaneously as smooth atoms according to Theorem 1.19 and as kernels of local means such that Theorem 1.10 can be applied. But this requires some adaptations. Let $\Psi^G = \Psi^G_m$ with $m = 0$ according to (3.3) and (3.4). Then it follows by Theorem 1.61 that

$$\widehat{\Psi^G}(0) = (2\pi)^{-n/2} \qquad \text{if} \qquad G \in G^0 \tag{3.11}$$

and for $G \in \{F, M\}^{n*}$,

$$(D^\alpha \widehat{\Psi^G})(0) = \int_{\mathbb{R}^n} x^\alpha \, \Psi^G(x) \, \mathrm{d}x = 0 \quad \text{for} \quad |\alpha| \le k. \tag{3.12}$$

But then it is quite clear that $2^{-j(s-n/p)}\, 2^{-jn/2}\, \Psi^{j,G}_m$ are atoms

$$\text{in } B^s_{pq}(\mathbb{R}^n) \quad \text{if } k > \max(s, \sigma_p - s) \tag{3.13}$$

and

$$\text{in } F^s_{pq}(\mathbb{R}^n) \quad \text{if } k > \max(s, \sigma_{pq} - s) \tag{3.14}$$

(ignoring inconsequential constants). As for local means the situation is not so clear for several reasons. We use one of the above functions Ψ^G as a kernel in (1.41) and get for $t > 0$ as there (at least formally)

$$\begin{aligned} \Psi^G(t, f)(x) &= \int_{\mathbb{R}^n} \Psi^G(y)\, f(x + ty)\, \mathrm{d}y \\ &= t^{-n} \int_{\mathbb{R}^n} \Psi^G\left(\frac{y - x}{t}\right) f(y)\, \mathrm{d}y. \end{aligned} \tag{3.15}$$

With $G \in \{F, M\}^{n*}$, $j \in \mathbb{N}$, $t = 2^{-j+1}$ and $x = 2^{-j+1}m$ where $m \in \mathbb{Z}^n$, one obtains that

$$\begin{aligned} \Psi^G(2^{-j+1}, f)(2^{-j+1}m) &= 2^{n(j-1)} \int_{\mathbb{R}^n} \Psi^G(2^{j-1}y - m)\, f(y)\, \mathrm{d}y \\ &= 2^{\frac{j-1}{2}n} \int_{\mathbb{R}^n} \Psi^{j,G}_m(y)\, f(y)\, \mathrm{d}y. \end{aligned} \tag{3.16}$$

Similarly if $j = 0$ and $G \in G^0$. Hence the Fourier coefficients of f with respect to the orthonormal basis in (3.5) can be represented in terms of local means. But one must be sure that the right-hand side of (3.16) makes sense if $f \in B^s_{pq}(\mathbb{R}^n)$ or $f \in F^s_{pq}(\mathbb{R}^n)$. This is the case if Ψ^G and hence $\Psi^{j,G}_m$ belongs to the corresponding dual spaces. There is a complete duality theory for these spaces. We refer to [Triβ], Section 2.11, and to [RuS96], Section 2.1.5. For our purpose it is sufficient to deal with the case $q = p$, hence $B^s_{pp}(\mathbb{R}^n)$. By [Triβ], Theorems 2.11.2 and 2.11.3, one has for the dual space of $B^s_{pp}(\mathbb{R}^n)$ that

$$B^s_{pp}(\mathbb{R}^n)' = B^{-s+\sigma_p}_{p'p'}(\mathbb{R}^n), \qquad s \in \mathbb{R}, \quad 0 < p < \infty, \tag{3.17}$$

where σ_p has the same meaning as in (2.6),

$$\frac{1}{p} + \frac{1}{p'} = 1 \text{ if } 1 \le p \le \infty \quad \text{and} \quad p' = \infty \text{ if } 0 < p < 1. \tag{3.18}$$

This assertion can be extended to $p = \infty$ if one replaces $\mathcal{C}^s(\mathbb{R}^n) = B^s_{\infty\infty}(\mathbb{R}^n)$ on the left-hand side of (3.17) by $\mathring{\mathcal{C}}^s(\mathbb{R}^n)$, the completion of $S(\mathbb{R}^n)$ in $\mathcal{C}^s(\mathbb{R}^n)$, [Tri$\beta$], Section 2.11.2, Remark 2 on p. 180. Functions belonging to $C^k(\mathbb{R}^n)$ with compact support are also elements of $B^{-s+\sigma_p}_{p'p'}(\mathbb{R}^n)$ if $k > \sigma_p - s$. But this is the case for the above k-wavelets if (3.13) or (3.14) is assumed. Then (3.16) makes sense for $f \in B^s_{pq}(\mathbb{R}^n)$ or $f \in F^s_{pq}(\mathbb{R}^n)$. One may also consult Section 5.1.7 where we give an elementary justification of this dual pairing. Compared with Theorem 1.10 there remain two problems. First one has to find a substitute for the Tauberian conditions of type (1.42). This can be done. Secondly, and more seriously, the above kernels in (3.16) have only a limited smoothness, whereas all theorems about local means assume that the kernels are C^∞ functions with compact support. This applies to the original version in [Triγ], Sections 2.4.6 and 2.5.3, and to all the later improvements as described in Section 1.4, where one finds also the corresponding references. But the reason is simple: There was no interest and no motivation to deal with non-smooth kernels. We discussed this point briefly in Remark 1.14. But the situation is now different. The k-wavelets have only a limited smoothness and there do not exist compactly supported C^∞ wavelets of the above type. The latter assertion is well known in wavelet theory and may be found, for example, in [Mal99], Proposition 7.4, p. 251, going back to Daubechies. We circumvent these difficulties by returning to our original theory of generalised characterisations of the spaces $F^s_{pq}(\mathbb{R}^n)$ and $B^s_{pq}(\mathbb{R}^n)$ as presented in [Triγ], Sections 2.4 and 2.5. This might be rather crude as far as local means are concerned. But it is sufficient for our purposes.

Let $\Psi^G(t, f)$ be the local means according to (3.15) where we now always assume that k satisfies at least (3.13) if $f \in B^s_{pq}(\mathbb{R}^n)$ or $f \in F^s_{pq}(\mathbb{R}^n)$. Let for $j \in \mathbb{N}$ and $G \in \{F, M\}^{n*}$,

$$\Psi^{G+}_j f(x) = \sup_{|y| \le \sqrt{n}\, 2^{-j+1}} \left| \Psi^G(2^{-j+1}, f)(x - y) \right| \tag{3.19}$$

be related maximal functions, obviously modified if $j = 0$ and $G \in G^0$. Recall that summations over j, G, m must always be understood as above, for example in (3.6).

Proposition 3.3.

(i) *Let* $0 < p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$ *and*

$$\mathbb{N} \ni k > k(s,p) = \max\left(s, \frac{2n}{p} + \frac{n}{2} - s\right). \tag{3.20}$$

Let $\Psi_j^{G+} f$ *be the maximal functions according to* (3.19) *based on the local means* (3.15) *and the* k*-wavelets as introduced in Section* 3.1.1. *Then*

$$\|f\,|B^s_{pq}(\mathbb{R}^n)\| \sim \left(\sum_{j=0}^{\infty} 2^{jsq} \sum_{G\in G^j} \big\|\Psi_j^{G+} f\,|L_p(\mathbb{R}^n)\big\|^q\right)^{1/q} \tag{3.21}$$

(*with the usual modification if* $q=\infty$) *are equivalent quasi-norms in* $B^s_{pq}(\mathbb{R}^n)$.

(ii) *Let* $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$ *and*

$$\mathbb{N} \ni k > k(s,p,q) = \max\left(s, \frac{2n}{\min(p,q)} + \frac{n}{2} - s\right). \tag{3.22}$$

Let $\Psi_j^{G+} f$ *be as above, now based on* k*-wavelets with* (3.22). *Then*

$$\|f\,|F^s_{pq}(\mathbb{R}^n)\| \sim \left\| \left(\sum_{j,G,m} 2^{jsq} \Psi_j^{G+} f(\cdot)^q\right)^{1/q} |L_p(\mathbb{R}^n)\right\| \tag{3.23}$$

(*with the usual modification if* $q=\infty$) *are equivalent quasi-norms in* $F^s_{pq}(\mathbb{R}^n)$.

Proof. Step 1. We prove (ii). We rely on [Triγ], Corollary 2, pp. 108/109. First we check that the notation used there and now results in the same expressions. Let

$$\varphi_0 = (\Psi^G)^\vee \text{ if } G \in G^0, \qquad \varphi = (\Psi^G)^\vee \text{ if } G \in \{F,M\}^{n*}, \tag{3.24}$$

and $\varphi_j(x) = \varphi(2^{-j+1}x)$, $j \in \mathbb{N}$. In the notation of [Triγ], p. 100, formula (4), we have for $j \in \mathbb{N}$,

$$\begin{aligned}
(\varphi_j(D)f)(x) &= \left(\varphi(2^{-j+1}\cdot)\widehat{f}\right)^\vee(x) \\
&= c\,2^{jn} \int_{\mathbb{R}^n} \varphi^\vee(2^{j-1}\xi)\, f(x-\xi)\,\mathrm{d}\xi \\
&= c\,2^{jn} \int_{\mathbb{R}^n} \Psi^G(2^{j-1}\xi)\, f(x+\xi)\,\mathrm{d}\xi \\
&= c'\,\Psi^G(2^{-j+1}, f)(x).
\end{aligned} \tag{3.25}$$

Similarly for $j = 0$. Hence the maximal functions in (3.19) are modifications of formula (55) in [Triγ], p. 109, and we can apply the indicated Corollary 2 if the conditions (50), (51) (52) and (54) on p. 108 in [Triγ] are satisfied and one has a substitute for the Tauberian condition (6), (7) on p. 101. First we mention that the condition $s_1 > \sigma_p$ in (54) is simply not needed. This came out later on and it is covered by the literature in Remark 1.8, especially [Ry99a] and [Mou01b], Section 1.3. As for (50) one can rely on the product structure of the analytic function φ in (3.24), based on (3.3) with $m = 0$ and $G \in \{F, M\}^{n*}$. If, for example, $G_1 = M$, then $\varphi(x) = x_1^k \widetilde{\varphi}(x)$ where $\widetilde{\varphi}$ is again analytic. Furthermore one can check by the corresponding proofs in [Triγ] that one can replace $|x|^{s_1}$ in (50) by x_1^k. This shows that (50) is satisfied. But instead of this explicit argument one can use the proof in [Mou01b], Theorem 1.10, saying that (50) is not necessary. We shift the proof of (51), (52) on p. 108 in [Triγ] with

$$s_0 + \frac{n}{\min(p,q)} < s \quad \text{and} \quad k > k(s,p,q) \tag{3.26}$$

where s_0 has the same meaning as there, to the next step. By (3.11) we have the Tauberian condition (6) on p. 101 in [Triγ] at least if $|x| \le 2\varepsilon$, for some $\varepsilon > 0$, preferring now the version given in (1.33). As for the counterpart of (7) on p. 101 in [Triγ], or, better, (1.34) with the same $\varepsilon > 0$ as above we rely on the modification given in [Triγ], Section 2.4.4, Proposition 2, p. 120, saying that it is sufficient to prove that

$$\sum_{G \in \{F,M\}^{n*}} |(\Psi^G)^\vee(\xi)| > 0 \quad \text{if} \quad \varepsilon/2 < |\xi| < 2\varepsilon. \tag{3.27}$$

But this is the case even if one restricts the sum in (3.27) to those $G \in \{F, M\}^{n*}$ where only one of the components of G is an M. Then $(\Psi^G)^\vee(\xi)$ is the product of (one-dimensional) analytic functions of type $\widehat{\psi_F}$ and $\widehat{\psi_M}$ according to Theorem 1.61.

Step 2. To complete the proof of part (ii) it remains to check (51), (52) on p. 108 in [Triγ] if s_0 and k are restricted by (3.26), where s_0 has the same meaning as there. Let, as there, $H \in S(\mathbb{R}^n)$ with

$$H(x) = 1 \text{ if } \varepsilon/2 \le |x| \le 2\varepsilon \quad \text{and} \quad H(x) = 0 \text{ if } |x| \le \varepsilon/4 \text{ or } |x| \ge 4\varepsilon \tag{3.28}$$

for some $\varepsilon > 0$, now adapted to (1.34). Let $g = \varphi_0$ or $g = \varphi$ according to (3.24). With $g_\beta = D^\beta g$ it follows that

$$\widehat{g_\beta} \in C^k(\mathbb{R}^n) \quad \text{has compact support.} \tag{3.29}$$

In particular, it follows from

$$x^\alpha\, g_\beta(x) = c \int_{\mathbb{R}^n} e^{ix\xi}\, D^\alpha \widehat{g_\beta}(\xi)\, \mathrm{d}\xi, \quad |\alpha| \le k, \tag{3.30}$$

that

$$|(D^\beta g)(x)| \le c\, 2^{-kl} \quad \text{if} \quad |x| \sim 2^l \varepsilon,\ l \in \mathbb{N},\ |\beta| \le \sigma + 1, \tag{3.31}$$

with $n/2 + n/\min(p,q) < \sigma \in \mathbb{R}$. Let $H^\sigma(\mathbb{R}^n) = H^\sigma_2(\mathbb{R}^n)$ be the usual Sobolev spaces according to (1.7)–(1.9) with $p = 2$. With s_0 as in (3.26) and σ as above one gets by (3.31) that

$$2^{-ls_0} \, \|g(2^l \cdot) \, H(\cdot) \, | H^\sigma(\mathbb{R}^n)\| \le c 2^{-l(s_0 - \sigma + k)}, \quad l \in \mathbb{N}, \tag{3.32}$$

where c is independent of l. This is clear if $\sigma \in \mathbb{N}$ and follows otherwise by interpolation. If $k > \sigma - s_0$ then the left-hand side of (3.32) is uniformly bounded with respect to $l \in \mathbb{N}_0$. This is the case if k is chosen according to (3.26) with σ as above. But this coincides with (51), (52) on p. 108 in [Triγ] and completes the proof of part (ii).

Step 3. The proof for the B-spaces is the same, now based on the two corollaries on p. 134 in [Triγ]. This means that one can replace $n/\min(p,q)$ by n/p in (3.22) which results in (3.20). This proves part (i). □

Remark 3.4. The above proof and the crude estimates for k in (3.20), (3.22) are not satisfactory. It would be desirable to extend the elegant Theorems 1.7, 1.10 and also Corollary 1.12 from C^∞ kernels to C^k kernels with natural restrictions for k, maybe as in (3.13), (3.14), and to incorporate the very peculiar modification in the above proposition. Since nothing of this is available we relied on the heavy machinery of [Triγ], Theorem 2.4.1, pp. 100/101, and its Corollaries on pp. 108/109, which apply to much more general situations neglecting all peculiarities of the above cases.*

3.1.3 The main assertion

First we clarify the convergence of

$$\sum_{j,G,m} \lambda^{j,G}_m \, 2^{-jn/2} \, \Psi^{j,G}_m, \qquad \lambda \in b^s_{pq}, \tag{3.33}$$

where $\Psi^{j,G}_m$ are k-wavelets according to (3.5) and b^s_{pq} is given by (3.7), (3.9). The index set $\{j, G, m\}$ is the same as there. Let $0 < p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$ and $k \in \mathbb{N}$ as in (3.13). Then it follows by the previous discussion in (3.12), (3.13) that (3.33) is an atomic decomposition according to Theorem 1.19, where we shifted now the normalising factors $2^{-j(s-n/p)}$ from the atoms to the coefficients. In particular it follows from Theorem 1.19 and the explanations given in the Remarks 1.20 and 1.65 that (3.33) *converges unconditionally in* $S'(\mathbb{R}^n)$. If $p < \infty$, $q < \infty$ then one gets also by Theorem 1.19 that (3.33) converges unconditionally in $B^s_{pq}(\mathbb{R}^n)$, whereas one has in the general case $0 < p \le \infty$, $0 < q \le \infty$, that (3.33) converges unconditionally at least in $B^\sigma_{p,q}(K)$ for any ball K in $\mathbb{R}^n$ and any $\sigma < s$ (called *local convergence in* $B^\sigma_{pq}(\mathbb{R}^n)$). Here we used the

*__Added in proof:__ There are now assertions of the desired type which will be published later on.

quite obvious notation that a converging series in a quasi-Banach space is said to *converge unconditionally* if any rearrangement converges too, and the outcome is the same. This justifies the abbreviation

$$\sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} = \sum_{j,G,m} . \tag{3.34}$$

In other words, the unconditional convergence of the series in the theorem below is ensured by the assumptions about the coefficients and not an extra condition.

Let $\Psi_m^{j,G}$ be always the n-dimensional k-wavelets according to (3.5) and let b_{pq}^s and f_{pq}^s be the sequence spaces as introduced in Definition 3.1. In Definition 1.56 we said what is meant by an unconditional (Schauder) basis in a quasi-Banach space.

Theorem 3.5.

(i) *Let $0 < p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$ and let $\Psi_m^{j,G}$ be the above k-wavelets with*

$$\mathbb{N} \ni k > \max\left(s, \frac{2n}{p} + \frac{n}{2} - s\right). \tag{3.35}$$

Let $f \in S'(\mathbb{R}^n)$. Then $f \in B_{pq}^s(\mathbb{R}^n)$ if, and only if, it can be represented as

$$f = \sum_{j,G,m} \lambda_m^{j,G}\, 2^{-jn/2}\, \Psi_m^{j,G}, \qquad \lambda \in b_{pq}^s, \tag{3.36}$$

unconditional convergence being in $S'(\mathbb{R}^n)$ and locally in any $B_{pq}^\sigma(\mathbb{R}^n)$ with $\sigma < s$. The representation (3.36) is unique,

$$\lambda_m^{j,G} = 2^{jn/2}\, \left(f, \Psi_m^{j,G}\right) \tag{3.37}$$

and

$$I \,:\, f \mapsto \left\{2^{jn/2}\left(f, \Psi_m^{j,G}\right)\right\} \tag{3.38}$$

is an isomorphic map of $B_{pq}^s(\mathbb{R}^n)$ **onto** *b_{pq}^s. If, in addition, $p < \infty$, $q < \infty$ then $\{\Psi_m^{j,G}\}$ is an unconditional basis in $B_{pq}^s(\mathbb{R}^n)$.*

(ii) *Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$ and let $\Psi_m^{j,G}$ be the above k-wavelets with*

$$\mathbb{N} \ni k > \max\left(s, \frac{2n}{\min(p,q)} + \frac{n}{2} - s\right). \tag{3.39}$$

Let $f \in S'(\mathbb{R}^n)$. Then $f \in F_{pq}^s(\mathbb{R}^n)$ if, and only if, it can be represented as

$$f = \sum_{j,G,m} \lambda_m^{j,G}\, 2^{-jn/2}\, \Psi_m^{j,G}, \qquad \lambda \in f_{pq}^s, \tag{3.40}$$

unconditional convergence being in $S'(\mathbb{R}^n)$ and locally in any $F_{pq}^\sigma(\mathbb{R}^n)$ with $\sigma < s$. The representation (3.40) is unique with (3.37) and I in (3.38) is an isomorphic map of $F_{pq}^s(\mathbb{R}^n)$ **onto** *f_{pq}^s. If, in addition, $q < \infty$, then $\{\Psi_m^{j,G}\}$ is an unconditional basis in $F_{pq}^s(\mathbb{R}^n)$.*

Proof. Step 1. By the above explanations, (3.36) is an atomic representation. This clarifies also all assertions about convergence. We have

$$\|f\,|B^s_{pq}(\mathbb{R}^n)\| \le c\,\|\lambda\,|b^s_{pq}\|, \tag{3.41}$$

where c is independent of λ. Similarly in the F-case.

Step 2. Let $f \in B^s_{pq}(\mathbb{R}^n)$ and let $\lambda^{j,G}_m(f)$ be the coefficients according to (3.37) indicating now f. Since the functions $\Psi^{j,G}_m$ are real it follows by (3.16) that

$$\lambda^{j,G}_m(f) = c\,\Psi^G\left(2^{-j+1}, f\right)(2^{-j+1}m), \qquad j \in \mathbb{N}, \tag{3.42}$$

and a corresponding assertion if $j = 0$. (Here $c = 2^{n/2}$). Then one obtains by (3.19) and Proposition 3.3 that

$$\|\lambda(f)\,|b^s_{pq}\| \le c\,\|f\,|B^s_{pq}(\mathbb{R}^n)\| < \infty, \tag{3.43}$$

where c is independent of f. Now it follows by Step 1 that

$$g = \sum_{j,G,m} \lambda^{j,G}_m(f)\,2^{-jn/2}\,\Psi^{j,G}_m \in B^s_{pq}(\mathbb{R}^n). \tag{3.44}$$

We use the duality (3.17). Since $k > \sigma_p - s$ the dual pairing of g and any k-wavelet $\Psi^{j',G'}_{m'}$ makes sense. An elementary argument in connection with this dual pairing may be found in Section 5.1.7. Since (3.5) is an orthonormal basis in $L_2(\mathbb{R}^n)$ one gets

$$\begin{aligned}\left(g, \Psi^{j',G'}_{m'}\right) &= \sum_{j,G,m} \lambda^{j,G}_m(f)\,2^{-jn/2}\left(\Psi^{j,G}_m, \Psi^{j',G'}_{m'}\right)\\ &= \left(f, \Psi^{j',G'}_{m'}\right).\end{aligned} \tag{3.45}$$

This can be extended to finite linear combinations of $\Psi^{j',G'}_{m'}$. If $\varphi \in S(\mathbb{R}^n)$ then one has the unique $L_2(\mathbb{R}^n)$-representation (3.36), (3.37). By the above considerations it follows that this representation converges also in the dual space of $B^s_{pq}(\mathbb{R}^n)$ according to (3.17). Applied to (3.45) one gets

$$(g, \varphi) = (f, \varphi) \qquad \text{for all} \quad \varphi \in S(\mathbb{R}^n) \tag{3.46}$$

and hence $g = f$. Similarly for the F-spaces.

Step 3. Hence $f \in S'(\mathbb{R}^n)$ belongs to $B^s_{pq}(\mathbb{R}^n)$ if, and only if, it can be represented by (3.36). This representation is unique with the coefficients (3.37). By (3.41), (3.44) with $g = f$, and (3.43) it follows that

$$\|f\,|B^s_{pq}(\mathbb{R}^n)\| \sim \|\lambda(f)\,|b^s_{pq}\|, \tag{3.47}$$

where $\lambda(f)$ are the coefficients according to (3.42). Hence I in (3.38) is an isomorphic map from $B^s_{pq}(\mathbb{R}^n)$ into b^s_{pq}. It remains to prove that this map is onto. Let $\lambda \in b^s_{pq}$. Then it follows by the above considerations that

$$f = \sum_{j,G,m} \lambda^{j,G}_m\,2^{-jn/2}\,\Psi^{j,G}_m \in B^s_{pq}(\mathbb{R}^n). \tag{3.48}$$

But this representation is unique and $\lambda_m^{j,G} = \lambda_m^{j,G}(f)$ according to (3.42). This proves that I is a map onto. Furthermore if $p < \infty$, $q < \infty$ then $\{\Psi_m^{j,G}\}$ is an unconditional basis. Similarly for the F-spaces. □

Remark 3.6. We followed essentially [Tri04a]. Accepting Proposition 3.3 as a comment on the corresponding assertions in [Triγ], which has little to do with the specific situation of Theorem 3.5, the proof is surprisingly simple. But the use of maximal functions spoils the constants. In particular, the restrictions for k in (3.35) and (3.39) are not optimal. One may ask for conditions of type (3.13) and (3.14)*. As we indicated at the end of Remark 1.65 a more direct way to assertions of the above type may be found in [Kyr03] (even for more general orthogonal and bi-orthogonal L_2-systems to begin with). Then one gets also more natural restrictions for k. On the other hand, to have the same k in Theorem 1.61(ii) both for smoothness and vanishing moments is not typical for compactly supported wavelets. Just on the contrary. The interplay between the size of the supports of ψ_F, ψ_M, the number of vanishing moments and the (resulting) smoothness is rather delicate. We refer to [Mal99], pp. 244, 251, 253, 254, and [Dau92].

Remark 3.7. In Section 1.7 we described the general background for wavelets in function spaces and gave in Remarks 1.63, 1.65 and 1.66 some specific references which will not be repeated here. Wavelets in function spaces is nowadays a fashionable subject and there is a huge literature. We return later on in Chapters 5 and 6 to anisotropic function spaces and weighted function spaces in $\mathbb{R}^n$ of the above type including some wavelet representations. Then we give in Remarks 5.25 and 6.17 corresponding specific references. Section 4.2 deals with the tricky problem of wavelets in bounded Lipschitz domains. At this moment we restrict ourselves to a few complements. Real and complex interpolation may extend the property of being a wavelet basis in two spaces to the corresponding interpolation spaces. Starting from $L_p(\mathbb{R}^n)$, $1 < p < \infty$, and, say, from the above k-wavelets with $k \in \mathbb{N}$ one gets by interpolation bases in some Lorentz spaces $L_{p,u}$ and Zygmund spaces $L_p(\log L)_a$ (both briefly mentioned after (1.210), (1.211)) and in more general rearrangement invariant spaces. We refer to [Soa97]. In Section 1.9.5 we described some Besov spaces $B^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ and $B^{\boldsymbol{\sigma}}_{pq}(\mathbb{R}^n)$ with $\boldsymbol{\sigma} = (\sigma_j)$ of generalised smoothness. There one finds also the necessary references. Some of these spaces can be obtained by real interpolation with functional parameters from the above Besov spaces $B^s_{pq}(\mathbb{R}^n)$. Based on this observation, part (i) of Theorem 3.5 has been extended in [Alm05] to some Besov spaces of generalised smoothness. A counterpart of Theorem 3.5 for corresponding spaces with dominating smoothness has been proved recently in [Vyb05a], [Vyb06].

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*__Added in proof:__ Based on an adapted version of Proposition 3.3 one can replace (3.35) by (3.13) and (3.39) by (3.14).

3.1.4 Two applications

Again we follow [Tri04a] and describe two simple applications of Theorem 3.5. If A and B are two quasi-Banach spaces then $A \cong B$ means that A is isomorphic to B, hence there is a linear isomorphic map of A onto B. Let $0 < p \le \infty$. Then ℓ_p is the usual quasi-Banach space of all sequences $\lambda = (\lambda_j)_{j=0}^\infty$, $\lambda_j \in \mathbb{C}$, such that

$$\|\lambda \,|\ell_p\| = \left(\sum_{j=0}^\infty |\lambda_j|^p \right)^{1/p} < \infty \tag{3.49}$$

(with the obvious modification if $p = \infty$).

Corollary 3.8. *Let* $0 < p \le \infty$ *and* $s \in \mathbb{R}$. *Then*

$$B^s_{pp}(\mathbb{R}^n) \cong \ell_p. \tag{3.50}$$

Proof. By Definition 3.1 one has $b^s_{pp} \cong \ell_p$. Now the corollary follows immediately from Theorem 3.5(i). □

Remark 3.9. Assertions of this type have some history. One gets as a special case

$$\mathcal{C}^s(\mathbb{R}^n) = B^s_{\infty\infty}(\mathbb{R}^n) \cong \ell_\infty \tag{3.51}$$

for the Hölder-Zygmund spaces according to (1.10). We refer to [Cie60] for a corresponding assertion on intervals. The isomorphism (3.50) with $1 < p < \infty$ goes back to [Tri73], [Triα], Section 2.11.1, and [Pee76], pp. 180/190. The corresponding assertion for $0 < p \le 1$ seems to be new, at least we could not find a reference. By Theorem 3.5 the isomorphic structure of the spaces $B^s_{pq}(\mathbb{R}^n)$ is independent of s (quite obvious), and of n. The situation for the spaces $F^s_{pq}(\mathbb{R}^n)$ seems to be more complicated. Especially for the Hardy spaces $h_1(\mathbb{R}^n) = F^0_{1,2}(\mathbb{R}^n)$ and related H_1-spaces this question attracted some attention, [Mau80], [Bog82], [Bog83], with the outcome that the spaces $h_1(\mathbb{R}^n)$ are not isomorphic to each other for different values of $n \in \mathbb{N}$.

For elements f with (1.618) we introduced in (1.620) the Besov characteristics $s_f(t)$ where $0 \le t = 1/p < \infty$. According to Theorem 1.199 it is an increasing (which means non-decreasing) concave function in the (t, s)-diagram in Figure 1.17.1 of slope smaller than or equal to n and for any such function $s(t)$ one finds an element f with (1.618) such that $s_f(t) = s(t)$. We formulated these assertions in Section 1.18 and return to this subject in detail in Chapter 7. Our proof will be based on (fractal) compactly supported Radon measures. But one may ask whether one can employ wavelet expansions of distributions and, in particular, Theorem 3.5 for this purpose. Our way is different, but we hinted in this context in Remark 1.200 to [Jaf00] and [Jaf01]. We restrict ourselves here to a simple example how to use Theorem 3.5 for this purpose. If f is given by (1.618) then it follows for $s_f(t)$ in (1.620) by elementary embedding that

$$s_f(t) = \sup\big\{s:\ f \in B^s_{pq}(\mathbb{R}^n)\big\} = \sup\big\{s:\ f \in F^s_{pq}(\mathbb{R}^n)\big\} \tag{3.52}$$

where again $0 \le t = 1/p < \infty$ ($p < \infty$ in the F-case) and $0 < q \le \infty$. What happens directly on such curves $s = s(t)$? Here is a simple example in the case when $s(t)$ is constant.

Corollary 3.10. *Let $0 < p \le \infty$ ($p < \infty$ in the F-case), $s \in \mathbb{R}$ and*

$$f_s = \sum_{j,G,m} \lambda^{j,G}_m \, 2^{-jn/2} \, \Psi^{j,G}_m \tag{3.53}$$

according to (3.36) where $\Psi^{j,G}_m$ are k-wavelets with $k > \max(s, \frac{n}{2} - s)$ and

$$\lambda^{j,G}_m = 2^{-js} \text{ if } |m| \le 2^j \quad \text{and} \quad \lambda^{j,G}_m = 0 \text{ otherwise.} \tag{3.54}$$

Let $0 < q \le \infty$. Then

$$f_s \in \begin{cases} B^s_{pq}(\mathbb{R}^n) & \text{if, and only if, } q = \infty, \\ F^s_{pq}(\mathbb{R}^n) & \text{if, and only if, } q = \infty. \end{cases} \tag{3.55}$$

Proof. Step 1. We can apply Theorem 3.5 if p, and in case of the F-spaces also q, are sufficiently large. For $j \in \mathbb{N}_0$ (and G) we have that

$$2^{j(s-n/p)} \bigg(\sum_m |\lambda^{j,G}_m|^p \bigg)^{1/p} \sim 1 \tag{3.56}$$

where the equivalence constants are independent of j. Hence for large p it follows by Theorem 3.5 and Definition 3.1 that

$$f_s \in B^s_{pq}(\mathbb{R}^n) \quad \text{if, and only if,} \quad q = \infty. \tag{3.57}$$

Recall that for all $s \in \mathbb{R}$, $0 < p < \infty$, and $0 < q \le \infty$,

$$F^s_{pq}(\mathbb{R}^n) \hookrightarrow B^s_{p,\max(p,q)}(\mathbb{R}^n). \tag{3.58}$$

Then one obtains for large $p < \infty$ and all $0 < q \le \infty$ that

$$f_s \in F^s_{pq}(\mathbb{R}^n) \quad \text{if, and only if,} \quad q = \infty, \tag{3.59}$$

using (3.8) with $q = \infty$ and (3.57) with $0 < q < \infty$. This proves the corollary if p is large.

Step 2. If $g \in F^s_{p_0 q}(\mathbb{R}^n)$ has compact support then $g \in F^s_{p_1 q}(\mathbb{R}^n)$ for any p_1 with $0 < p_1 \le p_0$. This is well known and essentially a consequence of characterisations in terms of local means according to Theorem 1.10. But it follows also immediately from Theorem 3.5 and (3.8) (the integration in connection with $L_p(\mathbb{R}^n)$ is restricted to a bounded set). Applied to f_s one obtains by (3.59), (3.58) that

$$f_s \in F^s_{p\infty}(\mathbb{R}^n) \hookrightarrow B^s_{p\infty}(\mathbb{R}^n) \tag{3.60}$$

for all $0 < p < \infty$, hence the if-part of the corollary. By (3.58) it is sufficient to prove the only-if-part for the B-spaces. We assume that

$$f_s \in B^s_{pq}(\mathbb{R}^n) \quad \text{for some } 0 < p < \infty,\ 0 < q < \infty. \tag{3.61}$$

By $f_s \in \mathcal{C}^s(\mathbb{R}^n) = B^s_{\infty\infty}(\mathbb{R}^n)$ and Hölder's inequality applied to (2.12) it follows that

$$f_s \in B^s_{p_\theta q_\theta}(\mathbb{R}^n) \quad \text{for } 0 < \theta < 1 \text{ and } p_\theta = p/\theta,\ q_\theta = q/\theta. \tag{3.62}$$

But for small θ this contradicts Step 1. □

Remark 3.11. According to (3.52) one has $s_{f_s}(t) = s$ for all $0 \le t < \infty$. The corollary is a refinement of this assertion.

3.1.5 Further wavelet isomorphisms

The arguments in the proofs of Proposition 3.3 and Theorem 3.5 are qualitative. For the system $\{\Psi^{j,G}_m\}$ in (3.5) of wavelets one needs essentially three ingredients:

- $\{\Psi^{j,G}_m\}$ is an (orthogonal) basis in $L_2(\mathbb{R}^n)$,
- $\Psi^{j,G}_m$ are related to atoms in the spaces considered,
- $\Psi^{j,G}_m$ may serve as kernels of corresponding local means.

Nowadays there are many orthogonal and bi-orthogonal systems of wavelets which fit in this scheme. We refer to [Woj97] and [Mal99], Chapter VII. We preferred so far the Daubechies wavelets. They have compact supports but only a limited smoothness. The latter effect cannot be avoided, [Mal99], Section 7.2.1, p. 244. The n-dimensional version of the Meyer wavelets according to Theorem 1.61(i) is much better adapted to the original Fourier analytical definition of $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$, and the arguments to arrive at the corresponding counterpart of Theorem 3.5 are simpler in many respects. We formulate the outcome and indicate its proof.

But first we complement Theorem 1.61(i). We choose for the real scaling function $\psi_F \in S(\mathbb{R}^n)$ and the associated real wavelet $\psi_M \in S(\mathbb{R})$ the explicit example in [Woj97], Exercise 3.2, p. 71, Proposition 3.2, p. 49, and formula (3.15), p. 50. Then one gets

$$|\widehat{\psi_F}(\xi)| > 0 \quad \text{if, and only if,} \quad |\xi| < 4\pi/3, \tag{3.63}$$

and

$$|\widehat{\psi_M}(\xi)| > 0 \quad \text{if, and only if,} \quad 2\pi/3 < |\xi| < 8\pi/3. \tag{3.64}$$

This fits perfectly in the scheme of Theorem 1.7 with $\varphi_0(\xi) = \widehat{\psi_F}(\xi)$ and $\varphi^0(\xi) = \widehat{\psi_M}(2\xi)$ where one may choose $\varepsilon = 2\pi/3$ in (1.33)–(1.35) (and $<$ in place of $\le$

which is immaterial). The extra factor 2 comes from the index-shifting $j \in \mathbb{N}$ in Theorem 1.7 compared with $j-1$ in (3.1). In particular the one-dimensional Tauberian condition (3.27) is satisfied.

We use the same notation as in connection with Theorem 3.5. In particular with $n \in \mathbb{N}$ the sequence spaces b^s_{pq} and f^s_{pq} have the same meaning as in Definition 3.1. In Definition 1.56 we said what is meant by an unconditional (Schauder) basis in a quasi-Banach space.

Theorem 3.12. *Let $n \in \mathbb{N}$ and let $\Psi^{j,G}_m$ be the n-dimensional Meyer wavelets according to* (3.1)–(3.5)*, where $\psi_F \in S(\mathbb{R})$ is the real scaling function and $\psi_M \in S(\mathbb{R})$ the real associated wavelet according to Theorem* 1.61(i) *and* (3.63), (3.64).

- (i) *Let $0 < p \le \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. Let $f \in S'(\mathbb{R}^n)$. Then $f \in B^s_{pq}(\mathbb{R}^n)$ if, and only if, it can be represented by* (3.36)*, now with the above n-dimensional Meyer wavelets, unconditional convergence being in $S'(\mathbb{R}^n)$ and locally in any $B^\sigma_{pq}(\mathbb{R}^n)$ with $\sigma < s$. This representation is unique where $\lambda^{j,G}_m$ is given by* (3.37)*. Furthermore, I in* (3.38) *is an isomorphic map of $B^s_{pq}(\mathbb{R}^n)$* **onto** *b^s_{pq}. If, in addition, $p < \infty$, $q < \infty$, then $\{\Psi^{j,G}_m\}$ is an unconditional basis in $B^s_{pq}(\mathbb{R}^n)$.*
- (ii) *Let $0 < p < \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. Let $f \in S'(\mathbb{R}^n)$. Then $f \in F^s_{pq}(\mathbb{R}^n)$ if, and only if, it can be represented by* (3.40) *now with the above n-dimensional Meyer wavelets $\Psi^{j,G}_m$, unconditional convergence being in $S'(\mathbb{R}^n)$ and locally in any $F^\sigma_{pq}(\mathbb{R}^n)$ with $\sigma < s$. This representation is unique where $\lambda^{j,G}_m$ is given by* (3.37)*. Furthermore, I in* (3.38) *is an isomorphic map of $F^s_{pq}(\mathbb{R}^n)$* **onto** *f^s_{pq}. If, in addition, $q < \infty$, then $\{\Psi^{j,G}_m\}$ is an unconditional basis in $F^s_{pq}(\mathbb{R}^n)$.*

Proof. First we remark that there is an immediate counterpart of Proposition 3.3 where everything is now simpler because one does not need the extra considerations caused by the limited smoothness of the Daubechies wavelets. Also the counterpart of the n-dimensional Tauberian condition (3.27) follows from the arguments given there and the one-dimensional case according to (3.63), (3.64). As for the counterpart of the first step of the proof of Theorem 3.5 we remark that the above n-dimensional Meyer wavelets $\Psi^{j,G}_m$ are molecules in all spaces considered. Recall that molecules have all the properties of atoms as used in Theorem 1.19 with the exception of the support conditions (1.56), (1.58) which are generalised by sufficiently strong decay conditions with respect to the distance to the cubes $Q_{\nu m}$. We refer to [FJW91], Section 5, especially p. 48. However in the above case it is not difficult to reduce the needed molecular decompositions to corresponding atomic decompositions. One may find some details in a slightly different context in Section 3.2.3 and (3.133) below. The other arguments in the Steps 2 and 3 of the proof of Theorem 3.5 are the same. □

3.2 Wavelet frames

3.2.1 Preliminaries and definitions

We described in Theorem 1.39 and Corollary 1.42 quarkonial frames based on approximate lattices. This high flexibility might be considered as the main advantage of this approach. It paves the way to a substantial theory of function spaces on rough structures such as (compact) fractal sets in $\mathbb{R}^n$ and abstract quasi-metric spaces. The theories described in Sections 1.12–1.18 depend at least partly on these possibilities. We return to problems of this type at the end of this book in the Chapters 7–9. If $\mathbb{R}^n$ is the underlying structure then the pure lattices $2^{-j}\mathbb{Z}^n$ are the first choice and the corresponding β-quarks simplify to (1.107). We return now to this subject but in a modified way compared with Section 1.8. We repeat, modify and formalise some definitions given there and it will be our first aim to prove Theorems 1.69 and 1.71. We follow essentially [Tri03a].

First we complement the notation as introduced in Sections 2.1.2 and 2.1.3. Let

$$\mathbb{R}^n_{++} = \{y \in \mathbb{R}^n : \ y = (y_1, \ldots, y_n), \ y_j > 0\} \tag{3.65}$$

and let k be a non-negative C^∞ function in $\mathbb{R}^n$ with

$$\operatorname{supp} k \subset \{y \in \mathbb{R}^n : \ |y| < 2^J\} \cap \mathbb{R}^n_{++} \tag{3.66}$$

for some $J \in \mathbb{N}$, such that

$$\sum_{m \in \mathbb{Z}^n} k(x - m) = 1 \qquad \text{where} \quad x \in \mathbb{R}^n. \tag{3.67}$$

One may fix $J = n$ (which is luxurious). Then

$$k^\beta(x) = (2^{-J}x)^\beta \, k(x) \geq 0 \quad \text{if } x \in \mathbb{R}^n \text{ and } \beta \in \mathbb{N}^n_0. \tag{3.68}$$

We wish to compare what follows with the wavelet bases and wavelet isomorphisms according to Section 3.1: gain and loss. We restrict ourselves to the spaces $B^s_{pq}(\mathbb{R}^n)$ with $p = q$, again abbreviated by

$$B^s_p(\mathbb{R}^n) = B^s_{pp}(\mathbb{R}^n) \quad \text{where } 0 < p \leq \infty, \, s \in \mathbb{R}, \tag{3.69}$$

as in (2.20) and with the Hölder-Zygmund spaces

$$\mathcal{C}^s(\mathbb{R}^n) = B^s_\infty(\mathbb{R}^n), \qquad s \in \mathbb{R}, \tag{3.70}$$

as a special case.

Definition 3.13. *Let* $0 < p \leq \infty$, $s \in \mathbb{R}$, $\varrho \in \mathbb{R}$,

$$\lambda = \left\{ \lambda^\beta_{jm} \in \mathbb{C} : \ j \in \mathbb{N}_0, \ \beta \in \mathbb{N}^n_0, \ m \in \mathbb{Z}^n \right\} \tag{3.71}$$

and

$$\|\lambda \,|b^{s,\varrho}_p\| = \left(\sum_{\beta \in \mathbb{N}^n_0} \sum_{j=0}^\infty \sum_{m \in \mathbb{Z}^n} 2^{\varrho|\beta|p + j(s-n/p)p} \, |\lambda^\beta_{jm}|^p \right)^{1/p} \tag{3.72}$$

(*with the usual modification if* $p=\infty$). *Then*

$$b_p^{s,\varrho} = \big\{ \lambda : \ \| \lambda \, | b_p^{s,\varrho} \| < \infty \big\} . \tag{3.73}$$

Let $b_p^s = b_p^{s,0}$.

Remark 3.14. One may compare this definition with Definitions 2.11 and 3.1. Since $\varrho \geq 0$ in Theorem 3.21 below can be chosen arbitrarily large one has now an exponential decay with respect to $\beta \in \mathbb{N}_0^n$ instead of the finite sums over G in (3.7). Compared with Definition 2.11 we insert now the normalising factors in the sequence spaces in contrast to the normalised atoms (but of course this is only a matter of convenience). In the sequel we always use the abbreviation

$$\sum_{\beta,j,m} = \sum_{\beta \in \mathbb{N}_0^n} \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} . \tag{3.74}$$

Next we introduce the counterparts of the father wavelet (or scaling function) $\Psi^G = \Psi^G_m$ with $G \in G^0$ and $m=0$ in (3.4) and of the (mother) wavelets $\Psi^G = \Psi^G_m$ with $G \in \{F,M\}^{n*}$ and $m=0$ in (3.3). Instead of G we have now $\beta \in \mathbb{N}_0^n$. Again we call them (father and mother) wavelets.

Definition 3.15. *Let* ω *be a* C^∞ *function in* $\mathbb{R}^n$ *with*

$$\operatorname{supp} \omega \subset (-\pi,\pi)^n \quad \textit{and} \quad \omega(x) = 1 \ \textit{if} \ |x| \leq 2. \tag{3.75}$$

Let for the same $J \in \mathbb{N}$ *as in* (3.66),

$$\omega^\beta(x) = \frac{i^{|\beta|}\, 2^{J|\beta|}}{(2\pi)^n \beta!}\, x^\beta \omega(x), \quad x \in \mathbb{R}^n, \quad \beta \in \mathbb{N}_0^n, \tag{3.76}$$

and

$$\Omega^\beta(x) = \sum_{m \in \mathbb{Z}^n} (\omega^\beta)^\vee(m)\, e^{-imx}, \qquad x \in \mathbb{R}^n. \tag{3.77}$$

Let φ_0 *be a* C^∞ *function in* $\mathbb{R}^n$ *with*

$$\varphi_0(x) = 1 \ \textit{if} \ |x| \leq 1 \quad \textit{and} \quad \varphi_0(x) = 0 \ \textit{if} \ |x| \geq 3/2\,, \tag{3.78}$$

and let $\varphi(x) = \varphi_0(x) - \varphi_0(2x)$. *Then for* $\beta \in \mathbb{N}_0^n$ *the father wavelets* Φ_F^β *and the mother wavelets* Φ_M^β *are given by*

$$(\Phi_F^\beta)^\vee(\xi) = \varphi_0(\xi)\, \Omega^\beta(\xi), \qquad \xi \in \mathbb{R}^n, \tag{3.79}$$

$$(\Phi_M^\beta)^\vee(\xi) = \varphi(\xi)\, \Omega^\beta(\xi), \qquad \xi \in \mathbb{R}^n. \tag{3.80}$$

Remark 3.16. Obviously, $|\beta|$ and x^β have the same meaning as in (2.2) and (2.3). Expanding $\omega^\beta(x)$ in the cube $(-\pi,\pi)^n$ in its Fourier series one gets

$$\begin{aligned} \omega^\beta(x) &= (2\pi)^{-n} \sum_{m \in \mathbb{Z}^n} e^{-imx} \int_{\mathbb{R}^n} e^{im\xi}\, \omega^\beta(\xi)\, \mathrm{d}\xi = (2\pi)^{-n/2} \sum_{m \in \mathbb{Z}^n} (\omega^\beta)^\vee(m)\, e^{-imx} \\ &= (2\pi)^{-n/2}\, \Omega^\beta(x), \qquad x \in (-\pi,\pi)^n, \end{aligned} \tag{3.81}$$

where we used the normalisation (2.7) of the Fourier transform and its inverse. Hence Ω^β is a periodic C^∞ function in $\mathbb{R}^n$ which coincides in $(-\pi,\pi)^n$ with $(2\pi)^{n/2}\,\omega^\beta$. By the support properties of φ_0 and φ,

$$\operatorname{supp}\varphi \subset \{x\in\mathbb{R}^n \;:\; 1/2 \le |x| \le 3/2\}, \tag{3.82}$$

it follows that both $\Phi_F^\beta \in S(\mathbb{R}^n)$ and $\Phi_M^\beta \in S(\mathbb{R}^n)$ are entire analytic functions with

$$\Phi_F^\beta(x) = \sum_{m\in\mathbb{Z}^n} (\omega^\beta)^\vee(m)\,\widehat{\varphi_0}(x+m), \qquad x\in\mathbb{R}^n, \tag{3.83}$$

$$\Phi_M^\beta(x) = \sum_{m\in\mathbb{Z}^n} (\omega^\beta)^\vee(m)\,\widehat{\varphi}(x+m), \qquad x\in\mathbb{R}^n. \tag{3.84}$$

By (3.79)–(3.81) and the properties of φ_0, φ and ω^β it follows that

$$\int_{\mathbb{R}^n} \Phi_M^\beta(\xi)\,\xi^\alpha\,\mathrm{d}\xi = 0, \qquad \alpha\in\mathbb{N}_0^n, \tag{3.85}$$

$$\int_{\mathbb{R}^n} \Phi_F^\beta(\xi)\,\xi^\alpha\,\mathrm{d}\xi \ne 0 \Longleftrightarrow \alpha=\beta, \tag{3.86}$$

and, in particular,

$$\widehat{\Phi_F^0}(0) = \Omega^0(0) = (2\pi)^{n/2}\omega^0(0) = (2\pi)^{-n/2}. \tag{3.87}$$

These properties, including the normalisation (3.87) are similar as for the smooth wavelets of Meyer type or Daubechies type. We refer to Theorem 1.61 and (3.11), (3.12). Next we introduce the counterpart of the wavelets $\Psi_m^{j,G}$ in (3.5) where $G\in G^0$ and $G=\{F,M\}^{n*}$ is now replaced by $\beta\in\mathbb{N}_0^n$ with an (even more than) exponential decay with respect to β, originating from $\beta!$ in the denominator in (3.76).

Definition 3.17. *Let $\beta\in\mathbb{N}_0^n$ and $m\in\mathbb{Z}^n$.*

(i) *Let k^β be the functions according to* (3.66)–(3.68) *for some admitted $J\in\mathbb{N}$. Then*

$$k_{jm}^\beta(x) = k^\beta(2^jx-m) \quad \text{where } j\in\mathbb{N}_0 \text{ and } x\in\mathbb{R}^n. \tag{3.88}$$

(ii) *Let Φ_F^β and Φ_M^β be the functions as introduced in Definition* 3.15 *with the same $J\in\mathbb{N}$ as in part* (i). *Then*

$$\Phi_{jm}^\beta(x) = \begin{cases} \Phi_F^\beta(x-m) & \text{if } j=0,\\ \Phi_M^\beta(2^jx-m) & \text{if } j\in\mathbb{N}. \end{cases} \tag{3.89}$$

Remark 3.18. The structure of (3.89) is the same as in (3.5). Whereas the orthonormal system $\{\Psi_m^{j,G}\}$ is dual to itself it comes out that $\{k_{jm}^\beta\}$ and $\{2^{jn}\Phi_{jm}^\beta\}$ are dual to each other, where $J\in\mathbb{N}$ is assumed to be the same for both systems.

3.2.2 Subatomic decompositions

Again let σ_p be as in (2.6), where $0 < p \le \infty$. Let

$$B^+_p(\mathbb{R}^n) = \bigcup_{s>\sigma_p} B^s_p(\mathbb{R}^n), \qquad 0 < p \le \infty, \tag{3.90}$$

where we used the abbreviation (3.69). The following slight improvement of Proposition 1.45, especially of (1.123) with $p = q$ will be of some service for us.

Proposition 3.19. *Let* $\{\varphi_j\}_{j=0}^\infty$ *be the dyadic resolution of unity according to* (2.8)–(2.10)*. Let* $0 < p \le \infty$ *and* $s \in \mathbb{R}$*. Then*

$$\|f \,|B^s_p(\mathbb{R}^n)\| \sim \Big(\sum_{j=0}^\infty \sum_{m\in\mathbb{Z}^n} 2^{j(s-n/p)p}\, |(\varphi_j \widehat{f})^\vee (2^{-j}m)|^p \Big)^{1/p} \tag{3.91}$$

(*with the usual modification if* $p = \infty$) *where the equivalence constants are independent of* s *and* f*.*

Remark 3.20. In other words, (1.123) with $p = q$ remains valid if one strengthens $2^{-j-J}m$ in (1.122) by $2^{-j}m$. Otherwise we have the previous assertions with the references given in Remark 1.46. In particular, for given $f \in S'(\mathbb{R}^n)$ the right-hand side of (3.91) is finite if, and only if, $f \in B^s_p(\mathbb{R}^n)$. This follows from (2.12) and the equivalence assertion in [Triδ], (14.56), p. 102, where we referred in turn to [Triβ], pp. 19–22. There one finds also a discussion with the outcome that there are functions φ_0 with (2.8) (which coincides with (3.78)) and φ_j according to (2.9) such that one has (3.91) for all p with $0 < p \le \infty$ (again with a reference to the literature, especially [Tri77]), which is tacitly assumed in the sequel.

The next theorem deals with series

$$\sum_{\beta,j,m} \lambda^\beta_{jm}\, k^\beta_{jm}, \qquad \lambda \in b^{s,\varrho}_p, \tag{3.92}$$

for $\varrho \ge 0$, $0 < p \le \infty$ and $s > \sigma_p$, where k^β_{jm} and $b^{s,\varrho}_p$ have the same meaning as in the Definitions 3.17 and 3.13. We used the abbreviation (3.74) which is also justified by the following comment. By (3.66) the support of k is also contained in an open ball centred at the origin of radius $2^{J-\varepsilon}$ for some $\varepsilon > 0$. Then by (3.68), (3.88), Definition 2.5 and Theorem 1.19,

$$\Big\{ 2^{\varepsilon|\beta|}\, 2^{-j(s-n/p)}\, k^\beta_{jm} : \ j \in \mathbb{N}_0,\, m \in \mathbb{Z}^n \Big\}, \quad \beta \in \mathbb{N}^n_0, \tag{3.93}$$

are admitted systems of atoms in $B^s_p(\mathbb{R}^n)$ with $s > \sigma_p$, ignoring constants which are independent of j, m, β. By the same arguments as in Remark 2.12 it follows that

$$\sum_{j=0}^\infty \sum_{m\in\mathbb{Z}^n} \lambda^\beta_{jm}\, k^\beta_{jm} \tag{3.94}$$

converges absolutely in $L_{\bar{p}}(\mathbb{R}^n)$ with $\bar{p} = \max(p,1)$ (and the indicated modification if $p = \infty$) to, say, some $g^\beta \in L_{\bar{p}}(\mathbb{R}^n)$ and that

$$\|g^\beta \,|L_{\bar{p}}(\mathbb{R}^n)\| \le c\, 2^{-\varepsilon|\beta|}, \tag{3.95}$$

where $c > 0$ is independent of β. Then the series in (3.92) converges also absolutely in $L_{\bar{p}}(\mathbb{R}^n)$ (indicated modification if $p = \infty$), and hence unconditionally in $L_{\bar{p}}(\mathbb{R}^n)$ (indicated modification if $p = \infty$) and in $S'(\mathbb{R}^n)$. (As for the notion of unconditional convergence we refer to the beginning of Section 3.1.3 and the references given there.) In particular, (3.74) makes sense. We use the above notation and

$$\lambda^\beta_{jm}(f) = 2^{jn}\,(f, \Phi^\beta_{jm}), \qquad f \in S'(\mathbb{R}^n). \tag{3.96}$$

To avoid any misunderstanding we remark that in the following theorem k^β_{jm} and Φ^β_{jm} according to Definition 3.17 are fixed, based on fixed admitted k, J, φ_0 and ω.

Theorem 3.21. *Let $0 < p \le \infty$, $\bar{p} = \max(1,p)$, $s > \sigma_p$ and $\varrho \ge 0$.*

(i) *Then $f \in S'(\mathbb{R}^n)$ is an element of $B^s_p(\mathbb{R}^n)$ if, and only if, it can be represented as*

$$f = \sum_{\beta,j,m} \lambda^\beta_{jm}\, k^\beta_{jm}, \qquad \lambda \in b^{s,\varrho}_p, \tag{3.97}$$

absolute convergence being in $L_{\bar{p}}(\mathbb{R}^n)$ if $\bar{p} < \infty$ and in $L_\infty(\mathbb{R}^n, w)$ with $w(x) = (1+|x|^2)^{\sigma/2}$ where $\sigma < 0$ if $p = \infty$. Furthermore,

$$\|f \,|B^s_p(\mathbb{R}^n)\| \sim \inf \|\lambda \,|b^{s,\varrho}_p\|, \tag{3.98}$$

where the infimum is taken over all admissible representations (3.97)*.*

(ii) *Let $\lambda^\beta_{jm}(f)$ be given by* (3.96)*. Then any $f \in B^+_p(\mathbb{R}^n)$ can be represented as*

$$f = \sum_{\beta,j,m} \lambda^\beta_{jm}(f)\, k^\beta_{jm}, \tag{3.99}$$

absolute convergence being in $L_{\bar{p}}(\mathbb{R}^n)$ if $\bar{p} < \infty$ and in $L_\infty(\mathbb{R}^n, w)$ with $w(x) = (1+|x|^2)^{\sigma/2}$ where $\sigma < 0$ if $p = \infty$. Furthermore,

$$B^s_p(\mathbb{R}^n) = \{f \in B^+_p(\mathbb{R}^n) : \ \|\lambda(f)\,|b^{s,\varrho}_p\| < \infty\} \tag{3.100}$$

(*equivalent quasi-norms*)*.*

Proof. Step 1. Let f be given by (3.97) and let

$$f = \sum_\beta f^\beta \qquad \text{with} \qquad f^\beta = \sum_{j,m} \lambda^\beta_{jm}\, k^\beta_{jm}. \tag{3.101}$$

As remarked above, (3.97) converges absolutely in $L_{\bar{p}}(\mathbb{R}^n)$ (with the indicated modification if $p = \infty$) and (3.93) are atoms in $B^s_p(\mathbb{R}^n)$. Then it follows by Theorem 1.19 that

$$\|f^\beta \,|B^s_p(\mathbb{R}^n)\| \le c\, 2^{-\varepsilon|\beta|} \Big(\sum_{j,m} 2^{j(s-n/p)p}\, |\lambda^\beta_{jm}|^p \Big)^{1/p}, \tag{3.102}$$

where c is independent of ε, and consequently

$$\|f\,|B^s_p(\mathbb{R}^n)\| \le c\,\|\lambda\,|b^{s,\varrho}_p\| \tag{3.103}$$

for any $\varrho \ge 0$.

Step 2. We prove that any $f \in B^s_p(\mathbb{R}^n)$ can be represented by (3.99) with (3.96) and

$$\|\lambda(f)\,|b^{s,\varrho}_p\| \le c\,\|f\,|B^s_p(\mathbb{R}^n)\| < \infty \tag{3.104}$$

where c is independent of $f \in B^s_p(\mathbb{R}^n)$. Then one gets as a by-product the converse of (3.103) and hence part (i). We adapt the relevant arguments in [Triϵ], pp. 17–24, to our situation. Let $f \in B^s_p(\mathbb{R}^n)$ with $0 < p \le \infty$ and $s > \sigma_p$. Let Q_j with $j \in \mathbb{N}_0$ be cubes in $\mathbb{R}^n$ centred at the origin and with side-length $2\pi 2^j$. In particular $\operatorname{supp} \varphi_j \subset Q_j$ where φ_j are given by (2.8), (2.9). We have

$$\widehat{f}(\xi) = \sum_{j=0}^\infty \varphi_j(\xi)\,\widehat{f}(\xi), \qquad \xi \in \mathbb{R}^n. \tag{3.105}$$

We expand $\varphi_j \widehat{f}$ in Q_j in its trigonometric series and obtain that

$$(\varphi_j \widehat{f})(\xi) = \sum_{m\in\mathbb{Z}^n} b_{jm} \exp\big(-i\, 2^{-j}\, m\xi\big)\, \mathrm{d}\xi, \qquad \xi \in Q_j, \tag{3.106}$$

with

$$\begin{aligned} b_{jm} &= (2\pi)^{-n}\, 2^{-jn} \int_{Q_j} (\varphi_j \widehat{f})(\xi) \exp\big(i\, 2^{-j}\, m\xi\big)\, \mathrm{d}\xi \\ &= (2\pi)^{-n/2}\, 2^{-jn}\, (\varphi_j \widehat{f})^\vee(2^{-j}m). \end{aligned} \tag{3.107}$$

By Proposition 3.19 we have

$$\|f\,|B^s_p(\mathbb{R}^n)\| \sim \Big(\sum_{m\in\mathbb{Z}^n} \sum_{j=0}^\infty 2^{jsp}\, 2^{jn(1-1/p)p}\, |b_{jm}|^p \Big)^{1/p}, \quad 0 < p \le \infty, \tag{3.108}$$

(with the usual modification if $p = \infty$). By (3.75) it follows that $\omega_j(x) = \omega(2^{-j}x)$ has compact support in Q_j and $\omega_j(x) = 1$ if $x \in \operatorname{supp} \varphi_j$. Then it follows from (3.106) that

$$\begin{aligned} (\varphi_j \widehat{f})^\vee(x) &= \sum_{m\in\mathbb{Z}^n} b_{jm}\, \omega_j^\vee(x - 2^{-j}m) \\ &= 2^{jn} \sum_{m\in\mathbb{Z}^n} b_{jm}\, \omega^\vee(2^j x - m), \qquad x \in \mathbb{R}^n. \end{aligned} \tag{3.109}$$

Let k be given by (3.66). Expanding the analytic function $\omega^\vee(2^j x - m)$ at $2^{-j}l$ with $l \in \mathbb{Z}^n$ one gets

$$
\begin{aligned}
&k(2^j x - l)\, \omega^\vee(2^j x - m) \\
&= \sum_{\beta \in \mathbb{N}_0^n} \frac{2^{j|\beta|}}{\beta!}\, (D^\beta \omega^\vee)(l-m)\, (x - 2^{-j}l)^\beta\, k(2^j x - l) \\
&= \sum_{\beta \in \mathbb{N}_0^n} \frac{2^{J|\beta|}}{\beta!}\, (D^\beta \omega^\vee)(l-m)\, k^\beta_{jl}(x),
\end{aligned}
\tag{3.110}
$$

where we used (3.68) and (3.88). By (3.67) and (3.109), (3.110) one obtains that

$$
\begin{aligned}
(\varphi_j \widehat{f})^\vee(x) &= \sum_{m \in \mathbb{Z}^n} 2^{jn}\, b_{jm} \sum_{l \in \mathbb{Z}^n} k(2^j x - l)\, \omega^\vee(2^j x - m) \\
&= \sum_{\beta \in \mathbb{N}_0^n} \sum_{l \in \mathbb{Z}^n} k^\beta_{jl}(x) \sum_{m \in \mathbb{Z}^n} 2^{jn}\, b_{jm}\, \frac{2^{J|\beta|}}{\beta!}\, (D^\beta \omega^\vee)(l-m).
\end{aligned}
\tag{3.111}
$$

Hence,

$$
f = \sum_{j=0}^{\infty} \sum_{\beta \in \mathbb{N}_0^n} \sum_{l \in \mathbb{Z}^n} \lambda^\beta_{jl}\, k^\beta_{jl} \tag{3.112}
$$

with

$$
\lambda^\beta_{jl} = \sum_{m \in \mathbb{Z}^n} 2^{jn}\, b_{jm}\, \frac{2^{J|\beta|}}{\beta!}\, (D^\beta \omega^\vee)(l-m). \tag{3.113}
$$

First we check that λ^β_{jl} are optimal coefficients. In other words, if $\varrho \geq 0$ is given we must find a constant $c = c_\varrho$ such that

$$
\|\lambda \,|b^{s,\varrho}_p\| \leq c\, \|f \,|B^s_p(\mathbb{R}^n)\| \qquad \text{for all } f \in B^s_p(\mathbb{R}^n) \tag{3.114}
$$

with $b^{s,\varrho}_p$ according to Definition 3.13. We claim that for any given $a > 0$ there are constants $C > 0$ and $c_a > 0$ such that

$$
|D^\beta \omega^\vee(x)| \leq c_a\, 2^{C|\beta|}\, (1 + |x|^2)^{-a}, \quad x \in \mathbb{R}^n, \quad \beta \in \mathbb{N}_0^n, \tag{3.115}
$$

where C is independent of x,a,β, and c_a is independent of x,β. We shift the proof of this assertion to Remark 3.22 below and take it temporarily for granted. Let $p \geq 1$. Interpreting (3.113) as a convolution in ℓ_p it follows that for any $\varrho \geq 0$ there is a constant $c(\varrho)$ such that

$$
\Big(\sum_{l \in \mathbb{Z}^n} |\lambda^\beta_{jl}|^p \Big)^{1/p} \leq c(\varrho)\, 2^{-(\varrho+1)|\beta|} \Big(\sum_{l \in \mathbb{Z}^n} |2^{jn} b_{jl}|^p \Big)^{1/p}. \tag{3.116}
$$

The term with $|\beta|$ follows from Stirling's formula for the Γ-function

$$\Gamma(t) = e^{-t}\, t^{t-1/2}\, \sqrt{2\pi}\, e^{\theta(t)/t} \quad \text{with} \quad 0 < \theta(t) < 1/2, \tag{3.117}$$

where $t > 0$, and

$$n! = \Gamma(n+1) \sim e^{-n}\, n^{n+1/2}, \qquad n \in \mathbb{N}. \tag{3.118}$$

This may be found in [ET96], p. 98, with a reference to [WiW52], 12.33. If $p < 1$ then one can use the p-triangle inequality. Now (3.114) follows (3.72), (3.108), (3.116). This completes also the proof of part (i).

Step 3. It remains to prove that the optimal coefficients λ^β_{jm} according to (3.113) can be written as (3.96). By (3.107) and well-known properties of the Fourier transform we have

$$2^{jn}\, b_{jm} = (2\pi)^{-n} \int_{\mathbb{R}^n} (\varphi_j)^\vee (2^{-j}m - y)\, f(y)\, \mathrm{d}y, \qquad j \in \mathbb{N}_0. \tag{3.119}$$

With φ as in Definition 3.15 we have $\varphi_j(x) = \varphi(2^{-j}x)$ if $j \in \mathbb{N}$. Then it follows that

$$2^{jn}\, b_{jm} = (2\pi)^{-n}\, 2^{jn} \int_{\mathbb{R}^n} \varphi^\vee (m - 2^j y)\, f(y)\, \mathrm{d}y, \qquad j \in \mathbb{N}. \tag{3.120}$$

Recall that

$$(D^\beta \omega^\vee)(\xi) = i^{|\beta|}\, (x^\beta \omega(x))^\vee (\xi).$$

Inserting (3.120) in (3.113) one obtains by (3.76) that

$$\lambda^\beta_{jl} = 2^{jn} \int_{\mathbb{R}^n} f(y) \sum_{m \in \mathbb{Z}^n} (\omega^\beta)^\vee (l-m)\, \varphi^\vee (m - 2^j y)\, \mathrm{d}y, \qquad j \in \mathbb{N}. \tag{3.121}$$

Replacing $l - m$ by m and using $\varphi^\vee(z) = \widehat{\varphi}(-z)$ one gets

$$\begin{aligned} \lambda^\beta_{jl} &= 2^{jn} \int_{\mathbb{R}^n} f(y) \sum_{m \in \mathbb{Z}^n} (\omega^\beta)^\vee (m)\, \widehat{\varphi}\, (2^j y - l + m)\, \mathrm{d}y \\ &= 2^{jn} \left(f, \Phi^\beta_{jl} \right), \qquad j \in \mathbb{N}, \end{aligned} \tag{3.122}$$

where we used (3.84) and (3.89). If $j = 0$ then one has to use (3.83) and (3.89). This completes the proof of part (ii). □

Remark 3.22. We prove (3.115) where we only need that ω in (3.75) has a compact support. We may assume that $a = K \in \mathbb{N}$. Let $\alpha \in \mathbb{N}^n_0$ with $|\alpha| \le 2K$. Then

$$\begin{aligned} x^\alpha\, D^\beta \omega^\vee(x) &= x^\alpha i^{|\beta|} (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\xi}\, \xi^\beta\, \omega(\xi)\, \mathrm{d}\xi \\ &= i^{|\beta| - |\alpha|} (2\pi)^{-n/2} \int_{\mathbb{R}^n} D^\alpha_\xi \left(e^{ix\xi} \right) \xi^\beta \omega(\xi)\, \mathrm{d}\xi \\ &= i^{|\alpha| + |\beta|} (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\xi}\, D^\alpha_\xi \left[\xi^\beta \omega(\xi) \right] \mathrm{d}\xi. \end{aligned} \tag{3.123}$$

Since ω has a compact support it follows for some $c \geq 0$ that

$$\begin{aligned} \left| D_\xi^\alpha \left[\xi^\beta \omega(\xi) \right] \right| &\leq c_K \sum_{\gamma+\eta=\alpha} |D^\gamma \xi^\beta| \cdot |D^\eta \omega(\xi)| \\ &\leq c'_K (1+|\beta|)^{2K}\, 2^{c|\beta|}. \end{aligned} \tag{3.124}$$

Inserting (3.124) in (3.123) and using again that ω has a compact support it follows that

$$|x^\alpha D^\beta \omega^\vee(x)| \leq c_K\, 2^{C|\beta|}, \qquad x \in \mathbb{R}^n. \tag{3.125}$$

This proves (3.115).

Remark 3.23. The representation (3.99) with (3.96) might be considered as a special case of (1.116), (1.117) and Corollary 1.42. As there we call $\{k^\beta_{jm}\}$ a *frame*. Originally, the notation *frame* comes from Hilbert space theory. Its extension to Banach spaces, then called *Banach frames*, goes back to [Gro91]. We refer for further details to [Gro01] and also to the survey part in [Tri02c]. We use here the word *frame* in a wider sense for quasi-Banach function spaces, preferably of type B^s_{pq} and F^s_{pq},

as a system of distinguished elements, in the above case $\{k^\beta_{jm}\}$, *such that any element of this space can be expanded by this system where the corresponding complex coefficients*, in our case $\lambda^\beta_{jm}(f)$, *depend linearly on* f *and produce an equivalent quasi-norm when measured in a suitable sequence space*, in our case $b^{s,\varrho}_p$.

Then the corresponding coefficients, in our case $\lambda^\beta_{jm}(f)$, are linear and bounded functionals on the space considered, generating a *dual frame*, in our case $\left\{2^{jn}\Phi^\beta_{jm}\right\}$ according to (3.96).

Remark 3.24. In contrast to the general quarkonial representations for the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ according to Section 1.6 and to [Triϵ], Sections 2 and 3, we restricted ourselves in the above considerations to the B-spaces and to the special case $B^s_p(\mathbb{R}^n) = B^s_{pq}(\mathbb{R}^n)$ with $q = p$. There is hardly any doubt that the above theory can be extended to the spaces $B^s_{pq}(\mathbb{R}^n)$ and maybe also to the spaces $F^s_{pq}(\mathbb{R}^n)$. But this has not yet been done. But the main reason for this restriction is the following. After presenting further decompositions in Sections 3.2.3, 3.2.4 based on Theorem 3.21 we develop in Section 3.2.5 a local smoothness theory as an application. But in this context one does not need possible generalisations of the above theory to more general spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$.

Remark 3.25. The above system $\{k^\beta_{jm}\}$ comes from (3.66)–(3.68) and (3.88). This fits pretty well in the *g-philosophy* as outlined in Section 1.19, so far restricted to (1.627)–(1.629). One may even identify the above function k with

$$k(x) = g(x - m^0) \left(\sum_{m\in\mathbb{Z}^n} g(x-m) \right)^{-1}, \qquad x \in \mathbb{R}^n, \tag{3.126}$$

where g is the bad guy of calculus according to (1.626) promoted to the king of function spaces and where $m^0 \in \mathbb{Z}^n$ is appropriately chosen ensuring (3.66). Incidentally, k in (3.126) does not refer to *king* but to *kernel* (of local means). Similarly, qu in (1.102) comes from *quark* and not from *queen* as one might guess after reading Section 1.19. (By the way, F in F^s_{pq} refers to nothing. It is a free invention of mine around 1970 looking for a letter not used so far in connection with function spaces).

3.2.3 Wavelet frames for distributions

Roughly speaking we dualise Theorem 3.21 resulting in spaces $B^s_p(\mathbb{R}^n)$ according to the abbreviation (3.69) with $1 < p \le \infty$ and $s < 0$. The counterpart of $B^+_p(\mathbb{R}^n)$ in (3.90) is now given by

$$B^{-\infty}_p(\mathbb{R}^n) = \bigcup_{s<0} B^s_p(\mathbb{R}^n), \qquad 0 < p \le \infty. \tag{3.127}$$

If $0 < p_1 \le p_2 \le \infty$ then one has by well-known embedding theorems,

$$B^{-\infty}_{p_1}(\mathbb{R}^n) \subset B^{-\infty}_{p_2}(\mathbb{R}^n) \subset B^{-\infty}_{\infty}(\mathbb{R}^n) = \mathcal{C}^{-\infty}(\mathbb{R}^n), \tag{3.128}$$

where the latter is reminiscent of the Hölder-Zygmund scale (1.10). If $f \in S'(\mathbb{R}^n)$ has compact support then f belongs to any space $B^{-\infty}_p(\mathbb{R}^n)$ with $0 < p \le \infty$. This follows from

$$|(f,\varphi)| \le c\, \|\varphi\, |C^K(\mathbb{R}^n)\| \le c'\, \|\varphi\, |B^{-s}_{p'}(\mathbb{R}^n)\|, \qquad -s > K + n/p', \tag{3.129}$$

for some $c > 0$, $K \in \mathbb{N}$, all $\varphi \in S(\mathbb{R}^n)$, $1 \le p' < \infty$, duality as in (3.17), resulting in $f \in B^s_p(\mathbb{R}^n)$ if $1 < p \le \infty$, and the indicated monotonicity assertion at the beginning of Step 2 in the proof of Corollary 3.10 with B in place of F. Hence the local behavior of any $f \in S'(\mathbb{R}^n)$ is captured by $B^{-\infty}_p(\mathbb{R}^n)$. But globally the situation is different. For example, if $f = P$ is a polynomial then one has for φ_0 in (2.8) that $(\varphi_0 \widehat{P})^\vee = P$. Hence, P does not belong to $\mathcal{C}^{-\infty}(\mathbb{R}^n)$ with exception of constants. To incorporate global assertions and to cover all $f \in S'(\mathbb{R}^n)$ one needs weighted spaces. We return to this subject (in the context of wavelet bases) in Chapter 6.

It is the main aim of this subsection to deal with universal wavelet expansions

$$f = \sum_{\beta,j,m} \lambda^\beta_{jm}\, \Phi^\beta_{jm}, \qquad f \in \mathcal{C}^{-\infty}(\mathbb{R}^n), \tag{3.130}$$

with the same functions Φ^β_{jm} as in Definition 3.17 and the abbreviation (3.74). In analogy to (3.92)–(3.95) we discuss first the meaning of

$$\sum_{\beta,j,m} \lambda^\beta_{jm}\, \Phi^\beta_{jm}, \qquad \lambda \in b^s_p, \quad s \in \mathbb{R},\ 0 < p \le \infty, \tag{3.131}$$

assuming $\varrho = 0$ in the sequence spaces $b^{s,\varrho}_p$ according to Definition 3.13. For any $\varepsilon > 0$,

$$\Big\{ 2^{\varepsilon|\beta|}\, 2^{-j(s-n/p)}\, \Phi^\beta_{jm} \; : \; j \in \mathbb{N}_0, \; m \in \mathbb{Z}^n \Big\}, \qquad \beta \in \mathbb{N}^n_0, \tag{3.132}$$

are admitted systems of normalised molecules in $B^s_p(\mathbb{R}^n)$, ignoring constants which may be assumed to be independent of j, m, β. As for the factor $2^{\varepsilon|\beta|}$ we refer to Definitions 3.15, 3.17 and the arguments in connection with (3.116) based on (3.117), (3.118). Normalised molecules have all the properties of normalised atoms according to Definition 1.15 with exception of the compactness assumption in (1.58) which is replaced by a sufficiently strong decay. Then Theorem 1.19 remains valid. This applies especially to the system (3.132) which are analytic functions belonging to $S(\mathbb{R}^n)$ satisfying the moment conditions (3.85). We refer for details about molecular decompositions to [FJW91], Section 5, especially p. 48. But in the above case it is not difficult to reduce the needed molecular decompositions to corresponding atomic decompositions: Let $L \in \mathbb{N}$ and let $(-\Delta)^L$ be the Lth power of the Laplacian in $\mathbb{R}^n$. Then it follows from (3.80), (3.67) that such a reduction can be based on

$$\Phi^\beta_M = \sum_{m \in \mathbb{Z}^n} (-\Delta)^L \, \Big[k(\cdot - m)\, (-\Delta)^{-L} \Phi^\beta_M \Big] \tag{3.133}$$

with L large enough such that the required moment conditions are satisfied. Each series

$$\sum_{j,m} \lambda^\beta_{jm}\, \Phi^\beta_{jm}, \qquad \beta \in \mathbb{N}^n_0, \tag{3.134}$$

converges unconditionally in $S'(\mathbb{R}^n)$ to some $g^\beta \in B^s_p(\mathbb{R}^n)$ with

$$\| g^\beta \,|B^s_p(\mathbb{R}^n)\| \le c\, 2^{-\varepsilon|\beta|} \bigg(\sum_{j,m} 2^{j(s-n/p)p}\, |\lambda^\beta_{jm}|^p \bigg)^{1/p} \tag{3.135}$$

where c is independent of β. This follows by the same arguments as in Remark 1.20 which can be extended from atoms to molecules with the same references as given there and by the indicated molecular version of Theorem 1.19. Then the full sum in (3.131) converges unconditionally in $S'(\mathbb{R}^n)$ to

$$g = \sum_{\beta \in \mathbb{N}^n_0} g^\beta \in B^s_p(\mathbb{R}^n) \quad \text{with} \quad \| g \,|B^s_p(\mathbb{R}^n)\| \le c\, \|\lambda \,|b^s_p\| \tag{3.136}$$

for some $c > 0$ which is independent of $\lambda \in b^s_p$.

The role of the local means (3.15), (3.16) is now taken over by k^β according to (3.66)–(3.68) in place of Ψ^G, hence

$$\begin{aligned} k^\beta(t,f)(x) &= \int_{\mathbb{R}^n} k^\beta(y)\, f(x+ty)\,\mathrm{d}y \\ &= t^{-n}\int_{\mathbb{R}^n} k^\beta\left(\frac{y-x}{t}\right) f(y)\,\mathrm{d}y \end{aligned} \tag{3.137}$$

with $t>0$ and $x\in\mathbb{R}^n$, and

$$\begin{aligned} k^\beta(2^{-j},f)(2^{-j}m) &= 2^{jn}\int_{\mathbb{R}^n} k^\beta(2^j y - m)\, f(y)\,\mathrm{d}y \\ &= 2^{jn}\int_{\mathbb{R}^n} k^\beta_{jm}(y)\, f(y)\,\mathrm{d}y \end{aligned} \tag{3.138}$$

where we used (3.88) with $j\in\mathbb{N}_0, m\in\mathbb{Z}^n, \beta\in\mathbb{N}_0^n$. As in (3.96) we abbreviate (3.138) by

$$\lambda^\beta_{jm}(f) = 2^{jn}\left(f, k^\beta_{jm}\right), \qquad f\in S'(\mathbb{R}^n). \tag{3.139}$$

To avoid any misunderstanding we again remark that in the following theorem k^β_{jm} and Φ^β_{jm} according to Definition 3.17 are fixed, based on fixed admitted k, J, φ_0 and ω.

Theorem 3.26. *Let $1<p\le\infty$ and $s<0$.*

(i) *Then $f\in S'(\mathbb{R}^n)$ is an element of $B^s_p(\mathbb{R}^n)$ if, and only if, it can be represented as*

$$f=\sum_{\beta,j,m}\lambda^\beta_{jm}\,\Phi^\beta_{jm}, \qquad \lambda\in b^s_p, \tag{3.140}$$

unconditional convergence being in $S'(\mathbb{R}^n)$. Furthermore,

$$\|f\,|B^s_p(\mathbb{R}^n)\| \sim \inf \|\lambda\,|b^s_p\|, \tag{3.141}$$

where the infimum is taken over all admissible representations (3.140).

(ii) *Let $\lambda^\beta_{jm}(f)$ be given by* (3.139). *Then any $f\in\mathcal{C}^{-\infty}(\mathbb{R}^n)$ can be represented as*

$$f=\sum_{\beta,j,m}\lambda^\beta_{jm}(f)\,\Phi^\beta_{jm}, \tag{3.142}$$

unconditional convergence being in $S'(\mathbb{R}^n)$. Furthermore,

$$B^s_p(\mathbb{R}^n) = \left\{f\in\mathcal{C}^{-\infty}(\mathbb{R}^n):\ \|\lambda(f)\,|b^s_p\|<\infty\right\} \tag{3.143}$$

(*equivalent norms*).

Proof. Step 1. By the above considerations, especially (3.134)–(3.136), it follows that

$$\|f\,|B^s_p(\mathbb{R}^n)\| \le c\,\|\lambda\,|b^s_p\| \tag{3.144}$$

for any representation (3.140), where c is independent of f and λ. This covers also the unconditional convergence.

Step 2. Let $\lambda(f)$ be the sequence with the components (3.139). We wish to prove that there is a number $c > 0$ such that for all $f \in B^s_p(\mathbb{R}^n)$,

$$\|\lambda(f)\,|b^s_p\| \le c\,\|f\,|B^s_p(\mathbb{R}^n)\| < \infty. \tag{3.145}$$

Using the duality $\ell_p = (\ell_{p'})'$ with $\frac{1}{p} + \frac{1}{p'} = 1$ (hence $1 \le p' < \infty$) it follows from (3.139) and (3.72) with $\varrho = 0$ that

$$\|\lambda(f)\,|b^s_p\| = \sup \sum_{\beta,j,m} \mu^\beta_{jm}\,\left(f, k^\beta_{jm}\right) \tag{3.146}$$

where the supremum is taken over all $\mu = \{\mu^\beta_{jm}\}$ such that the terms in (3.146) are non-negative and

$$\|\mu\,|b^{-s}_{p'}\| \le 1. \tag{3.147}$$

Here we used that $2^{jn}\,2^{j(s-n/p)} = 2^{j(s+n/p')}$. Then

$$\|\lambda(f)\,|b^s_p\| \le \sup|(f,g)| \quad \text{where} \quad g = \sum_{\beta,j,m} \mu^\beta_{jm}\,k^\beta_{jm} \tag{3.148}$$

with (3.147). Hence by Theorem 3.21,

$$\|\lambda(f)\,|b^s_p\| \le \sup\Big\{|(f,g)| :\ g \in B^{-s}_{p'}(\mathbb{R}^n),\ \|g\,|B^{-s}_{p'}(\mathbb{R}^n)\| \le c\Big\} \tag{3.149}$$

for some $c > 0$ which is independent of g. Then (3.145) follows from (3.149) and the duality (3.17).

Step 3. We prove part (ii) of the theorem. Then one gets as a by-product the converse of (3.144) and hence part (i). By Theorem 3.21 any $\psi \in S(\mathbb{R}^n)$ can be represented as

$$\psi = \sum_{\beta,j,m} 2^{jn}\,\left(\psi, \Phi^\beta_{jm}\right)\,k^\beta_{jm}, \tag{3.150}$$

converging unconditionally in any space $B^\sigma_{p'}(\mathbb{R}^n)$ with $\sigma > 0$. Then it follows for $f \in B^s_p(\mathbb{R}^n)$ with $1 < p \le \infty$ and $s < 0$ that

$$(f,\psi) = \left(\sum_{\beta,j,m} 2^{jn}\,\left(f, k^\beta_{jm}\right)\,\Phi^\beta_{jm}, \psi\right). \tag{3.151}$$

Here we used Step 2 and the above considerations about unconditional convergence in connection with (3.134)–(3.136). But this proves (3.142) with (3.139) and also (3.143). □

Remark 3.27. By the Theorems 3.21, 3.26 and the explanations given in Remark 3.23 one gets the dual frames

$$\{k^\beta_{jm}\}, \{2^{jn}\Phi^\beta_{jm}\} \qquad \text{and} \qquad \{\Phi^\beta_{jm}\}, \{2^{jn}k^\beta_{jm}\}$$

in the corresponding spaces. By (3.138), (3.139) one needs only a knowledge of f near a given point, here $2^{-j}m$, to calculate the corresponding coefficients $\lambda^\beta_{jm}(f)$. This gives the possibility to develop a local smoothness theory. We return to this point in Section 3.2.5. But first we extend the underlying above approach from $s<0$ to $s>0$.

Remark 3.28. In contrast to Theorem 3.21 we restricted our considerations in Theorem 3.26 to $\varrho = 0$. This is sufficient for what follows. But one can extend the above assertions, in particular (3.140), (3.141), (3.143) from b^s_p (which means $\varrho = 0$) to $b^{s,\varrho}_p$ now with $\varrho \le 0$ according to Definition 3.13. In particular,

$$\|\lambda(f)\,|b^{s,\varrho}_p\| \sim \|f\,|B^s_p(\mathbb{R}^n)\|, \qquad f \in B^s_p(\mathbb{R}^n), \tag{3.152}$$

where $\lambda(f)$ is given by (3.139) and $1<p\le\infty$, $s<0$, $\varrho\le 0$ (the equivalence constants depend on ϱ, maybe on s,p, but not on f): The estimate of the left-hand side of (3.152) from above by the right-hand side is a consequence of (3.145). The converse assertion follows from the above considerations, in particular from (3.135) with, say, $\varepsilon = |\varrho| + \eta$, $\eta > 0$. Finally we mention that (3.143) and (3.152) are essentially discrete versions of characterisations of the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ with $s<0$ as described in Corollary 1.12. In particular for $0<p\le\infty$ and $s<0$ there is a number $r_0>0$ such that for all r with $0<r\le r_0$,

$$\begin{aligned} \|f\,|B^s_p(\mathbb{R}^n)\| &\sim \Big(\sum_{j=0}^\infty 2^{jsp}\,\|k(2^{-j},f)\,|L_p(\mathbb{R}^n)\|^p\Big)^{1/p} \\ &\sim \Big(\sum_{j=0}^\infty \sum_{m\in\mathbb{Z}^n} 2^{j(s-n/p)p}\,|k(2^{-j},f)(r2^{-j}m)|^p\Big)^{1/p} \end{aligned} \tag{3.153}$$

(with the usual modification if $p=\infty$) where $k(t,f)$ are the local means (3.137) with $\beta=0$. This follows from (1.54) where the second equivalence in (3.153) can be obtained from the corresponding characterisation in terms of maximal functions. An explicit formulation may be found in [Win95], Theorem 4, pp. 16/17. Obviously, (3.138), (3.139) with $\beta=0$ and corresponding terms in (3.143), (3.152) are closely related to the discrete version in (3.153).

3.2.4 Wavelet frames for functions

Let $L\in\mathbb{N}$. Then

$$D_L = \mathrm{id} + (-\Delta)^L, \qquad \text{where} \quad \Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} \tag{3.154}$$

is the Laplacian in $\mathbb{R}^n$. It is well known that

$$D_L B^s_{pq}(\mathbb{R}^n) = B^{s-2L}_{pq}(\mathbb{R}^n) \qquad \text{is an isomorphism} \tag{3.155}$$

for all parameters $s \in \mathbb{R}, 0 < p \le \infty, 0 < q \le \infty$. Similarly for $F^s_{pq}(\mathbb{R}^n)$. Let $1 < p \le \infty, s > 0$ and $L \in \mathbb{N}$ such that $s - 2L < 0$. Then one can try to lift the characterisations and representations in Theorem 3.26 from $B^{s-2L}_{pq}(\mathbb{R}^n)$ via D_L^{-1} to $B^s_{pq}(\mathbb{R}^n)$. We have done this in detail in [Tri03a]. What follows is a simplified and modified version of some aspects of this paper. First we fix some notation.

Definition 3.29. *Let $\beta \in \mathbb{N}^n_0$ and $m \in \mathbb{Z}^n$.*

(i) *Let k and k^β be the same functions as in* (3.66)–(3.68)*. Then*

$$k^{\beta,L}(x) = (-\Delta)^L\, k^\beta(x), \qquad L \in \mathbb{N}_0, \quad x \in \mathbb{R}^n, \tag{3.156}$$

and

$$k^{\beta,L}_{jm}(x) = k^{\beta,L}(2^j x - m), \qquad j \in \mathbb{N}_0, \quad x \in \mathbb{R}^n. \tag{3.157}$$

(ii) *Let Ω^β, φ_0 and φ be the same functions as in Definition* 3.15*. Let $L \in \mathbb{N}_0$ and $l \in \mathbb{N}_0$. Then $\Phi^{\beta,L}_F$ and $\Phi^{\beta,L,l}_M$ are given by*

$$(\Phi^{\beta,L}_F)^\vee(\xi) = \frac{\varphi_0(\xi)}{1+|\xi|^{2L}}\, \Omega^\beta(\xi), \qquad \xi \in \mathbb{R}^n, \tag{3.158}$$

$$(\Phi^{\beta,L,l}_M)^\vee(\xi) = \frac{\varphi(\xi)}{2^{-2lL}+|\xi|^{2L}}\, \Omega^\beta(\xi), \qquad \xi \in \mathbb{R}^n. \tag{3.159}$$

Furthermore

$$\Phi^{\beta,L}_{jm}(x) = \begin{cases} \Phi^{\beta,L}_F(x-m) & \text{if} \quad j = 0, \\ \Phi^{\beta,L,j}_M(2^j x - m) & \text{if} \quad j \in \mathbb{N}. \end{cases} \tag{3.160}$$

Remark 3.30. This is the L-version of Definitions 3.15, 3.17. If $L = 0$ then (3.157) coincides with (3.88) and in case of part (ii) we have

$$2\Phi^{\beta,0}_F = \Phi^\beta_F \qquad \text{and} \qquad 2\Phi^{\beta,0,l}_M = \Phi^\beta_M. \tag{3.161}$$

Otherwise we modified Definition 3 in [Tri03a] where we split the mother wavelets (3.159) into a homogeneous part without the summand 2^{-2lL} and a remainder part. This has some advantages but will not be done here. We use $k^{\beta,L}$ and $k^{\beta,L}_{jm}$ similarly as in (3.137), (3.138), as kernels of local means, hence

$$\begin{aligned} k^{\beta,L}(t,f)(x) &= \int_{\mathbb{R}^n} k^{\beta,L}(y)\, f(x+ty)\, \mathrm{d}y \\ &= t^{-n} \int_{\mathbb{R}^n} k^{\beta,L}\left(\frac{y-x}{t}\right) f(y)\, \mathrm{d}y \end{aligned} \tag{3.162}$$

and

$$
\begin{aligned}
k^{\beta,L}(2^{-j},f)(2^{-j}m) &= 2^{jn}\int_{\mathbb{R}^n} k^{\beta,L}(2^j y-m)\, f(y)\,\mathrm{d}y \\
&= 2^{jn}\int_{\mathbb{R}^n} k^{\beta,L}_{jm}(y)\, f(y)\,\mathrm{d}y.
\end{aligned} \tag{3.163}
$$

In particular, if $0<s<2L$ with $L\in\mathbb{N}$, then $k^{\beta,L}(t,f)$ are local means as used in Theorem 1.10 satisfying (1.42). For fixed $\beta\in\mathbb{N}_0^n$ and $L\in\mathbb{N}_0$,

$$
\left\{(\Phi_M^{\beta,L,l})^\vee:\ l\in\mathbb{N}_0\right\} \quad \text{and, hence,} \quad \left\{\Phi_M^{\beta,L,l}:\ l\in\mathbb{N}_0\right\}, \tag{3.164}
$$

are bounded sets with respect to the topology in $S(\mathbb{R}^n)$. Then it follows by the discussion and the references at the beginning of Section 3.2.3, especially in connection with (3.131), (3.132) that for any ε,

$$
\left\{2^{\varepsilon|\beta|}\, 2^{-j(s-n/p)}\, \Phi_M^{\beta,L,l}(2^j\cdot -m):\ j\in\mathbb{N},\ m\in\mathbb{Z}^n\right\}, \quad \beta\in\mathbb{N}_0^n, \tag{3.165}
$$

are systems of normalised molecules in $B^s_p(\mathbb{R}^n)$ (but also in $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$), ignoring constants which are independent not only of j,m, but also of β,l. This applies in particular to the 'diagonal' functions $\Phi_M^{\beta,L,j}(2^jx-m)$ where $l=j$. Hence for any $\varepsilon>0$,

$$
\left\{2^{\varepsilon|\beta|}\, 2^{-j(s-n/p)}\, \Phi^{\beta,L}_{jm}:\ j\in\mathbb{N}_0,\ m\in\mathbb{Z}^n\right\}, \qquad \beta\in\mathbb{N}_0^n, \tag{3.166}
$$

are systems of normalised molecules in $B^s_p(\mathbb{R}^n)$, ignoring constants which are independent of j,m,β. In analogy to Theorem 3.26 and (3.130) we are interested now in wavelet expansions

$$
f=\sum_{\beta,j,m}\lambda^\beta_{jm}\,\Phi^{\beta,L}_{jm}, \qquad f\in L_p(\mathbb{R}^n), \quad \lambda\in b^s_p, \tag{3.167}
$$

where $1<p\le\infty$ and $0<s<2L$. Here $b^s_p=b^{s,\varrho}_p$ with $\varrho=0$ are the sequence spaces according to Definition 3.13. As for the convergence of the series

$$
\sum_{\beta,j,m}\lambda^\beta_{jm}\,\Phi^{\beta,L}_{jm}, \qquad 1<p\le\infty,\ 0<s<2L,\ \lambda\in b^s_p, \tag{3.168}
$$

we are in the same position as in (3.131), (3.132). By the arguments as given there in connection with (3.132)–(3.136) it follows that (3.168) converges unconditionally in $S'(\mathbb{R}^n)$, say, to g, with a counterpart of (3.136),

$$
\|g\,|B^s_p(\mathbb{R}^n)\|\le c\,\|\lambda\,|b^s_p\|, \tag{3.169}
$$

where c is independent of λ. (By the special properties of $\Phi^{\beta,L}_{jm}$ it follows that the series in (3.168) converges unconditionally in $S'(\mathbb{R}^n)$ for any p,s with $0<p\le\infty$, $s\in\mathbb{R}$, and $\lambda\in b^s_p$ with (3.169) for the limit element.) But if $1<p\le\infty$ and $s>0$, then one is in the same situation as at the beginning of Section 3.2.2, especially in connection with (3.93)–(3.95) and with a reference to Remark 2.12. The arguments in Remark 2.12 can be extended from compactly supported atoms to the above rapidly decreasing molecules $\Phi^{\beta,L}_{jm}$ belonging to $S(\mathbb{R}^n)$. Then it follows that the series in (3.168) converges absolutely in $L_p(\mathbb{R}^n)$ if $1 \le p < \infty$ and in $L_\infty(\mathbb{R}^n, w)$ if $p = \infty$, where $L_\infty(\mathbb{R}^n, w)$ with $w(x) = (1 + |x|^2)^{\sigma/2}$, $\sigma < 0$, is the same weighted L_∞-space as at the end of Remark 2.12. This argument works for all $0 < s < \infty$. The full restrictions for p and s in (3.168) are needed when it comes to the question as to whether any $f \in B^s_p(\mathbb{R}^n)$ can be represented by (3.167). Next we formulate the counterpart of Theorem 3.26 with the optimal coefficients in (3.139). Let $0 < s < 2L$ with $L \in \mathbb{N}$ and let $k^{\beta,L}_{jm}$ and $k^\beta_{jm} = k^{\beta,0}_{jm}$ be as in (3.157) and (3.88). Then

$$\varkappa^{\beta,L}_{jm}(f) = 2^{jn} \left(f, k^{\beta,L}_{jm} \right), \qquad f \in S'(\mathbb{R}^n), \tag{3.170}$$

and

$$\lambda^{\beta,L}_{jm}(f) = \varkappa^{\beta,L}_{jm}(f) + 2^{jn-2jL} \left(f, k^\beta_{jm} \right), \qquad f \in S'(\mathbb{R}^n), \tag{3.171}$$

is the adequate counterpart of (3.139). Here the two terms are the same as in (3.163) and in (3.138), the latter multiplied with 2^{-2jL}. Otherwise we use in the following theorem the above notation. In particular, $k^{\beta,L}_{jm}, k^\beta_{jm}$ and $\Phi^{\beta,L}_{jm}$ have the same meaning as in Definitions 3.29 and 3.17(i), based on fixed admitted k, J, φ_0, ω and, now, L. The sequence spaces b^s_p are the same as in Definition 3.13. Let $B^+_p(\mathbb{R}^n)$ be as in (3.90).

Theorem 3.31. *Let $1 < p \le \infty$, $L \in \mathbb{N}$ and $0 < s < 2L$.*

(i) *Then $f \in L_p(\mathbb{R}^n)$ is an element of $B^s_p(\mathbb{R}^n)$ if, and only if, it can be represented as*

$$f = \sum_{\beta,j,m} \lambda^\beta_{jm} \, \Phi^{\beta,L}_{jm}, \qquad \lambda \in b^s_p, \tag{3.172}$$

absolute, and hence unconditional, convergence being in $L_p(\mathbb{R}^n)$ (with the indicated modification for $p = \infty$). Furthermore,

$$\|f \,|B^s_p(\mathbb{R}^n)\| \sim \inf \|\lambda \,|b^s_p\| \tag{3.173}$$

where the infimum is taken over all admissible representations (3.172).

(ii) *Let $\varkappa^{\beta,L}_{jm}(f)$ and $\lambda^{\beta,L}_{jm}(f)$ be given by (3.170), (3.171). Then any $f \in B^+_p(\mathbb{R}^n)$ can be represented as*

$$f = \sum_{\beta,j,m} \lambda^{\beta,L}_{jm}(f) \, \Phi^{\beta,L}_{jm}, \tag{3.174}$$

absolute convergence being in $L_p(\mathbb{R}^n)$ (above modification if $p = \infty$). Furthermore,

$$\begin{aligned} B^s_p(\mathbb{R}^n) &= \left\{ f \in L_p(\mathbb{R}^n) : \ \|\lambda^L(f) \,|b^s_p\| < \infty \right\} \\ &= \left\{ f \in L_p(\mathbb{R}^n) : \ \|\varkappa^L(f) \,|b^s_p\| + \|f \,|L_p(\mathbb{R}^n)\| < \infty \right\} \end{aligned} \tag{3.175}$$

(equivalent norms).

Proof. Step 1. It follows by the above considerations that (3.172) is a molecular decomposition in $B^s_p(\mathbb{R}^n)$. It converges as indicated and

$$\|f\,|B^s_p(\mathbb{R}^n)\| \le c\,\|\lambda\,|b^s_p\|, \qquad \lambda \in b^s_p\,, \tag{3.176}$$

where $c > 0$ is independent of λ.

Step 2. We prove that $f \in B^s_p(\mathbb{R}^n)$ can be represented by (3.174) with

$$\|\lambda^L(f)\,|b^s_p\| \le c\,\|f\,|B^s_p(\mathbb{R}^n)\|, \qquad f \in B^s_p(\mathbb{R}^n), \tag{3.177}$$

where $c > 0$ is independent of f. Then one gets as a by-product the converse of (3.176) and hence part (i). Let $f \in B^s_p(\mathbb{R}^n)$. By (3.154), (3.155) we can apply Theorem 3.26 to $D_L f \in B^{s-2L}_p(\mathbb{R}^n)$ and one gets by (3.142), (3.139)

$$f(x) = \sum_{\beta,j,m} 2^{jn}\left((\mathrm{id} + (-\Delta)^L)f, k^\beta_{jm}\right) \cdot D_L^{-1}\left[\Phi^\beta_G(2^j\cdot - m)\right](x) \tag{3.178}$$

with $G = F$ if $j = 0$ and $G = M$ if $j \in \mathbb{N}$ as an optimal decomposition. By (3.88) and (3.156), (3.157) it follows that

$$\begin{aligned} &2^{jn}\left((\mathrm{id} + (-\Delta)^L)f, k^\beta_{jm}\right)\\ &= 2^{jn}\int_{\mathbb{R}^n}\left(1 + (-\Delta)^L\right) f(x)\cdot k^\beta(2^jx - m)\,\mathrm{d}x\\ &= 2^{jn}\int_{\mathbb{R}^n} f(x)\,k^\beta(2^jx - m)\,\mathrm{d}x + 2^{jn+2Lj}\int_{\mathbb{R}^n} f(x)\,k^{\beta,L}(2^jx - m)\,\mathrm{d}x\\ &= 2^{2jL}\lambda^{\beta,L}_{jm}(f) = \widetilde{\lambda}^{\beta,L}_{jm}(f), \end{aligned} \tag{3.179}$$

where we used (3.170), (3.171). By Theorem 3.26 it follows that

$$\begin{aligned} \|\lambda^L(f)\,|b^s_p\| &= \|\widetilde{\lambda}^L(f)\,|b^{s-2L}_p\|\\ &\le c\,\|D_L f\,|B^{s-2L}_p(\mathbb{R}^n)\|\\ &\le c'\,\|f\,|B^s_p(\mathbb{R}^n)\|. \end{aligned} \tag{3.180}$$

This proves (3.177). Recall that

$$D_L^{-1}g = \left(\frac{g^\vee(\xi)}{1 + |\xi|^{2L}}\right)^\wedge, \qquad g \in S'(\mathbb{R}^n). \tag{3.181}$$

Choosing $g = \Phi^\beta_F(\cdot - m)$ one gets by (3.79),

$$g^\vee(\xi) = e^{im\xi}\,(\Phi^\beta_F)^\vee(\xi) = e^{im\xi}\,\varphi_0(\xi)\,\Omega^\beta(\xi) \tag{3.182}$$

and by (3.158), (3.160)

$$D_L^{-1}\left[\Phi_F^\beta(\cdot - m)\right] = \Phi_F^{\beta,L}(\cdot - m) = \Phi_{0,m}^{\beta,L}. \tag{3.183}$$

Let $g = \Phi_M^\beta(2^j \cdot - m)$ with $j \in \mathbb{N}$ and $m \in \mathbb{Z}^n$. Then

$$\begin{aligned} g^\vee(\xi) &= 2^{-jn}\left(\Phi_M^\beta(\cdot - m)\right)^\vee(2^{-j}\xi) \\ &= 2^{-jn} e^{i2^{-j}m\xi}\left(\Phi_M^\beta\right)^\vee(2^{-j}\xi). \end{aligned} \tag{3.184}$$

By (3.80) we have

$$\frac{g^\vee(\xi)}{1+|\xi|^{2L}} = 2^{-jn}\, 2^{-2jL}\, \frac{\varphi(2^{-j}\xi)\,\Omega^\beta(2^{-j}\xi)}{2^{-2jL}+|2^{-j}\xi|^{2L}}\; e^{i2^{-j}m\xi} \tag{3.185}$$

and by (3.159), (3.160) that

$$\left(\frac{g^\vee(\xi)}{1+|\xi|^{2L}}\right)^\wedge(x) = 2^{-2jL}\Phi_M^{\beta,L,j}(2^j x - m) \tag{3.186}$$

and hence

$$D_L^{-1}\left[\Phi_M^\beta(2^j \cdot - m)\right](x) = 2^{-2jL}\,\Phi_{jm}^{\beta,L}(x). \tag{3.187}$$

Inserting (3.183), (3.187), and (3.179) in (3.178) one gets the representation (3.174). Now (3.176) and (3.177) prove part (i) and also the first characterisation in (3.175).

Step 3. We prove the second characterisation in (3.175) and rewrite (3.171) as

$$\lambda_{jm}^{\beta,L}(f) = \varkappa_{jm}^{\beta,L}(f) + \nu_{jm}^\beta(f), \qquad \widetilde{\nu}_{jm}^\beta(f) = 2^{2jL}\nu_{jm}^\beta(f). \tag{3.188}$$

Since $s - 2L < 0$ it follows from Theorem 3.26 that

$$\|\nu(f)\,|b_p^s\| \sim \|\widetilde{\nu}(f)\,|b_p^{s-2L}\| \sim \|f\,|B_p^{s-2L}(\mathbb{R}^n)\| \le c\,\|f\,|L_p(\mathbb{R}^n)\|. \tag{3.189}$$

Then one gets

$$\begin{aligned} \|f\,|B_p^s(\mathbb{R}^n)\| &\sim \|\lambda^L(f)\,|b_p^s\| \\ &\lesssim \|\varkappa^L(f)\,|b_p^s\| + \|f\,|L_p(\mathbb{R}^n)\| \\ &\lesssim \|\lambda^L(f)\,|b_p^s\| + \|f\,|L_p(\mathbb{R}^n)\| \\ &\lesssim \|f\,|B_p^s(\mathbb{R}^n)\| \end{aligned} \tag{3.190}$$

since $B_p^s(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n)$. This proves the second assertion in (3.175). □

Remark 3.32. We used $s > 0$ only to make sure that the series in (3.172) converges absolutely in $L_p(\mathbb{R}^n)$ (with the indicated modification if $p = \infty$) and in connection with the above Step 3 proving the second characterisation in (3.175). But otherwise we did not need this assumption in the above Steps 1 and 2. As remarked, (3.165) are universal systems of normalised molecules in all spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$, including the unconditional convergence of series of type (3.172) in $S'(\mathbb{R}^n)$. Then one gets by Steps 1 and 2 the following generalisation of Theorems 3.26 and 3.31. We use the above notation. We only recall that we introduced $\mathcal{C}^{-\infty}(\mathbb{R}^n)$ in (3.128).

Corollary 3.33. *Let $1 < p \le \infty$, $L \in \mathbb{N}_0$ and $-\infty < s < 2L$.*

(i) *Then $f \in S'(\mathbb{R}^n)$ is an element of $B^s_p(\mathbb{R}^n)$ if, and only if, it can be represented as*

$$f = \sum_{\beta,j,m} \lambda^\beta_{jm}\, \Phi^{\beta,L}_{jm}, \qquad \lambda \in b^s_p, \tag{3.191}$$

unconditional convergence being in $S'(\mathbb{R}^n)$. Furthermore,

$$\|f \,|B^s_p(\mathbb{R}^n)\| \sim \inf \|\lambda \,|b^s_p\|, \tag{3.192}$$

where the infimum is taken over all admissible representations (3.191).

(ii) *Let $\lambda^{\beta,L}_{jm}$ be given by* (3.171). *Then any $f \in \mathcal{C}^{-\infty}(\mathbb{R}^n)$ can be represented as*

$$f = \sum_{\beta,j,m} \lambda^{\beta,L}_{jm}(f)\, \Phi^{\beta,L}_{jm}, \tag{3.193}$$

unconditional convergence being in $S'(\mathbb{R}^n)$. Furthermore,

$$B^s_p(\mathbb{R}^n) = \big\{ f \in \mathcal{C}^{-\infty}(\mathbb{R}^n) : \ \|\lambda^L(f) \,|b^s_p\| < \infty \big\} \tag{3.194}$$

(*equivalent norms*).

Proof. As mentioned above, the proof is covered by Steps 1 and 2 of the proof of Theorem 3.31. □

Remark 3.34. This corollary extends Theorem 3.26 from $-\infty < s < 0$ to $-\infty < s < 2L$, where $L \in \mathbb{N}_0$. If $L = 0$ then it follows from (3.171) and (3.139) that

$$\lambda^{\beta,0}_{jm}(f) = 2\lambda^\beta_{jm}(f). \tag{3.195}$$

The additional factor 2 is compensated by (3.161). Hence (3.193) with $L = 0$ coincides with (3.142).

3.2.5 Local smoothness theory

We wish to use Theorems 3.26, 3.31 and, in particular, Corollary 3.33, to say something about the local behavior of a given element $f \in B^s_p(\mathbb{R}^n)$ in dependence on the coefficients, say, $\lambda^{\beta,L}_{jm}(f)$ in (3.193). Again we follow [Tri03a], but in a simplified and more qualitative version. For this purpose we introduce some notation. Let Ω be a domain in $\mathbb{R}^n$. Then

$$B^s_p(\Omega), \qquad 0 < p \le \infty, \quad s \in \mathbb{R}, \tag{3.196}$$

is the restriction of

$$B^s_p(\mathbb{R}^n) = B^s_{pp}(\mathbb{R}^n), \qquad 0 < p \le \infty, \quad s \in \mathbb{R}, \tag{3.197}$$

to Ω. We use again the abbreviation (3.69) and rely on Definition 1.95 and the explanations given there. As usual,

$$B^{s,\mathrm{loc}}_p(\Omega), \qquad 0 < p \le \infty, \quad s \in \mathbb{R}, \tag{3.198}$$

is the collection of all $f \in D'(\Omega)$ such that the restriction of f to any bounded domain ω with $\bar{\omega} \subset \Omega$ belongs to $B^s_p(\omega)$, hence $f \in B^s_p(\omega)$ (with the usual abuse of notation).

We need the counterpart of these local function spaces on the sequence side for the coefficients $\lambda^{\beta,L}_{jm}(f)$ in (3.171), hence

$$\begin{aligned} \lambda^{\beta,L}_{jm}(f) = {} & 2^{jn} \int_{\mathbb{R}^n} k^{\beta,L}(2^j y - m)\, f(y)\, \mathrm{d}y \\ & + 2^{jn-2jL} \int_{\mathbb{R}^n} k^{\beta}(2^j y - m)\, f(y)\, \mathrm{d}y, \end{aligned} \tag{3.199}$$

where $f \in S'(\mathbb{R}^n)$, with the C^∞ function k according to (3.66), (3.67), and the abbreviations

$$k^\beta(x) = (2^{-J}x)^\beta k(x), \qquad k^{\beta,L}(x) = (-\Delta)^L k^\beta(x), \tag{3.200}$$

as introduced in (3.68) and (3.156). As before $\beta \in \mathbb{N}^n_0$, $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$. In particular,

$$\operatorname{supp} k^{\beta,L} \subset \operatorname{supp} k^\beta = \operatorname{supp} k \subset \{y \in \mathbb{R}^n : \ |y| < 2^J\} \tag{3.201}$$

for some $J \in \mathbb{N}$. We assume that J is fixed, maybe $J = n$ (which is luxurious). We do not indicate the dependence on J in what follows. As before, $B(x,r)$ is the ball centred at $x \in \mathbb{R}^n$ and of radius $r > 0$. The kernels of the local means in (3.199) have supports in balls $B(2^{-j}m, 2^{J-j})$. After these reminders setting the stage of what follows we introduce now the local counterparts of the sequence spaces b^s_p according to Definition 3.13.

Definition 3.35. *Let Ω be a domain in $\mathbb{R}^n$ and let (j,Ω) for $j \in \mathbb{N}_0$ be the collection of all $m \in \mathbb{Z}^n$ such that*

$$B\left(2^{-j}m,\, 2^{J-j}\right) \subset \Omega. \tag{3.202}$$

Let

$$\lambda = \left\{ \lambda^\beta_{jm} \in \mathbb{C} : \ j \in \mathbb{N}_0, \ \beta \in \mathbb{N}^n_0, \ m \in (j, \Omega) \right\} \tag{3.203}$$

and

$$\| \lambda \,| b^s_p(\Omega) \| = \left(\sum_{\beta \in \mathbb{N}^n_0} \sum_{j=0}^{\infty} \sum_{m \in (j,\Omega)} 2^{j(s-n/p)p} \, |\lambda^\beta_{jm}|^p \right)^{1/p} \tag{3.204}$$

(*with the usual modification if* $p = \infty$). *Then*

$$b^s_p(\Omega) = \{ \lambda : \ \| \lambda \,| b^s_p(\Omega) \| < \infty \}. \tag{3.205}$$

Furthermore, $b^s_p(\Omega)^{\mathrm{loc}}$ *is the collection of all* λ *according to* (3.203) *with* $\lambda \in b^s_p(\omega)$ *for any bounded domain* ω *with* $\bar{\omega} \subset \Omega$.

Remark 3.36. Recall that we do not indicate the dependence on J. One may replace 2^{J-j} in (3.202) by 2^{n-j}.

If $1 < p \le \infty$, $L \in \mathbb{N}_0$ and $-\infty < s < 2L$, then we have for any $f \in \mathcal{C}^{-\infty}(\mathbb{R}^n)$ the representation (3.193) with the explicit coefficients $\lambda^{\beta,L}_{jm}(f)$ according to (3.199) and the entire analytic functions

$$\Phi^{\beta,L}_{jm}(x) = \Phi^{\beta,L,j}_{M}(2^j x - m), \qquad x \in \mathbb{R}^n, \quad j \in \mathbb{N}, \quad m \in \mathbb{Z}^n, \tag{3.206}$$

according to (3.160) (modification if $j = 0$), which are elements of $S(\mathbb{R}^n)$. The local behavior of f in a given domain Ω in $\mathbb{R}^n$ will be described by

$$f_\Omega = \sum_{\beta, j, m \in (j,\Omega)} \lambda^{\beta,L}_{jm}(f) \, \Phi^{\beta,L}_{jm}, \tag{3.207}$$

with (3.199), (3.206) (modification if $j = 0$) and the same summations as in (3.203), (3.204). Otherwise we use the notation as introduced in Definition 3.35 naturally adapted to the sequences in (3.193), (3.207), hence with $\lambda^L(f)$ in place of λ. In connection with (3.198) we again write $f \in B^{\sigma,\mathrm{loc}}_p(\Omega)$ if $f \in S'(\mathbb{R}^n)$ and its restriction $f|\Omega$ to Ω belongs to $B^{\sigma,\mathrm{loc}}_p(\Omega)$. Let $C^\infty(\Omega)$ be the collection of all complex-valued C^∞ functions in Ω (nothing is assumed near the boundary). To avoid any misunderstanding we recall that the kernels $k^{\beta,L}$, k^β in (3.199) and $\Phi^{\beta,L}_{jm}$ have the same meaning as in the Definitions 3.29, 3.17 based on fixed admitted k, J, φ_0, ω and L.

Theorem 3.37. *Let* Ω *be a domain in* $\mathbb{R}^n$. *Let* $1 < p \le \infty$, $L \in \mathbb{N}_0$ *and* $-\infty < s < 2L$. *Let* $f \in B^s_p(\mathbb{R}^n)$.

(i) *Let* f *and* f_Ω *be represented by* (3.193) *and* (3.207). *Then*

$$f - f_\Omega \in C^\infty(\Omega). \tag{3.208}$$

(ii) *Let* $-\infty < s < \sigma < 2L$. *Then*

$$f \in B^{\sigma,\mathrm{loc}}_p(\Omega) \quad \textit{if, and only if,} \quad \lambda^L(f) \in b^\sigma_p(\Omega)^{\mathrm{loc}}. \tag{3.209}$$

Proof. Step 1. We begin with a preparation. Let

$$\langle x\rangle = (1+|x|^2)^{1/2}, \qquad x \in \mathbb{R}^n. \tag{3.210}$$

Let $\alpha \in \mathbb{N}_0^n$, $\varrho > 0$, and $d > 0$ be given. Let Φ^β be either $\Phi_F^{\beta,L}$ or $\Phi_M^{\beta,L,l}$ according to Definition 3.29. Then we claim that there is a constant $c = c(\alpha, \varrho, d, L)$ which is independent of β, l and x such that

$$|D^\alpha \Phi^\beta(x)| \le c\, 2^{-\varrho|\beta|}\, \langle x\rangle^{-d}, \qquad x \in \mathbb{R}^n, \quad \beta \in \mathbb{N}_0^n. \tag{3.211}$$

First we remark that according to (3.76) and (3.115), (3.117), (3.118),

$$|(\omega^\beta)^\vee(y)| \le c\, 2^{-\varkappa|\beta|}\, |D^\beta \omega^\vee(y)| \le c' 2^{-\varrho|\beta|} \langle y\rangle^{-a}, \qquad y \in \mathbb{R}^n, \tag{3.212}$$

where $\varkappa > 0$, and hence $\varrho > 0$, and $a > 0$ are at our disposal, and c, c' are independent of β. By (3.158), (3.159) and for fixed $\beta \in \mathbb{N}_0^n$, the functions $\Phi_F^{\beta,L}$ and $\Phi_M^{\beta,L,l}$ form a bounded set in $S(\mathbb{R}^n)$ with respect to l. We apply (3.83), (3.84), obviously modified to these functions and obtain by (3.212) uniformly with respect to l that

$$\begin{aligned} |D^\alpha \Phi^\beta(x)| &\le c\, 2^{-\varrho|\beta|} \sum_{m\in\mathbb{Z}^n} \langle m\rangle^{-a}\, \langle x+m\rangle^{-a} \\ &\le c' 2^{-\varrho|\beta|}\, \langle x\rangle^{-d}, \qquad x \in \mathbb{R}^n. \end{aligned} \tag{3.213}$$

This proves (3.211).

Step 2. We need a second preparation. Let f be given by (3.193) and let $f_{\beta,j}$ be the sum over $m \in \mathbb{Z}^n$ for fixed $\beta \in \mathbb{N}_0^n$ and $j \in \mathbb{N}_0$. Then with Φ^β being either $\Phi_F^{\beta,L}$ or $\Phi_M^{\beta,L,j}$ and (3.206),

$$|D^\alpha f_{\beta,j}(x)| \le \Big| \sum_{m\in\mathbb{Z}^n} \lambda_{jm}^{\beta,L}(f)\cdot (D^\alpha\Phi^\beta)(2^j x - m)\, 2^{j|\alpha|} \Big| \tag{3.214}$$

and by (3.194), (3.211),

$$|D^\alpha f_{\beta,j}(x)| \le c\, 2^{-j(s-n/p)+j|\alpha|}\, 2^{-\varrho|\beta|} \sum_{m\in\mathbb{Z}^n} \langle 2^j x - m\rangle^{-d}, \tag{3.215}$$

where $\varrho > 0$ and $d > 0$ are at our disposal. Then $f_{\beta,j}$ and also

$$f_j = \sum_{\beta\in\mathbb{N}_0^n} f_{\beta,j} \qquad \text{are } C^\infty \text{ in } \mathbb{R}^n, \quad j \in \mathbb{N}_0. \tag{3.216}$$

Step 3. We prove (3.208). Let $f = f_\Omega + f^\Omega$. We may assume that $0 \in \Omega$. Then it is sufficient to show that f^Ω is C^∞ in a neighborhood of 0. The sum in f^Ω is restricted to those (j,m) for which we do not have (3.202). In particular, $|m| \ge a2^j$ for some $a > 0$ and all j with $j \ge j_0$. However the terms f_j with $j < j_0$ are C^∞

and, hence, to prove (3.208) we may assume $j \geq j_0$ in what follows. Let $|x| \leq \varepsilon$ with $0 < \varepsilon < a$. Then it follows by (3.215) that

$$|D^\alpha f^\Omega_{\beta,j}(x)| \leq c\, 2^{-j(s-n/p)+j|\alpha|-\varrho|\beta|}\, 2^{-jd}, \qquad |x| \leq \varepsilon, \tag{3.217}$$

where $\varrho > 0$ and $d > 0$ are at our disposal. Now one is in the same position as in Step 2. Summation over $\beta \in \mathbb{N}^n_0$ and $j \geq j_0$ shows that f^Ω is C^∞ in a ball of radius ε centred at the origin. This proves (3.208).

Step 4. We prove (ii). Let $\lambda^L(f) \in b^\sigma_p(\Omega)^{\mathrm{loc}}$. Then $\lambda^L(f) \in b^\sigma_p(\omega)$ for any bounded domain ω with $\bar{\omega} \subset \Omega$. We split now f represented by (3.193) into $f = f_\omega + f^\omega$. Since $\sigma < 2L$ it follows by Corollary 3.33 that $f_\omega \in B^\sigma_p(\mathbb{R}^n)$. By the above considerations we have $f^\omega \in C^\infty(\omega)$. Both together gives $f \in B^{\sigma,\mathrm{loc}}_p(\Omega)$. Conversely, let $f \in B^{\sigma,\mathrm{loc}}_p(\Omega)$ and let ω as above. Then $f \in B^\sigma_p(\omega)$ and by Definition 1.95 there is an element $g \in B^\sigma_p(\mathbb{R}^n)$ such that $g|\omega = f|\omega$. One can apply Corollary 3.33 to g. Then one has the counterpart of (3.194) with g and b^σ_p in place of f and b^s_p. If the ball in (3.202) is a subset of ω then one has $\lambda^{\beta,L}_{jm}(g) = \lambda^{\beta,L}_{jm}(f)$. This applies to any domain ω with $\bar{\omega} \subset \Omega$ and, as a consequence,

$$\lambda^{\beta,L}_{jm}(g) = \lambda^{\beta,L}_{jm}(f) \qquad \text{if} \quad j \geq j(\omega). \tag{3.218}$$

However for $j < j(\omega)$ there is nothing to prove since the local improvement from s to σ is a matter of large j. One gets $\lambda^L(f) \in b^\sigma_p(\Omega)^{\mathrm{loc}}$. □

Remark 3.38. The above theorem is the qualitative version of the more quantitative considerations in [Tri03a] where we studied in greater detail the influence of the number J and where we expressed (3.209) more explicitly. But even the above version makes clear how to proceed

$$\textit{from global via local to pointwise smoothness.} \tag{3.219}$$

One may begin with the global Besov characteristics $s_f(t)$ according to (1.620) with $t = 1/p$, where one can replace $B^s_{p\infty}(\mathbb{R}^n)$ by $B^s_{pp}(\mathbb{R}^n) = B^s_p(\mathbb{R}^n)$ with the same outcome. Quite obviously there is a local counterpart, say, $s_f(\Omega, t)$ where Ω is a domain in $\mathbb{R}^n$. If $x^0 \in \mathbb{R}^n$ such that $x^0 \in \operatorname{sing\,supp} f$ according to (1.619) then

$$s_f(x^0, t) = \lim_{l \to \infty} s_f(\Omega_l, t) < \infty \quad \text{with} \quad \Omega_l = B(x^0, 2^{-l}) \tag{3.220}$$

makes sense. Theorem 3.37 suggests to calculate $s_f(\Omega, t)$, $s_f(\Omega_l, t)$ and $s_f(x^0, t)$ in terms of the local means (3.199) and the sequence spaces, say, $b^\sigma_p(\Omega_l)$. However there are some difficulties. Substantial progress has been made quite recently in [Scn05], [Scn06] converting these questions in a detailed study of related Besov spaces of varying pointwise smoothness. On the one hand, by the properties of $\Phi^{\beta,L}_{jm}$ both (3.193) and (3.207) are universal molecular representations for any $f \in S'(\mathbb{R}^n)$ (or at least $f \in \mathcal{C}^{-\infty}(\mathbb{R}^n)$ according to (3.128)), but on the other hand optimality of the coefficients (generating equivalent quasi-norms) according to Corollary 3.33 is only guaranteed if $1 < p \le \infty$ and $\sigma < 2L$. We refer to [Tri03a] where we dealt with these questions and problems in some detail. It might be of some interest that the flexibility of the basic kernel functions $k, k^\beta, k^{\beta,L}$ according to (3.66), (3.200) paves the way to deal with the more subtle

$$\textit{directional (or conical) local and pointwise smoothness.} \tag{3.221}$$

For this purpose one can replace Ω_l in (3.220) by the intersection of Ω_l and a given cone with vertex x^0. One can modify k in (3.66) in such a way that one needs in (3.199) only values of f within this cone. But this has not been done so far.

Remark 3.39. According to Theorems 1.64 and 3.5 one can expand any f belonging to $B^s_{pq}(\mathbb{R}^n)$or $F^s_{pq}(\mathbb{R}^n)$ uniquely in terms of the Daubechies wavelet basis $\{\Psi^{j,G}_m\}$. The coefficients $\lambda^{j,G}_m$ in (3.37) are again local means. The question whether one can extract from these wavelet coefficients an improved local smoothness of f compared with the given global one attracted a lot of attention. It is the same question as considered in part (ii) of the above Theorem 3.37 and Remark 3.38. We refer to [Jaf00], [Jaf01], [Jaf04], [Jaf05], [JaM96], [JMR01], [Mey98], [Mey01]. There one finds also further suggestions about how to measure local and pointwise smoothness and how questions of this type are related to problems in physics, science, and technology. Especially some basic ideas in [Jaf05] how to deal with problems of this type are similar as the above constructions.

Remark 3.40. One point seems to be of special interest when comparing Theorem 3.37, especially (3.209), with the literature mentioned in the preceding remark. Let $x^0 \in \mathbb{R}^n$ and let $\{\varphi_j\}$ be the same dyadic resolution of unity as in (2.8)–(2.10) and in Definition 2.1. Let $s \in \mathbb{R}$ and $s' \in \mathbb{R}$. Then the two-microlocal spaces

$$\mathcal{C}^{s,s'}(x^0, \mathbb{R}^n) = \Big\{ f \in S'(\mathbb{R}^n) : \ \|f \,|\mathcal{C}^{s,s'}(x^0, \mathbb{R}^n)\| < \infty \Big\} \tag{3.222}$$

with

$$\|f \,|\mathcal{C}^{s,s'}(x^0, \mathbb{R}^n)\| = \sup_{j \in \mathbb{N}_0, x \in \mathbb{R}^n} 2^{js} \left(1 + 2^j |x - x^0|\right)^{s'} |(\varphi_j \widehat{f})^\vee(x)| \tag{3.223}$$

are refinements of the Hölder-Zygmund spaces $\mathcal{C}^s(\mathbb{R}^n)$ according to Definition 2.1 and (1.10). These globally defined spaces have local counterparts $\mathcal{C}^{s,s',\mathrm{loc}}(x^0, \mathbb{R}^n)$ with respect to the same point x^0 similarly as in (3.198). We refer to [Mey98], Definitions 3.1 and 3.5, pp. 58, 67/68. Furthermore there are characterisations of these spaces in terms of wavelet coefficients based on Meyer or Daubechies wavelets as introduced in Section 1.7.3. In particular according to [Mey98], Theorem 3.6, p. 66, one can characterise the two-microlocal versions $\mathcal{C}^{s,s'}(x^0, \mathbb{R}^n)$ of the Hölder-Zygmund spaces $\mathcal{C}^s(\mathbb{R}^n) = B^s_{\infty\infty}(\mathbb{R}^n)$ as in Theorem 3.5(i) with $p = q = \infty$ and the coefficients

$$(1 + |2^j x^0 - m|)^{s'} \lambda^{j,G}_m = (1 + |2^j x^0 - m|)^{s'} \, 2^{jn/2} \left(f, \Psi^{j,G}_m\right). \tag{3.224}$$

In [JaM96] and [Mey98] these two-microlocal spaces of Hölder-Zygmund type have been used to study delicate local and pointwise regularity assertions for functions belonging to Sobolev-Besov spaces. Replacing the L_∞-norm in (3.223) by L_2-norms one gets the original version of two-microlocal spaces in [Bony84], [JaM96], p. 19. Recently corresponding two-microlocal spaces $B^{s,s'}_{pq}(x^0, \mathbb{R}^n)$ have been considered in [MoY04] again characterised as in Theorem 3.5(i) with the coefficients (3.224). On the other hand, Theorem 3.37(ii) is a satisfactory local smoothness theory based on the expansions (3.174), (3.207), with optimal coefficients originating from the simple local means (3.199)–(3.201). There are the factors $(2^j y - m)^\beta$ in the corresponding kernels which resemble the additional factors in (3.224). One could try to incorporate the extra factors in (3.224) in these kernels in the framework of a two-microlocal extension of Theorems 3.26, 3.31 and especially Corollary 3.33 and to develop a refined smoothness theory as in Theorem 3.37. But nothing has been done so far.

3.3 Complements

3.3.1 Gausslets

So far we dealt with several types of building blocks for the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$: atoms (molecules), quarks, wavelet bases and wavelet frames. One may ask to which extent other distinguished expansions in $\mathbb{R}^n$ fit in this scheme. First candidates might be the building blocks in the time-frequency analysis or Gabor analysis dealing with frames in $L_2(\mathbb{R}^n)$ of type

$$f(x) = \sum_{m\in\mathbb{Z}^n}\sum_{k\in\mathbb{Z}^n} c_{mk}\, e^{2\pi i b k\cdot x}\, g(x-am), \qquad x\in\mathbb{R}^n, \tag{3.225}$$

unconditional convergence being in $L_2(\mathbb{R}^n)$. Here g is a *window*,

$$g \in L_\infty(\mathbb{R}^n)\cap L_2(\mathbb{R}^n), \qquad a>0, \quad b>0, \tag{3.226}$$

with the most prominent example of the adapted Gauss-function $g(x) = e^{-\pi|x|^2}$. As explained in Remark 3.23 the question of whether (3.225) is a frame representation requires that there are distinguished coefficients $c_{mk}(f)$ depending linearly on f such that

$$\Big(\sum_{m,k} |c_{mk}(f)|^2\Big)^{1/2} \sim \|f\,|L_2(\mathbb{R}^n)\|. \tag{3.227}$$

We do not discuss these problems here and refer to [Gro01]. A short description of some aspects from the above point of view may also be found in [Tri02c], Section 2.1. We only mention that

$$f(x) = \sum_{m,k} c_{mk}\, e^{2\pi i bk\cdot x}\, e^{-\pi|x-am|^2}, \qquad x\in\mathbb{R}^n, \tag{3.228}$$

is a frame in $L_2(\mathbb{R}^n)$ if $ab > 0$ is small. One may ask whether this theory can be extended from $L_2(\mathbb{R}^n)$ to $L_p(\mathbb{R}^n)$ or to $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$. But there are serious obstacles which can be seen if one takes the Fourier transform of (3.225),

$$\widehat{f}(\xi) = \sum_{m,k} c'_{mk}\, e^{-iam\xi} \cdot \widehat{g}\,(\xi - 2\pi bk), \qquad \xi \in \mathbb{R}^n. \tag{3.229}$$

According to Definition 2.1 the structure of all spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ is characterised by dyadic resolutions on the Fourier side. This is also well reflected by all building blocks considered so far. But the above building blocks in (3.225), (3.226), (3.229) behave differently. Hence there is little hope to extend the Gabor analysis to spaces of type $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$. They do not fit in the g-philosophy as described in Section 1.19. However if one replaces on the Fourier side the dyadic annuli in Definition 2.1 by congruent cubes then one gets the so-called *modulation spaces* $M^s_{pq}(\mathbb{R}^n)$ where $s \in \mathbb{R}$ (or even a function), $0 < p \le \infty$, $0 < q \le \infty$, which are not the subject of this book. But (3.229) is well adapted to this type of space. The present state of art of these spaces and also references may be found in [Gro01], Sections 11-14. On the other hand one may ask whether one can take the Gauss-function subject to the procedures (1.627)–(1.629) to get building blocks in the space $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$. This can be done. We restrict ourselves here to a description of this theory referring for details and proofs to [Tri02c].

Definition 3.41. *Let*

$$H(x) = \sum_{m \in \mathbb{Z}^n} e^{-|x-m|^2/2} \tag{3.230}$$

and

$$G^\beta(x) = \frac{x^\beta}{\sqrt{\beta!}}\, e^{-|x|^2/2}\, H^{-1}(x), \qquad \beta \in \mathbb{N}^n_0, \quad x \in \mathbb{R}^n. \tag{3.231}$$

Let $s \in \mathbb{R}$, $0 < p \le \infty$, $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$. Then

$$G^\beta_{jm}(x) = 2^{-j(s-n/p)}\, G^\beta(2^j x - m), \qquad x \in \mathbb{R}^n, \tag{3.232}$$

are the (s,p)-β-Gausslets.

Remark 3.42. This is the direct counterpart of the regular (s,p)-β-quarks according to Definition 1.36 and (1.107) and of the functions k^β_{jm} in (3.88). As for basic notation we refer to Section 2.1.2. The Gausslets have no longer compact supports, but otherwise they behave similarly to the β-quarks. In particular one can prove by elementary calculation that

$$\max_{x \in \mathbb{R}^n} G^\beta(x) \sim \prod_{j=1}^n (1+\beta_j)^{-1/4}, \qquad \beta \in \mathbb{N}^n_0, \tag{3.233}$$

where the equivalence constants are independent of β, [Tri02c], p. 438. This can be complemented by

$$|D^\gamma G^\beta(x)| \le c\, 2^{\varepsilon|\beta|}, \qquad x \in \mathbb{R}^n, \quad |\gamma| \le K, \quad \beta \in \mathbb{N}^n_0, \tag{3.234}$$

where for given $\varepsilon > 0$ the constant c may depend on ε and K, but not on β and x. Some of the notation and considerations in connection with Definition 1.36, Theorem 1.39 and Remark 1.41 can be taken over. In particular, for $0 < p \le \infty$, $0 < q \le \infty$, $\varrho \in \mathbb{R}$, and

$$\lambda = \left\{ \lambda^\beta : \beta \in \mathbb{N}_0^n \right\} \quad \text{with} \quad \lambda^\beta = \left\{ \lambda^\beta_{jm} \in \mathbb{C} : j \in \mathbb{N}_0, \ m \in \mathbb{Z}^n \right\}, \tag{3.235}$$

let as in (1.108), (1.109),

$$\|\lambda \,|b_{pq}\|_\varrho = \sup_{\beta \in \mathbb{N}_0^n} 2^{\varrho|\beta|} \, \|\lambda^\beta \,|b_{pq}\| \tag{3.236}$$

and

$$\|\lambda \,|f_{pq}\|_\varrho = \sup_{\beta \in \mathbb{N}_0^n} 2^{\varrho|\beta|} \, \|\lambda^\beta \,|f_{pq}\|, \tag{3.237}$$

where $\| \cdot |b_{pq}\|$ and $\| \cdot |f_{pq}\|$ have been introduced in (1.64), (1.65). Let σ_p and σ_{pq} be as in (1.68) or (2.6). We are interested in representations of the type

$$f = \sum_{\beta,j,m} \lambda^\beta_{jm} \, G^\beta_{jm}, \qquad \|\lambda \,|b_{pq}\|_\varrho < \infty, \tag{3.238}$$

where G^β_{jm} are (s,p)-β-Gausslets with $0 < p \le \infty$, $s > \sigma_p$ and $\varrho > 0$. Here $\sum_{\beta,j,m}$ is the abbreviation introduced in (3.74) which must be justified. As in connection with (3.132), the related comments and references, and (3.234), for any $\varepsilon > 0$,

$$\left\{ 2^{-\varepsilon|\beta|} \, G^\beta_{jm} : j \in \mathbb{N}_0, \ m \in \mathbb{Z}^n \right\}, \qquad \beta \in \mathbb{N}_0^n, \tag{3.239}$$

are systems of normalised molecules, ignoring constants which are independent of j, m, β. Let $\bar{p} = \max(1,p)$. Now it follows by the same arguments (and the same interpretation) as in Remark 2.12, that the series in (3.238) converges absolutely in $L_{\bar{p}}(\mathbb{R}^n)$ (with the indicated modification if $p = \infty$) and, hence, unconditionally in $S'(\mathbb{R}^n)$. One has now the following counterpart of Theorem 1.39 and Corollary 1.42.

Theorem 3.43.

(i) *Let $\varrho > 0$,*

$$0 < p \le \infty, \quad 0 < q \le \infty, \quad s > \sigma_p. \tag{3.240}$$

Then $B^s_{pq}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ which can be represented as

$$f = \sum_{\beta,j,m} \lambda^\beta_{jm} \, G^\beta_{jm}, \qquad \|\lambda \,|b_{pq}\|_\varrho < \infty, \tag{3.241}$$

where G^β_{jm} are (s,p)-β-Gausslets, unconditional convergence being in $S'(\mathbb{R}^n)$. Furthermore,

$$\|f \,|B^s_{pq}(\mathbb{R}^n)\| \sim \inf \|\lambda \,|b_{pq}\|_\varrho \tag{3.242}$$

where the infimum is taken over all admitted representations (3.241)*. There are functions* $\Psi^{\beta,\varrho}_{jm} \in S(\mathbb{R}^n)$ *such that* f *can be represented by* (3.241) *with*

$$\lambda^{\beta}_{jm}(f) = \left(f, \Psi^{\beta,\varrho}_{jm}\right), \qquad j \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n, \quad \beta \in \mathbb{N}^n_0, \tag{3.243}$$

in place of λ^{β}_{jm} *and*

$$\|f \,|B^s_{pq}(\mathbb{R}^n)\| \sim \|\lambda(f)\,|b_{pq}\|_{\varrho}. \tag{3.244}$$

(ii) *Let* $\varrho > 0$,

$$0 < p < \infty, \quad 0 < q \le \infty, \quad s > \sigma_{pq}. \tag{3.245}$$

Then $F^s_{pq}(\mathbb{R}^n)$ *is the collection of all* $f \in S'(\mathbb{R}^n)$ *which can be represented as*

$$f = \sum_{\beta,j,m} \lambda^{\beta}_{jm}\, G^{\beta}_{jm}, \qquad \|\lambda\,|f_{pq}\|_{\varrho} < \infty, \tag{3.246}$$

where G^{β}_{jm} *are* (s,p)*-*β*-Gausslets, unconditional convergence being in* $S'(\mathbb{R}^n)$*. Furthermore,*

$$\|f \,|F^s_{pq}(\mathbb{R}^n)\| \sim \inf \|\lambda\,|f_{pq}\|_{\varrho} \tag{3.247}$$

where the infimum is taken over all admitted representations (3.246)*. Additionally,*

$$\|f \,|F^s_{pq}(\mathbb{R}^n)\| \sim \|\lambda(f)\,|f_{pq}\|_{\varrho} \tag{3.248}$$

for the representation of f *with* $\lambda^{\beta}_{jm} = \lambda^{\beta}_{jm}(f)$ *according to* (3.246) *and* (3.243)*.*

Remark 3.44. Since (3.239) are systems of normalised molecules one gets for some $c > 0$ the estimate

$$\|f \,|B^s_{pq}(\mathbb{R}^n)\| \le c\,\|\lambda\,|b_{pq}\|_{\varrho} \tag{3.249}$$

with f given by (3.241). As for the converse and the frame representation of f with (3.243), (3.244), we refer to [Tri02c]. Similarly for $F^s_{pq}(\mathbb{R}^n)$.

Remark 3.45. The restriction $s > \sigma_p$ for the B-spaces and $s > \sigma_{pq}$ for the F-spaces is the same as for the quarkonial decompositions in Theorem 1.39. Then one does not need moment conditions for the corresponding building blocks. As for the quarkonial decompositions we extended this theory in [Triϵ], Section 3, to all $s \in \mathbb{R}$. We refer also to Remark 1.47. This is the point where the procedure (1.630) (*differentiations*) is coming in, hence G^{β} in (3.231) must be complemented by

$$G^{\beta,L}(x) = (-\Delta)^L\, G^{\beta}(x) = e^{-|x|^2/2}\, P_{\beta,L}(x), \qquad \beta \in \mathbb{N}^n_0, \quad L \in \mathbb{N}, \tag{3.250}$$

where $P_{\beta,L}$ are distinguished polynomials. Then one can extend the above theorem to all $s \in \mathbb{R}$. This has been done in [Tri98] which is the first paper about this theory.

However it came out later on that there are problems with the convergence of some series. This has been corrected in [Tri02c], and Theorem 3.43 coincides with the main assertion of the latter paper. An extension to all $s \in \mathbb{R}$ based on (3.250) should be possible following [Tri98] appropriately modified. In connection with the above theorem we refer to [KyP01] where wavelet bases in homogeneous spaces of type B^s_{pq} and F^s_{pq} have been constructed which may have exponential decay or even a decay of type $e^{-|x|^2/2}$. There are also some applications to nonlinear approximation and one may ask whether the above theorem can be used for similar purposes.

3.3.2 Positivity

Using the lift (1.7) any element f of the Sobolev space $H^s_p(\mathbb{R}^n)$ with $s > 0$, $1 < p < \infty$, can be written as

$$f(x) = (I_{-s}\, I_s f)(x) = \int_{\mathbb{R}^n} G_{-s}(x-y)\,(I_s f)(y)\,\mathrm{d}y, \tag{3.251}$$

where $I_s f \in L_p(\mathbb{R}^n)$ and where $G_{-s}(y)$ are the positive Bessel potential kernels. As for the latter assertion we refer to [AdH96], Sections 1.2.4, 1.2.5, pp. 10–13. Then it follows from the corresponding property for the spaces $L_p(\mathbb{R}^n)$ that any $f \in H^s_p(\mathbb{R}^n)$ can be decomposed as

$$f = f_1 - f_2 + i f_3 - i f_4, \tag{3.252}$$

with $f_l \geq 0$, $f_l \in H^s_p(\mathbb{R}^n)$, and

$$\|f\,|H^s_p(\mathbb{R}^n)\| \sim \sum_{l=1}^{4} \|f_l\,|H^s_p(\mathbb{R}^n)\| \tag{3.253}$$

(equivalent norms). The question arises for which other spaces one has a similar decomposition property in non-negative elements.

Definition 3.46. *Let $s \in \mathbb{R}$, $0 < p \leq \infty$ ($p < \infty$ for the F-spaces) and $0 < q \leq \infty$. Let $A^s_{pq}(\mathbb{R}^n)$ be either $B^s_{pq}(\mathbb{R}^n)$ or $F^s_{pq}(\mathbb{R}^n)$ according to Definition* 2.1. *Then $A^s_{pq}(\mathbb{R}^n)$ is said to have the positivity property if any $f \in A^s_{pq}(\mathbb{R}^n)$ can be decomposed as*

$$f = f_1 - f_2 + i f_3 - i f_4 \qquad \text{with} \quad f_l \geq 0, \quad f_l \in A^s_{pq}(\mathbb{R}^n), \tag{3.254}$$

and

$$\|f\,|A^s_{pq}(\mathbb{R}^n)\| \sim \sum_{l=1}^{4} \|f_l\,|A^s_{pq}(\mathbb{R}^n)\|, \tag{3.255}$$

with equivalence constants which are independent of f.

Remark 3.47. Recall that $f \ge 0$ for $f \in S'(\mathbb{R}^n)$ means that

$$f(\varphi) \ge 0 \quad \text{for any real non-negative } \varphi \in S(\mathbb{R}^n). \tag{3.256}$$

By the above considerations the Sobolev spaces $H^s_p(\mathbb{R}^n)$ with $1 < p < \infty$ and $s \ge 0$ have the positivity property. We use σ_p and σ_{pq} as introduced in (2.6).

Theorem 3.48.

(i) *The spaces*

$$B^s_{pq}(\mathbb{R}^n) \qquad \textit{with} \quad 0 < p \le \infty, \quad 0 < q \le \infty, \quad s > \sigma_p, \tag{3.257}$$

and

$$F^s_{pq}(\mathbb{R}^n) \qquad \textit{with} \quad 0 < p < \infty, \quad 0 < q \le \infty, \quad s > \sigma_{pq}, \tag{3.258}$$

have the positivity property.

(ii) *The spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ with*

$$0 < p \le \infty \ (p < \infty \textit{ for the } F\textit{-spaces}), \quad 0 < q \le \infty, \quad s < \sigma_p, \tag{3.259}$$

do not have the positivity property.

Proof. Step 1. We prove (i). Let ψ be the same non-negative C^∞ function as in (1.105), (1.106) with the additional property

$$\operatorname{supp} \psi \subset \mathbb{R}^n_{++} = \{y \in \mathbb{R}^n : \ y = (y_1, \ldots, y_n), \ y_j > 0\}. \tag{3.260}$$

Then $(\beta\text{-qu})_{jm}(x) \ge 0$ in (1.107). We apply Theorem 1.39 and Corollary 1.42 to the spaces in (3.257), (3.258) and get

$$f = \sum_{\beta, j, m} \lambda^\beta_{jm}(f)\, (\beta\text{-qu})_{jm} \tag{3.261}$$

with

$$\|f \,|B^s_{pq}(\mathbb{R}^n)\| \sim \|\lambda(f)\,|b_{pq}\|_\varrho, \quad \|f\,|F^s_{pq}(\mathbb{R}^n)\| \sim \|\lambda(f)\,|f_{pq}\|_\varrho \tag{3.262}$$

in the notation used there. Recall that $a_+ = \max(a, 0)$ for $a \in \mathbb{R}$. Let f_1, f_2, f_3, f_4 be given by (3.261) with

$$(\operatorname{Re} \lambda^\beta_{jm}(f))_+, \quad (-\operatorname{Re} \lambda^\beta_{jm}(f))_+, \quad (\operatorname{Im} \lambda^\beta_{jm}(f))_+, \quad (-\operatorname{Im} \lambda^\beta_{jm}(f))_+ \tag{3.263}$$

in place of $\lambda^\beta_{jm}(f)$. Then (3.255) is an immediate consequence of Theorem 1.39.

Step 2. We prove (ii). Let $f \ge 0$ be a compactly supported element of $B^s_{pq}(\mathbb{R}^n)$ or $F^s_{pq}(\mathbb{R}^n)$ with (3.259). Then it follows from the Riesz representation theorem according to [Mat95], 1.16, p. 15 (and 1.11, p. 12) that there is a unique finite Radon measure μ with supp $\mu =$ supp f and

$$f(\varphi) = \int_{\mathbb{R}^n} \varphi(\gamma)\, \mu(\mathrm{d}\gamma), \qquad \varphi \in S(\mathbb{R}^n). \tag{3.264}$$

(More detailed versions of this famous assertion may be found in [Lang93], Chapter IX, Theorems 2.3 and 2.7, pp. 256, 264, and in [Mall95], II,2, Theorem 2.2, pp. 61/62.) We refer also to Section 1.12.2. It follows by Proposition 1.127 that $f \in B^0_{1,\infty}(\mathbb{R}^n)$. Then one obtains that any compactly supported element of a space $A^s_{pq}(\mathbb{R}^n)$ with the positivity property belongs also to $B^0_{1,\infty}(\mathbb{R}^n)$. But this is not possible if p, q and s are restricted by (3.259). This follows from Theorem 1.199(ii). □

Remark 3.49. We followed [Tri03e] where one finds also further results. In particular, all spaces $F^s_{pq}(\mathbb{R}^n)$ with

$$F^s_{pq}(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n) \tag{3.265}$$

have the positivity property. This applies to $s > n/p$ and to the limiting cases according to (1.204).

Chapter 4

Spaces on Lipschitz Domains, Wavelets and Sampling Numbers

4.1 Spaces on Lipschitz domains

4.1.1 Introduction

Recall that this book consists of two parts. Chapter 1 is a self-contained survey of some aspects of the recent theory of function spaces and its applications in continuation of [Triγ], Chapter 1, with the same heading. The second part covers the other chapters of this book. Both parts should be readable independently. This causes a mild overlapping as far as some basic definitions are concerned. But otherwise we use Chapter 1 as a source of references in the later chapters. The present Chapter 4 is an especially good example of this procedure. In Section 1.11 and in [Tri02a] we surveyed some aspects of the recent theory of function spaces on non-smooth domains in $\mathbb{R}^n$, especially on bounded Lipschitz domains. This will not be repeated here with the exception of a few basic definitions. We focus our attention now on wavelets and on the recovery problem of continuous functions in bounded Lipschitz domains and the corresponding optimal rates of convergence for sampling. In Section 4.1 we collect the underlying assertions for function spaces on bounded Lipschitz domains. Section 4.2 deals with wavelets. The indicated recovery problem will be considered in Section 4.3 in terms of sampling numbers. These results will be complemented in Section 4.4. In particular we compare the sampling numbers with other means to measure compactness such as approximation numbers and entropy numbers.

4.1.2 Definitions

First we recall some definitions from Sections 1.11.1, 1.11.4 for sake of completeness and independence as just explained in Section 4.1.1. Let Ω be an arbitrary domain in $\mathbb{R}^n$. As before we identify open sets with domains. Then $L_p(\Omega)$ with $0 < p < \infty$ is the usual quasi-Banach space of all complex-valued Lebesgue measurable functions in Ω such that

$$\|f\,|L_p(\Omega)\| = \left(\int_\Omega |f(x)|^p\,\mathrm{d}x\right)^{1/p} < \infty, \tag{4.1}$$

complemented by $L_\infty(\Omega)$, normed by

$$\begin{aligned}\|f\,|L_\infty(\Omega)\| &= \operatorname{ess\,sup}_{x\in\Omega} |f(x)| \\ &= \inf\{N:\ |\{x\in\Omega \text{ with } |f(x)|\ge N\}| = 0\}.\end{aligned} \tag{4.2}$$

As before, $|\Gamma|$ is the Lebesgue measure of a Lebesgue-measurable set Γ in $\mathbb{R}^n$. As usual, $D(\Omega) = C_0^\infty(\Omega)$ stands for the collection of all complex-valued infinitely differentiable functions in $\mathbb{R}^n$ with compact support in Ω. Let $D'(\Omega)$ be the dual space of all distributions on Ω. Let $g \in S'(\mathbb{R}^n)$, where $S'(\mathbb{R}^n)$ is the space of all tempered distributions according to Section 2.1.2. Then we denote by $g|\Omega$ its restriction on Ω,

$$g|\Omega \in D'(\Omega): \quad (g|\Omega)(\varphi) = g(\varphi) \quad \text{for } \varphi \in D(\Omega). \tag{4.3}$$

Let $A^s_{pq}(\mathbb{R}^n)$ be the spaces according to Definition 2.1 where either $A = B$ or $A = F$.

Definition 4.1. *Let Ω be a domain in $\mathbb{R}^n$ and let*

$$0 < p \le \infty, \quad 0 < q \le \infty, \quad s \in \mathbb{R}, \tag{4.4}$$

(with $p < \infty$ for the F-spaces). Then $A^s_{pq}(\Omega)$ is the collection of all $f \in D'(\Omega)$ such that there is a $g \in A^s_{pq}(\mathbb{R}^n)$ with $g|\Omega = f$. Furthermore,

$$\|f\,|A^s_{pq}(\Omega)\| = \inf \|g\,|A^s_{pq}(\mathbb{R}^n)\| \tag{4.5}$$

where the infimum is taken over all $g \in A^s_{pq}(\mathbb{R}^n)$ such that its restriction $g|\Omega$ to Ω coincides in $D'(\Omega)$ with f.

Remark 4.2. This definition coincides essentially with Definition 1.95(i). As remarked there $A^s_{pq}(\Omega)$ is a quasi-Banach space (Banach space if $p \ge 1$, $q \ge 1$). Otherwise one may consult Section 1.11 where we listed special cases, properties and references.

Next we introduce Lipschitz domains following essentially Section 1.11.4. Let $2 \le n \in \mathbb{N}$. Then

$$\mathbb{R}^{n-1} \ni x' \mapsto h(x') \in \mathbb{R} \tag{4.6}$$

is called a *Lipschitz function* (on $\mathbb{R}^{n-1}$) if there is a number $c > 0$ such that

$$|h(x') - h(y')| \le c\,|x' - y'| \quad \text{for all} \quad x' \in \mathbb{R}^{n-1}, \quad y' \in \mathbb{R}^{n-1}. \tag{4.7}$$

Definition 4.3. *Let $n \in \mathbb{N}$.*

(i) *A special Lipschitz domain in $\mathbb{R}^n$ with $n \geq 2$ is the collection of all points $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$ such that*

$$h(x') < x_n < \infty, \tag{4.8}$$

where $h(x')$ is a Lipschitz function according to (4.6), (4.7).

(ii) *A bounded Lipschitz domain in $\mathbb{R}^n$ with $n \geq 2$ is a bounded domain Ω in $\mathbb{R}^n$ where the boundary $\partial\Omega$ can be covered by finitely many open balls B_j in $\mathbb{R}^n$ with $j = 1, \ldots, J$, centred at $\partial\Omega$ such that*

$$B_j \cap \Omega = B_j \cap \Omega_j \quad \textit{for} \quad j = 1, \ldots, J, \tag{4.9}$$

where Ω_j are rotations of suitable special Lipschitz domains in $\mathbb{R}^n$.

(iii) *A bounded Lipschitz domain in the real line $\mathbb{R}$ is the interior of a finite union of disjoint bounded closed intervals.*

Remark 4.4. Of course we always assume that bounded Lipschitz domains are not empty. Again we refer to Section 1.11 for further information and in particular for properties and special cases of corresponding spaces $A^s_{pq}(\Omega)$ which will only be quoted when needed.

4.1.3 Further spaces, some embeddings

Sampling and the recovery of continuous functions comes from numerics. We deal here with these problems mainly within the scales of the spaces B^s_{pq} and F^s_{pq}. But there are a few other spaces which are of interest in this context and which are not covered by these scales. The most distinguished examples are L_1, L_∞, C and the corresponding smoothness spaces W^k_1, W^k_∞, C^k with $k \in \mathbb{N}$, built on them.

First we introduce these spaces on $\mathbb{R}^n$. Obviously, $L_p(\mathbb{R}^n)$ with $1 \leq p \leq \infty$ has the same meaning as above, normed by (4.1), (4.2) with $\Omega = \mathbb{R}^n$. Let $C(\mathbb{R}^n)$ be the collection of all complex-valued bounded uniformly continuous functions on $\mathbb{R}^n$, normed by

$$\| f \,| C(\mathbb{R}^n) \| = \sup_{x \in \mathbb{R}^n} |f(x)|. \tag{4.10}$$

Let $k \in \mathbb{N}_0$. Then

$$C^k(\mathbb{R}^n) = \{ f \in C(\mathbb{R}^n) : \; D^\alpha f \in C(\mathbb{R}^n), \; |\alpha| \leq k \}, \tag{4.11}$$

and for $1 \leq p \leq \infty$,

$$W^k_p(\mathbb{R}^n) = \{ f \in L_p(\mathbb{R}^n) : \; D^\alpha f \in L_p(\mathbb{R}^n), \; |\alpha| \leq k \}, \tag{4.12}$$

are the usual Sobolev spaces, always naturally normed. According to (1.3), (1.4),

$$W^k_p(\mathbb{R}^n) = F^k_{p,2}(\mathbb{R}^n) \quad \text{if} \quad 1 < p < \infty, \quad k \in \mathbb{N}_0. \tag{4.13}$$

Now we are also interested in the spaces with $p = 1$ and $p = \infty$. Then (4.13) does not hold.

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ according to Definition 4.3. Let $\bar{\Omega}$ be the closure of Ω. Then $C(\bar{\Omega})$ is the collection of all complex-valued continuous functions on $\bar{\Omega}$, normed by

$$\|f\,|C(\bar{\Omega})\| = \sup_{x\in\Omega} |f(x)| = \max_{x\in\bar{\Omega}} |f(x)|. \tag{4.14}$$

Then $C(\bar{\Omega})$ is the restriction of $C(\mathbb{R}^n)$ on Ω,

$$C(\bar{\Omega}) = C(\mathbb{R}^n)|\Omega. \tag{4.15}$$

If $k \in \mathbb{N}$, then

$$C^k(\bar{\Omega}) = \left\{ f \in C(\bar{\Omega}) :\ D^\alpha f \in C(\bar{\Omega}),\ |\alpha| \le k \right\}, \tag{4.16}$$

naturally normed. As for the Sobolev spaces $W^k_p(\Omega)$ with $1 \le p \le \infty$ and $k \in \mathbb{N}$ one has two possibilities, either as restriction of $W^k_p(\mathbb{R}^n)$ as in Definition 4.1, temporarily denoted by

$$W^k_p(\mathbb{R}^n)|\Omega, \quad 1 \le p \le \infty, \quad k \in \mathbb{N}, \tag{4.17}$$

or intrinsically,

$$W^k_p(\Omega) = \left\{ f \in L_p(\Omega) :\ \|f\,|W^k_p(\Omega)\| < \infty \right\} \tag{4.18}$$

with

$$\|f\,|W^k_p(\Omega)\| = \sum_{|\alpha|\le k} \|D^\alpha f\,|L_p(\Omega)\|. \tag{4.19}$$

Both possibilities coincide. In particular, one has according to (4.13) that

$$W^k_p(\Omega) = F^k_{p,2}(\Omega), \quad 1 < p < \infty, \quad k \in \mathbb{N}. \tag{4.20}$$

But the spaces with $p = 1$ and $p = \infty$ are also of interest. We formulate the outcome together with some consequences which will be useful later on.

Proposition 4.5. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ according to Definition 4.3. Let $1 \le p \le \infty$ and $k \in \mathbb{N}$. Then*

$$W^k_p(\mathbb{R}^n)|\Omega = W^k_p(\Omega). \tag{4.21}$$

Furthermore,

$$B^k_{1,1}(\Omega) \hookrightarrow W^k_1(\Omega) \hookrightarrow B^k_{1,\infty}(\Omega) \tag{4.22}$$

and

$$B^k_{\infty,1}(\Omega) \hookrightarrow C^k(\bar{\Omega}) \hookrightarrow W^k_\infty(\Omega) \hookrightarrow B^k_{\infty,\infty}(\Omega). \tag{4.23}$$

Proof. The assertion (4.21) is covered by Theorem 1.122 and the references given there. As for (4.22), (4.23) we first remark that

$$B^0_{1,1}(\mathbb{R}^n) \hookrightarrow L_1(\mathbb{R}^n) \hookrightarrow B^0_{1,\infty}(\mathbb{R}^n) \tag{4.24}$$

and

$$B^0_{\infty,1}(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n) \hookrightarrow B^0_{\infty,\infty}(\mathbb{R}^n). \tag{4.25}$$

We refer to [Triβ], Proposition 2.5.7, p. 89, [ET96], p. 44, with a reference to [SiT95]. This can be lifted from level 0 to level $k \in \mathbb{N}$. Restriction to Ω based on (4.21) and Definition 4.1 proves all inclusions in (4.22), (4.23) with exception of

$$C^k(\bar{\Omega}) \hookrightarrow W^k_\infty(\Omega). \tag{4.26}$$

But this is obvious by definition. □

Finally we collect a few basic assertions setting the stage for what follows. Recall that $A^s_{pq}(\Omega)$ are the spaces as introduced in Definition 4.1 and that σ_p is given by (2.6).

Proposition 4.6. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ according to Definition* 4.3.

(i) *Let $s_1 \in \mathbb{R}$, $s_2 \in \mathbb{R}$ and p_1,p_2,q_1,$q_2 \in (0,\infty]$ (with $p_l < \infty$ for the F-spaces). Then*

$$\mathrm{id}: \quad A^{s_1}_{p_1q_1}(\Omega) \hookrightarrow A^{s_2}_{p_2q_2}(\Omega) \tag{4.27}$$

is compact if, and only if,

$$s_1 - s_2 > n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ . \tag{4.28}$$

(ii) *Let $s \in \mathbb{R}$, $0 < p \le \infty$ (with $p < \infty$ for the F-spaces), $0 < q \le \infty$. Then*

$$\mathrm{id}: \quad A^s_{pq}(\Omega) \hookrightarrow C(\bar{\Omega}) \tag{4.29}$$

if, and only if,

$$\begin{cases} \text{either} & s > n/p, \\ \text{or} & A = B,\ s = n/p,\ 0 < q \le 1, \\ \text{or} & A = F,\ s = n/p,\ 0 < p \le 1. \end{cases} \tag{4.30}$$

(iii) *Let $0 < p \le \infty$ (with $p < \infty$ for the F-spaces), $0 < q \le \infty$ and $s > \sigma_p$. Then*

$$\mathrm{id}: \quad A^s_{pq}(\Omega) \hookrightarrow L_1(\Omega). \tag{4.31}$$

This embedding is compact.

Proof. The compactness of id in (4.27) and (4.28) is covered by Theorem 1.97 where one has even the sharper assertion (1.306) measuring the degree of compactness in terms of entropy numbers. We return to this point later on. The non-compactness assertion in part (i) means that id in (4.27) is not compact in all cases where id is continuous but (4.28) is not satisfied. But these are the limiting cases discussed at the end of Section 1.11.1. The corresponding embeddings are not compact as remarked at the beginning of Section 1.11.2. But one can also prove this assertion rather quickly shifting this question to sequence spaces. By (1.299) it is sufficient to care for the B-spaces. Then the non-compactness of the limiting embeddings follows now from the wavelet representation in Theorem 3.5, combined with Corollary 3.10 and its proof making clear that there is no compact embedding if $s_2 \geq s_1$. As for part (ii) with $s = n/p$ and $\mathbb{R}^n$ in place of Ω we refer to Theorem 1.73(ii). This can be extended to $s > n/p$ by elementary embedding. The restriction from $\mathbb{R}^n$ to Ω follows from Definition 4.1, (4.15) and the observation that the sharpness of these assertions is a local matter. As for the latter claim one may consult the references in Remark 1.74. The $\mathbb{R}^n$-counterpart of the continuous embedding in (4.31) can be found in [Triϵ], Theorem 11.2, pp. 168–169. Then one gets (4.31) by restriction. But (4.31) follows also from part (i) and Hölder's inequality including the compactness. □

Remark 4.7. If $s < \sigma_p$ then there is no continuous embedding of type (4.31). A final answer for the delicate limiting case $s = \sigma_p$ may be found in [Triϵ], Theorem 11.2, pp. 168–169. But this will not be needed here. As for further (sharp) embeddings one may consult the end of Section 1.11.1 and the references given there.

4.1.4 Intrinsic characterisations

For spaces $A^s_{pq}(\Omega)$ with $s > \sigma_p$ on bounded Lipschitz domains Ω in $\mathbb{R}^n$ we have the characterisations described in Theorem 1.118. We deal now with some modifications of the parts (ii) and (iii) of this theorem. Again for sake of completeness and independence we repeat first some notation introduced there and needed in what follows. Let

$$(\Delta^1_h f)(x) = f(x+h) - f(x), \qquad (\Delta^{l+1}_h f)(x) = \Delta^1_h \, (\Delta^l_h f)(x) \tag{4.32}$$

where $x \in \mathbb{R}^n$, $h \in \mathbb{R}^n$, $l \in \mathbb{N}$, are the iterated differences in $\mathbb{R}^n$.

Definition 4.8. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ according to Definition 4.3. Let*

$$M \in \mathbb{N}, \quad x \in \Omega, \quad 0 < t < \infty, \quad 0 < u \leq \infty. \tag{4.33}$$

Then

$$V^M_\Omega(x,t) = \{h \in \mathbb{R}^n : \ |h| < t, \ x + \tau h \in \Omega \text{ for } 0 \leq \tau \leq M\} \tag{4.34}$$

and for $x \in \Omega$,

$$d^{M,\Omega}_{t,u} f(x) = \begin{cases} \left(t^{-n} \int\limits_{h \in V^M_\Omega(x,t)} \left| \left(\Delta^M_h f\right)(x)\right|^u \, \mathrm{d}h \right)^{1/u} & \text{if } 0 < u < \infty, \\ \sup_{h \in V^M_\Omega(x,t)} \left| \left(\Delta^M_h f\right)(x)\right| & \text{if } u = \infty. \end{cases} \tag{4.35}$$

Remark 4.9. If one replaces Ω in (4.34), (4.35) by $\mathbb{R}^n$ then one has the ball means (1.377) in $\mathbb{R}^n$, and in Theorem 1.116 corresponding characterisations of some spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$. In case of bounded Lipschitz domains we have so far Theorem 1.118 where $V^M_\Omega(x,t)$ coincides with $V^M(x,t)$ in (1.385) (indicating now Ω). In particular, $V^M_\Omega(x,t)$ is the maximal star-shaped open subset of a ball of radius t, centred at the origin, such that $x + M \cdot V^M_\Omega(x,t) \subset \Omega$. Otherwise one may consult Remark 1.119 where we listed some papers dealing with characterisations of function spaces, preferably of B-spaces, in terms of differences. We wish to adapt the parts (ii) and (iii) of Theorem 1.118 to our later needs.

Let $M \in \mathbb{N}$. Let $\mathcal{P}^M(\mathbb{R}^n)$ be the space of all complex-valued polynomials in $\mathbb{R}^n$ of degree smaller than M and $\mathcal{P}^M(\Omega)$ be the restriction of $\mathcal{P}^M(\mathbb{R}^n)$ to the bounded (non-empty) Lipschitz domain Ω in $\mathbb{R}^n$. Let

$$\left\{ P^{\Omega,M}_j \right\}_{j=1}^{\dim^M} \quad \text{with} \quad \dim^M = \dim \mathcal{P}^M(\mathbb{R}^n) = \dim \mathcal{P}^M(\Omega) \tag{4.36}$$

be an $L_2(\Omega)$-orthonormal basis consisting of real polynomials in $\mathcal{P}^M(\Omega)$. Recall that $a_+ = \max(0,a)$ for $a \in \mathbb{R}$.

Theorem 4.10. *Let* Ω *be a bounded Lipschitz domain in* $\mathbb{R}^n$ *according to Definition* 4.3. *Let* Ω *be connected. Let* $d^{M,\Omega}_{t,u} f$ *be the means as introduced in Definition* 4.8 *and let* $\left\{ P^{\Omega,M}_j \right\}$ *be the above orthonormal polynomial basis of* $\mathcal{P}^M(\Omega)$.

(i) *Let* $0 < p \le \infty$, $0 < q \le \infty$, $1 \le u \le r \le \infty$, *and*

$$n \left(\frac{1}{p} - \frac{1}{r} \right)_+ < s < M \in \mathbb{N}. \tag{4.37}$$

Then $B^s_{pq}(\Omega)$ *is the collection of all* $f \in L_{\max(p,r)}(\Omega)$ *such that*

$$\begin{aligned} \|f \,|B^s_{pq}(\Omega)\|^*_{u,M} = {} & \sum_{j=1}^{\dim^M} \left| \int_\Omega f(x)\, P^{\Omega,M}_j(x)\, \mathrm{d}x \right| \\ & + \left(\int_0^1 t^{-sq} \left\| d^{M,\Omega}_{t,u} f \,|L_p(\Omega) \right\|^q \frac{\mathrm{d}t}{t} \right)^{1/q} < \infty \end{aligned} \tag{4.38}$$

in the sense of equivalent quasi-norms (*usual modification if* $q = \infty$).

(ii) *Let* $0 < p < \infty$, $0 < q \le \infty$, $1 \le u \le r \le \infty$ *and*

$$n\left(\frac{1}{\min(p,q)} - \frac{1}{r}\right)_+ < s < M \in \mathbb{N}. \tag{4.39}$$

Then $F^s_{pq}(\Omega)$ *is the collection of all* $f \in L_{\max(p,r)}(\Omega)$ *such that*

$$\begin{aligned} \|f \,|F^s_{pq}(\Omega)\|^*_{u,M} = &\sum_{j=1}^{\dim^M} \left| \int_\Omega f(x)\, P_j^{\Omega,M}(x)\,\mathrm{d}x \right| \\ &+ \left\| \left(\int_0^1 t^{-sq}\, (d^{M,\Omega}_{t,u} f)(\cdot)^q \,\frac{\mathrm{d}t}{t}\right)^{1/q} |L_p(\Omega)\right\| < \infty \end{aligned} \tag{4.40}$$

in the sense of equivalent quasi-norms (*usual modification if* $q = \infty$).

Proof. Step 1. By (4.40), (1.389) and Hölder's inequality we have

$$\|f \,|F^s_{pq}(\Omega)\|^*_{u,M} \preceq \|f \,|F^s_{pq}(\Omega)\|_{u,M} \tag{4.41}$$

where we used the abbreviation (1.390). Similarly for $B^s_{pq}(\Omega)$.

Step 2. We prove the converse of (4.41) by contradiction assuming that there is no positive constant c such that

$$\|f \,|L_{\bar p}(\Omega)\| \le c\, \|f \,|F^s_{pq}(\Omega)\|^*_{u,M}, \qquad f \in F^s_{pq}(\Omega). \tag{4.42}$$

Then there is a sequence of functions $\{f_j\}_{j=1}^\infty \subset F^s_{pq}(\Omega)$ such that

$$1 = \|f_j \,|L_{\bar p}(\Omega)\| > j\, \|f_j \,|F^s_{pq}(\Omega)\|^*_{u,M}, \qquad j \in \mathbb{N}. \tag{4.43}$$

By Proposition 4.6(i),(iii), the embedding of $F^s_{pq}(\Omega)$ into $L_{\bar p}(\Omega)$ is compact. In particular, the set $\{f_j\}$ is bounded in $F^s_{pq}(\Omega)$ and hence precompact in $L_{\bar p}(\Omega)$. We may assume that

$$f_j \to f \text{ in } L_{\bar p}(\Omega), \quad \text{hence} \quad \|f \,|L_{\bar p}(\Omega)\| = 1. \tag{4.44}$$

By (4.43) and (1.389) the sequence $\{f_j\}$ converges also in $F^s_{pq}(\Omega)$ and

$$\left(d^{M,\Omega}_{t,u} f\right)(x) = 0 \text{ in } \Omega, \qquad \int_\Omega f(x)\, P_l^{\Omega,M}(x)\,\mathrm{d}x = 0 \tag{4.45}$$

for $l = 1, \ldots, \dim^M$. Then we have also $\left(d^{N,\Omega}_{t,u} f\right)(x) = 0$ for all $M < N \in \mathbb{N}$. Since (1.389) is a characterisation it follows that $f \in F^\sigma_{pq}(\Omega)$ for any $\sigma \in \mathbb{R}$. By well-known embedding theorems of type (4.27), say, with $A^{s_2}_{p_2 q_2} = \mathcal{C}^{s_2}$, it follows that $D^\alpha f \in C(\bar\Omega)$ for all $\alpha \in \mathbb{N}_0^n$. Hence $f \in C^\infty(\bar\Omega)$ and we have $(\Delta^M_h f)(x) = 0$ locally. Then f must be locally a polynomial of degree less than M. We add a comment about this point in Remark 4.11 below. Since we assumed that Ω is connected it follows that f is also globally in Ω a polynomial of degree less than M, hence $f \in \mathcal{P}^M(\Omega)$. Now one gets from the second part of (4.45) that $f = 0$. This contradicts (4.44). Similarly for the B-spaces. $\square$

Remark 4.11. We used that functions $f \in C^\infty(\bar{\Omega})$ with $(\Delta_h^M f)(x) = 0$ locally must be (locally) polynomials of degree less than M. This follows from distinguished integral representations of functions and its derivatives in terms of differences as it may be found in [Triγ], Proposition 3.3.2, pp. 174/175, with the outcome $(D^\beta f)(x) = 0$ for all $|\beta| = M$. Hence f is locally a polynomial of degree less than M. We add a second comment about the additional assumption that Ω is assumed to be connected. By Definition 4.3 an arbitrary bounded Lipschitz domain in $\mathbb{R}^n$ has finitely many, say $L \in \mathbb{N}$, connected components. One can apply Theorem 4.10 to each of these components. Then one clips together the outcome by extending the polynomial basis (4.36) in each of these components by zero to the other components, if $L > 1$. One gets a basis consisting of $L \cdot \dim^M$ elements and corresponding characterisations of the spaces $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$ in the above theorem.

Corollary 4.12. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ according to Definition* 4.3. *Let Ω be connected. Let $d^{M,\Omega}_{t,u} f$ be the means as introduced in Definition* 4.8 *and let $\{P_j^{\Omega,M}\}$ be the orthonormal basis* (4.36) *of $\mathcal{P}^M(\Omega)$. Let $0 < p \le \infty$, $\bar{p} = \max(p,1)$ and let for $f \in L_{\bar{p}}(\Omega)$,*

$$g_f(x) = \sum_{j=1}^{\dim^M} a_j \, P_j^{\Omega,M}(x) \quad \textit{with} \quad a_j = \int_\Omega f(x)\, P_j^{\Omega,M}(x)\,\mathrm{d}x. \tag{4.46}$$

(i) *Let p, q, u, r and also s, M be as in Theorem* 4.10(i). *Let $f \in B^s_{pq}(\Omega)$. Then*

$$\begin{aligned} \inf_{g \in \mathcal{P}^M(\Omega)} \|f - g\,|B^s_{pq}(\Omega)\|^*_{u,M} &= \|f - g_f\,|B^s_{pq}(\Omega)\|^*_{u,M} \\ &= \left(\int_0^1 t^{-sq}\, \|d^{M,\Omega}_{t,u} f\,|L_p(\Omega)\|^q \,\frac{\mathrm{d}t}{t} \right)^{1/q}. \end{aligned} \tag{4.47}$$

(ii) *Let p, q, u, r and also s, M be as in Theorem* 4.10(ii). *Let $f \in F^s_{pq}(\Omega)$. Then*

$$\begin{aligned} &\inf_{g \in \mathcal{P}^M(\Omega)} \|f - g\,\|F^s_{pq}(\Omega)\|^*_{u,M} = \|f - g_f\,|F^s_{pq}(\Omega)\|^*_{u,M} \\ &= \left\| \left(\int_0^1 t^{-sq} \left(d^{M,\Omega}_{t,u} f \right)(\cdot)^q \,\frac{\mathrm{d}t}{t} \right)^{1/q} |L_p(\Omega) \right\|. \end{aligned} \tag{4.48}$$

Proof. This follows immediately from Theorem 4.10 and the assumption that $\{P_j^{\Omega,M}\}$ is a real orthonormal basis in $\mathcal{P}^M(\Omega)$. □

Corollary 4.13. *Let $d^{M,\tau}_{t,u} f$ be the means according to* (4.35) *with respect to the balls*

$$\Omega = \omega_\tau = \{x \in \mathbb{R}^n : \ |x| < \tau\}. \tag{4.49}$$

(i) *Let $0 < p \le \infty$, $0 < q \le \infty$, $1 \le u \le \infty$ and*

$$n/p < s < M \in \mathbb{N}. \tag{4.50}$$

Then there is a positive constant c such that for all τ with $0<\tau\le 1$ and all $f\in B^s_{pq}(\omega_\tau)$,

$$\inf_{g\in\mathcal{P}^M(\omega_\tau)}\sup_{|x|<\tau}|f(x)-g(x)| \le c\,\tau^{s-n/p}\left(\int_0^\tau t^{-sq}\,\|d^{M,\tau}_{t,u}f\,|L_p(\omega_\tau)\|^q\,\frac{\mathrm{d}t}{t}\right)^{1/q}. \tag{4.51}$$

(ii) *Let $0<p<\infty$, $0<q\le\infty$, $1\le u\le\infty$, and*

$$n/\min(p,q)<s<M\in\mathbb{N}. \tag{4.52}$$

Then there is a positive constant c such that for all τ with $0<\tau\le 1$ and all $f\in F^s_{pq}(\omega_\tau)$,

$$\inf_{g\in\mathcal{P}^M(\omega_\tau)}\sup_{|x|<\tau}|f(x)-g(x)| \le c\,\tau^{s-n/p}\left\|\left(\int_0^\tau t^{-sq}\,\left(d^{M,\tau}_{t,u}f\right)(\cdot)^q\,\frac{\mathrm{d}t}{t}\right)^{1/q}|L_p(\omega_\tau)\right\|. \tag{4.53}$$

Proof. We prove part (ii). The proof of part (i) is the same. Let $f\in F^s_{pq}(\omega_\tau)$. Then $f(\tau\cdot)\in F^s_{pq}(\omega)$ where $\omega=\omega_1$ is the unit ball. Let $g\in\mathcal{P}^M(\omega_\tau)$ be such that $g(\tau\cdot)\in\mathcal{P}^M(\omega)$ is the optimal polynomial according to (4.46), (4.48) for $f(\tau\cdot)$ and $\Omega=\omega$. It follows by (4.29), (4.30) and (4.48) that

$$\sup_{|x|<\tau}|f(x)-g(x)| = \sup_{|x|<1}|f(\tau x)-g(\tau x)| \preceq \left\|\left(\int_0^1 t^{-sq}\,d^{M,1}_{t,u}f(\tau\cdot)(\cdot)^q\,\frac{\mathrm{d}t}{t}\right)^{1/q}|L_p(\omega)\right\|. \tag{4.54}$$

By (4.35) we have for $|x|<1$ and $0<t<1$, $0<\tau<1$ (and $0<u<\infty$, usual modification if $u=\infty$),

$$\begin{aligned}\left(d^{M,1}_{t,u}f(\tau\cdot)\right)^u(x) &= t^{-n}\int\limits_{h\in V^M_\omega(x,t)}\left|\left(\Delta^M_h f(\tau\cdot)\right)(x)\right|^u\mathrm{d}h\\ &= (\tau t)^{-n}\int\limits_{\tau h\in V^M_\omega(\tau x,\tau t)}\left|\left(\Delta^M_{\tau h}f\right)(\tau x)\right|^u\tau^n\,\mathrm{d}h\\ &= \left(d^{M,\tau}_{\tau t,u}f\right)^u(\tau x).\end{aligned} \tag{4.55}$$

Inserting (4.55) in (4.54) one gets (4.53). □

Remark 4.14. By construction $g(\tau\cdot)$ are optimal polynomials for $f(\tau\cdot)$ according to (4.46). Hence they depend linearly on f. In this subsection we followed [NoT04].

4.2 Wavelet para-bases

4.2.1 Wavelets in Euclidean n-space, revisited

In Section 3.1 we dealt with wavelet bases in $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$. In the present Section 4.2 we try to shift this theory from $\mathbb{R}^n$ to bounded Lipschitz domains in $\mathbb{R}^n$ following partly [Tri06a]. Wavelet bases and wavelet frames on intervals, domains and other structures attracted a lot of attention. We commented briefly on this topic at the beginning of Section 1.8 including a (rather incomplete) list of relevant references. This will not be repeated here. We combine here what has been said in Section 3.1 about Daubechies wavelets in $\mathbb{R}^n$ with a few (more or less sophisticated) properties of function spaces on $\mathbb{R}^n$ and in domains characterised by such key words as *scaling properties, refined localisation* and (*boundary*) *atoms*, in a purely qualitative way. In other words one can replace the Daubechies wavelets by any other (orthogonal or bi-orthogonal) wavelet bases or wavelet frames as long as the corresponding building blocks meet the qualitative requirements. But we will not stress this point. Just the contrary. First we return to Section 3.1 adapting and modifying what has been said there to what follows. As at the beginning of Section 3.1.1 and in Theorem 1.61(ii) we always assume that

$$\psi_F \in C^k(\mathbb{R}) \quad \text{and} \quad \psi_M \in C^k(\mathbb{R}) \qquad \text{with} \quad k \in \mathbb{N} \tag{4.56}$$

are the real compactly supported scaling function ψ_F and the real compactly supported associated wavelet ψ_M on the real line $\mathbb{R}$. Now we adapt (3.1)–(3.5) to our later needs. Let $n \in \mathbb{N}$ and $l \in \mathbb{N}_0$. Let

$$G = (G_1, \ldots, G_n) \in G^{l,l} = \{F, M\}^n \tag{4.57}$$

if G_r is either F or M. Let

$$G = (G_1, \ldots, G_n) \in G^{j,l} = \{F, M\}^{n*}, \qquad l < j \in \mathbb{N}, \tag{4.58}$$

if G_r is either F or M and where * indicates that at least one of the components of G must be an M. The cardinal number of $G^{l,l}$ is 2^n and the cardinal number of $G^{j,l}$ with $j > l$ is $2^n - 1$. Let

$$\Psi_{G,m}(x) = 2^{Ln/2} \prod_{r=1}^{n} \psi_{G_r}\left(2^L x_r - m_r\right), \quad G \in G^{l,l}, \quad m \in \mathbb{Z}^n, \tag{4.59}$$

and

$$\Psi^{j,l}_{G,m}(x) = 2^{jn/2} \Psi_{G,m}(2^j x), \quad j \ge l, \quad G \in G^{j,l}, \quad m \in \mathbb{Z}^n. \tag{4.60}$$

Here $L \in \mathbb{N}_0$ and $l \in \mathbb{N}_0$ are (independently) at our disposal and will be fixed later on in connection with Whitney decompositions of bounded Lipschitz domains. If $L = l = 0$ then we have the orthonormal wavelet basis in $L_2(\mathbb{R}^n)$ according to (3.2)–(3.5). By the multiresolution analysis as described in Section 1.7.1 this assertion remains valid for all $L \in \mathbb{N}_0$ and all $l \in \mathbb{N}_0$. Hence, *for fixed* $L \in \mathbb{N}_0$ *and* $l \in \mathbb{N}_0$*,*

$$\left\{ \Psi^{j,l}_{G,m} : \ j \geq l, \ G \in G^{j,l}, \ m \in \mathbb{Z}^n \right\} \tag{4.61}$$

is an orthonormal basis in $L_2(\mathbb{R}^n)$ *of compactly supported real* n*-dimensional Daubechies wavelets.*

A little bit in contrast to (3.5) we clip together (not only notationally as we shall see later on) the n-dimensional scaling function and the wavelet $\Psi^{l,l}_{G,0}$ with $G \in G^{l,l}$ at the same basic level 2^{l+L} of dilation. The notation $\Psi^{j,l}_{G,m}$ is somewhat luxurious since these functions do not depend on l at least as far as scaling is concerned. But later on we fix L once and for all and deal simultaneously with all $l \in \mathbb{N}_0$. Then it might be good to know where the diverse wavelets are coming from. By the above comments we have for fixed L and l in $L_2(\mathbb{R}^n)$ the orthonormal wavelet expansions

$$f = \sum_{j \geq l} \sum_{G \in G^{j,l}} \sum_{m \in \mathbb{Z}^n} \lambda^{j,G,l}_m \, 2^{-jn/2} \, \Psi^{j,l}_{G,m} \tag{4.62}$$

with

$$\lambda^{j,G,l}_m = \lambda^{j,G,l}_m(f) = 2^{jn/2} \left(f, \Psi^{j,l}_{G,m} \right), \tag{4.63}$$

where again the latter is the scalar product in $L_2(\mathbb{R}^n)$. If $L = l = 0$ then we have Theorem 3.5. By the above comments one can extend these assertions to all $L \in \mathbb{N}_0$, $l \in \mathbb{N}_0$. This is quite clear, but it seems to be reasonable to give an explicit formulation. First we modify Definition 3.1 introducing related sequence spaces. Let

$$l \in \mathbb{N}_0, \quad s \in \mathbb{R}, \quad 0 < p \leq \infty, \quad 0 < q \leq \infty, \tag{4.64}$$

and

$$\lambda = \left\{ \lambda^{j,G}_m \in \mathbb{C} : \ j \geq l, \ G \in G^{j,l}, \ m \in \mathbb{Z}^n \right\}. \tag{4.65}$$

Then $b^s_{pq,l}$ is the collection of all sequences (4.65) quasi-normed by

$$\| \lambda \, | b^s_{pq,l} \| = \left(\sum_{j \geq l} 2^{j(s-n/p)q} \sum_{G \in G^{j,l}} \left(\sum_{m \in \mathbb{Z}^n} |\lambda^{j,G}_m|^p \right)^{q/p} \right)^{1/q} \tag{4.66}$$

(usual modification if p and/or q is infinite), and $f^s_{pq,l}$ is the collection of all sequences (4.65) quasi-normed by

$$\| \lambda \, | f^s_{pq,l} \| = \left\| \left(\sum_{j,G,m} 2^{jsq} \, |\lambda^{j,G}_m \chi_{jm}(\cdot)|^q \right)^{1/q} | L_p(\mathbb{R}^n) \right\| \tag{4.67}$$

(usual modification if $q = \infty$). Here χ_{jm} is the same characteristic function as in connection with (3.8). The summation over j, G, m in (4.67) is the same as in (4.66).

Proposition 4.15. *Let $L \in \mathbb{N}_0$ and $l \in \mathbb{N}_0$.*

(i) *Let*

$$0 < p \le \infty,\ 0 < q \le \infty,\ s \in \mathbb{R},\ k > \max\left(s, \frac{2n}{p} + \frac{n}{2} - s\right) \tag{4.68}$$

in (4.56). *Then $f \in S'(\mathbb{R}^n)$ is an element of $B^s_{pq}(\mathbb{R}^n)$ if, and only if, it can be represented by* (4.62) *with $\lambda \in b^s_{pq,l}$. Furthermore, if $f \in B^s_{pq}(\mathbb{R}^n)$ then the representation* (4.62) *is unique with $\lambda = \lambda(f) = \{\lambda^{j,G,l}_m(f)\}$ according to* (4.63) *and $f \mapsto \lambda(f)$ is an isomorphic map of $B^s_{pq}(\mathbb{R}^n)$* **onto** *$b^s_{pq,l}$.*

(ii) *Let*

$$0 < p < \infty,\ 0 < q \le \infty,\ s \in \mathbb{R},\ k > \max\left(s, \frac{2n}{\min(p,q)} + \frac{n}{2} - s\right) \tag{4.69}$$

in (4.56). *Then $f \in S'(\mathbb{R}^n)$ is an element of $F^s_{pq}(\mathbb{R}^n)$ if, and only if, it can be represented by* (4.62) *with $\lambda \in f^s_{pq,l}$. Furthermore, if $f \in F^s_{pq}(\mathbb{R}^n)$ then the representation* (4.62) *is unique with $\lambda = \lambda(f) = \{\lambda^{j,G,l}_m(f)\}$ according to* (4.63) *and $f \mapsto \lambda(f)$ is an isomorphic map of $F^s_{pq}(\mathbb{R}^n)$* **onto** *$f^s_{pq,l}$.*

Remark 4.16. This is a modification of Theorem 3.5 based on the above remarks about the multiresolution analysis. All other arguments remain unchanged. This applies in particular to the rather careful discussions in Section 3.1.3 and Theorem 3.5 about the unconditional convergence of (4.62). This will not be repeated here. If $p < \infty$, $q < \infty$ then (4.62), (4.63) are unconditional Schauder bases in the corresponding spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$. *

4.2.2 Scaling properties

Our approach to find suitable counterparts of Theorem 3.5 for spaces on bounded Lipschitz domains in $\mathbb{R}^n$ is based on scaling properties and refined localisations of some F-spaces. First we deal with scaling properties.

Preparing our later notation, let for $l \in \mathbb{N}_0$,

$$Q^0_l \subset Q^1_l \subset Q^2_l \subset Q_l, \tag{4.70}$$

be open cubes with sides parallel to the axes of coordinates, centred at the origin and with respective side-lengths $2^{-l}, 5 \cdot 2^{-l-2}, 6 \cdot 2^{-l-2}, 2^{-l+1}$. By (4.59), (4.60) the functions $\Psi^{j,l}_{G,m}$ have compact supports centred at $2^{-L-j}m$ and of diameter $\sim 2^{-L-j}$. Now we fix $L \in \mathbb{N}_0$ once and for all such that

$$\operatorname{supp} \Psi^{j,l}_{G,m} \subset Q_l \quad \text{if } 2^{-L-j}m \in Q^2_l \text{ for } l \in \mathbb{N}_0 \text{ and } j \ge l, \tag{4.71}$$

and

$$2^{-L-j}m \in Q^2_l \quad \text{if } Q^1_l \cap \operatorname{supp} \Psi^{j,l}_{G,m} \ne \emptyset \text{ for } l \in \mathbb{N}_0 \text{ and } j \ge l. \tag{4.72}$$

*As far as k in (4.68), (4.69) is concerned we refer to the footnote on p. 156.

This can be done first for $l = 0$. Then it follows for any $l \in \mathbb{N}_0$ by the structure of $\Psi^{j,l}_{G,m}$. Of course, L depends on the supports of ψ_F and ψ_M in (4.56) and on k. By the well-known dependence of the diameters of the supports of the Daubechies wavelets in $\mathbb{R}$ on $k \in \mathbb{N}$ one may even assume $L = bk$ for some $b \in \mathbb{N}$ which is independent of k. But this will not be needed. Let $F^s_{\infty\infty} = B^s_{\infty\infty}$.

Proposition 4.17. *Let*

$$0 < p \le \infty, \ 0 < q \le \infty, \ s > \sigma_{pq} = \left(\frac{1}{\min(p,q)} - 1 \right)_+ \tag{4.73}$$

with $q = \infty$ *if* $p = \infty$.

(i) *Let* $0 < t \le 1$ *and*

$$f \in F^s_{pq}(\mathbb{R}^n) \quad \textit{with} \quad \operatorname{supp} f \subset \{ y \in \mathbb{R}^n : \ |y| < t \}. \tag{4.74}$$

Then

$$\| f(t\cdot) \,|F^s_{pq}(\mathbb{R}^n)\| \sim t^{s-n/p} \, \| f \,|F^s_{pq}(\mathbb{R}^n)\|, \tag{4.75}$$

where the equivalence constants are independent of t *and* f.

(ii) *Let, in addition,* $k \in \mathbb{N}$ *be chosen as in* (4.69). *Let* (*afterwards*) L *be fixed as above. Then*

$$f \in F^s_{pq}(\mathbb{R}^n) \quad \textit{with} \quad \operatorname{supp} f \subset Q^1_l, \quad l \in \mathbb{N}_0, \tag{4.76}$$

can be represented by (4.62), (4.63) *with*

$$\| f \,|F^s_{pq}(\mathbb{R}^n)\| \sim \| \lambda(f) \,| f^s_{pq,l} \| \tag{4.77}$$

according to (4.67), *where the equivalence constants are independent of* l *and* f.

Proof. Step 1. Part (i) follows immediately from [Triϵ, Corollary 5.16, p. 66, and Proposition 5.5, p. 45], where the latter coincides essentially with (1.336)–(1.338) for $A = F$.

Step 2. For fixed $l \in \mathbb{N}_0$ we have (4.62), (4.63), and (4.77) for all $f \in F^s_{pq}(\mathbb{R}^n)$. Hence it remains to prove that under the restriction (4.76) the equivalence constants in (4.77) can be chosen independently of $l \in \mathbb{N}_0$ (and f). Let f be given by (4.76). Then we apply (4.62), (4.63) with 0 in place of l to $f(2^{-l}x)$, hence

$$f(2^{-l}x) = \sum_{j \ge 0} \sum_{G \in G^{j,0}} \sum_{m \in \mathbb{Z}^n} \lambda^{j,G,0}_m \big(f(2^{-l}\cdot) \big) \, 2^{-jn/2} \, \Psi^{j,0}_{G,m}(x) \tag{4.78}$$

and

$$f(x) = \sum_{j \ge 0} \sum_{G \in G^{j,0}} \sum_{m \in \mathbb{Z}^n} \lambda^{j,G,0}_m \big(f(2^{-l}\cdot) \big) \, 2^{-jn/2} \, \Psi^{j,0}_{G,m}(2^l x). \tag{4.79}$$

We have by (4.60) that

$$\Psi^{j,0}_{G,m}(2^l x) = 2^{-ln/2} \, \Psi^{j+l,l}_{G,m}(x), \quad G \in G^{j,0} = G^{j+l,l}, \tag{4.80}$$

and by (4.63), (4.80) that

$$\begin{aligned}\lambda_m^{j,G,0}\big(f(2^{-l}\cdot)\big) &= 2^{jn/2}\int_{\mathbb{R}^n} f(2^{-l}x)\,\Psi^{j,0}_{G,m}(x)\,\mathrm{d}x \\ &= 2^{n(j-l)/2}\int_{\mathbb{R}^n} f(2^{-l}x)\,\Psi^{j+l,l}_{G,m}(2^{-l}x)\,\mathrm{d}x \\ &= 2^{n(j+l)/2}\int_{\mathbb{R}^n} f(x)\,\Psi^{j+l,l}_{G,m}(x)\,\mathrm{d}x \\ &= \lambda_m^{j+l,G,l}(f).\end{aligned} \tag{4.81}$$

We insert (4.80), (4.81) in (4.79) and replace $j+l$ by j. Then one gets

$$f(x) = \sum_{j\ge l}\sum_{G\in G^{j,l}}\sum_{m\in\mathbb{Z}^n} \lambda_m^{j,G,l}(f)\,2^{-jn/2}\,\Psi^{j,l}_{G.m}(x), \tag{4.82}$$

arriving again at (4.62). But now we get by (4.67) and (4.81) that

$$\begin{aligned}&\|\lambda\big(f(2^{-l}\cdot)\big)\,|f^s_{pq,0}\| \\ &= \Bigg\| \Bigg(\sum_{j\ge 0}\sum_{G\in G^{j,0}}\sum_{m\in\mathbb{Z}^n} 2^{jsq}\,\big|\lambda_m^{j+l,G,l}(f)\,\chi_{j+l,m}(2^{-l}\cdot)\big|^q\Bigg)^{1/q} |L_p(\mathbb{R}^n)\Bigg\| \\ &= 2^{-l(s-n/p)}\Bigg\| \Bigg(\sum_{j\ge l}\sum_{G\in G^{j,l}}\sum_{m\in\mathbb{Z}^n} 2^{jsq}\,\big|\lambda_m^{j,l}(f)\,\chi_{j,m}(\cdot)\big|^q\Bigg)^{1/q} |L_p(\mathbb{R}^n)\Bigg\| \\ &= 2^{-l(s-n/p)}\,\|\lambda(f)\,|f^s_{pq,l}\|.\end{aligned} \tag{4.83}$$

By Proposition 4.15(ii) we have

$$\|f(2^{-l}\cdot)\,|F^s_{pq}(\mathbb{R}^n)\| \sim \|\lambda\big(f(2^{-l}\cdot)\big)\,|f^s_{pq,0}\| \tag{4.84}$$

independently of $l\in\mathbb{N}_0$. Then (4.77) follows from (4.75) with $t=2^{-l}$ and (4.83). □

4.2.3 Refined localisation

First we complement Definition 4.1. Let as there A^s_{pq} be either B^s_{pq} or F^s_{pq}.

Definition 4.18. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ according to Definition 4.3 and let*

$$0<p\le\infty,\quad 0<q\le\infty,\quad \max\Big(\frac1p-1,\,n(\frac1p-1)\Big)<s<\infty \tag{4.85}$$

(with $p<\infty$ for the F-spaces). Then $\widetilde{A}^s_{pq}(\Omega)$ is the closed subspaces of $A^s_{pq}(\mathbb{R}^n)$ given by

$$\widetilde{A}^s_{pq}(\Omega) = \big\{f\in A^s_{pq}(\mathbb{R}^n):\ \operatorname{supp} f\subset\overline{\Omega}\,\big\}. \tag{4.86}$$

Remark 4.19. This coincides with Definition 1.95(iv) where we have now (1.337) under the restriction (4.85). In particular, by the discussion in Section 1.11.6 the spaces $\widetilde{A}^s_{pq}(\Omega)$ can also be interpreted as subspaces of $D'(\Omega)$ according to Definition 1.95(iii) which can be quasi-normed by (1.338).

The refined localisation we have in mind is based on the well-known *Whitney decomposition* here applied to bounded Lipschitz domains in $\mathbb{R}^n$ in the version of Stein, [Ste70, Theorem 3, p. 16, Theorem 1, p. 167] adapted to our needs. In generalisation of (4.70) we introduce the concentric open cubes in $\mathbb{R}^n$ with sides parallel to the axes of coordinates,

$$Q^0_{lr} \subset Q^1_{lr} \subset Q^2_{lr} \subset Q_{lr}, \qquad l \in \mathbb{N}_0, \quad r = 1, \ldots, M^l, \tag{4.87}$$

centred at $2^{-l}m^r$ for some $m^r \in \mathbb{Z}^n$ and with the respective side-lengths $2^{-l}, 5 \cdot 2^{-l-2}, 6 \cdot 2^{-l-2}, 2^{-l+1}$. According to the Whitney decomposition there are pairwise disjoint cubes Q^0_{lr} of this type such that

$$\Omega = \bigcup_{l,r} \overline{Q^0_{lr}} \tag{4.88}$$

and

$$\text{dist } (Q_{lr}, \partial\Omega) \sim 2^{-l} \quad \text{with} \quad l \in \mathbb{N}_0; \quad r = 1, \ldots, M^l. \tag{4.89}$$

We note that $M^l \sim 2^{(n-1)l}$. By the construction in [Ste70] we may assume that $|l - l'| \le 1$ for any two admitted cubes Q^1_{lr}, $Q^1_{l'r'}$ having a non-empty intersection. Let $\{\varrho_{lr}\}$ be a related resolution of unity of non-negative C^∞ functions such that

$$\text{supp } \varrho_{lr} \subset Q^1_{lr}, \qquad |D^\gamma \varrho_{lr}(x)| \le c_\gamma \, 2^{l|\gamma|}, \quad \gamma \in \mathbb{N}^n_0, \tag{4.90}$$

for some $c_\gamma > 0$, and

$$\sum_{l=0}^{\infty} \sum_{r=1}^{M^l} \varrho_{lr}(x) = 1 \qquad \text{if} \quad x \in \Omega. \tag{4.91}$$

Let $\widetilde{F}^s_{pq}(\Omega)$ be the spaces according to Definition 4.18 notationally complemented by $\widetilde{F}^s_{\infty\infty}(\Omega) = \widetilde{B}^s_{\infty\infty}(\Omega)$. Let σ_{pq} be as in (4.73).

Proposition 4.20.

(i) *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ according to Definition* 4.3 *and let $\{\varrho_{lr}\}$ be the above resolution of unity. Let*

$$1 < p \le \infty, \; 1 < q \le \infty, \; s > 0 \qquad (\textit{with } q = \infty \textit{ if } p = \infty). \tag{4.92}$$

Then $\widetilde{F}^s_{pq}(\Omega)$ is the collection of all $f \in L_1(\Omega)$ such that

$$\left(\sum_{l=0}^{\infty} \sum_{r=1}^{M^l} \| \varrho_{lr} f \, | F^s_{pq}(\mathbb{R}^n) \|^p \right)^{1/p} < \infty \tag{4.93}$$

(*usual modification if* $p = q = \infty$). *Furthermore,* (4.93) *is an equivalent norm.*

(ii) *Let* Ω *be a bounded* C^∞ *in* $\mathbb{R}^n$ *and let*

$$0 < p \le \infty, \quad 0 < q \le \infty, \quad s > \sigma_{pq}, \tag{4.94}$$

(*with* $q = \infty$ *if* $p = \infty$). *Then* $\widetilde{F}^s_{pq}(\Omega)$ *is the collection of all* $f \in L_1(\Omega)$ *with* (4.93) (*equivalent quasi-norm*).

Proof. Part (ii) coincides essentially with [Triϵ, Theorem 5.14, p. 60/61]. Its (rather long and complicated) proof relies on the equivalent quasi-norms in Theorem 1.118(iii) with $r = 1$ which in case of C^∞ domains remain valid for all p, q, s with (4.94) and

$$0 < u < \min(1, p, q) \qquad \text{or} \qquad u = 1 < \min(p, q). \tag{4.95}$$

We refer also to Theorem 1.116(iii) and a corresponding explicit assertion in [Triϵ, Corollary 5.15, p. 66]. The latter part with $u = 1$ was not stated explicitly (there was no need to do so) but it is covered without any changes. Otherwise the proof relies on maximal inequalities based again on (4.95). In case of Lipschitz domains only (1.389) with $u = 1$ is available. Then we can apply only the second part of (4.95). Afterwards one can follow the arguments in [Triϵ] in order to prove part (i). □

Remark 4.21. The difference between bounded C^∞ domains in $\mathbb{R}^n$ on the one hand and bounded Lipschitz domains in $\mathbb{R}^n$ on the other hand comes from $u \ge 1$ in Theorem 1.118(iii) in contrast to $u > 0$ in Theorem 1.116(iii) and its counterpart for C^∞ domains. Whether part (i) remains valid for all p, q, s with (4.94) is unclear.*

4.2.4 Wavelets in domains: positive smoothness

We combine Propositions 4.17 and 4.20 in order to get decompositions of $f \in \widetilde{F}^s_{pq}(\Omega)$ by wavelets according to (4.60) having supports in Ω. But the technicalities are a little bit tricky. We are mainly interested in bounded Lipschitz domains Ω in $\mathbb{R}^n$ furnished with Whitney decompositions and related resolutions of unity according to (4.87)–(4.91). Let $\{F, M\}^n$ and $\{F, M\}^{n*}$ as in (4.57), (4.58) and let L be as in (4.71) and (4.72). Let for $j \in \mathbb{N}_0$,

$$S^{\Omega,1}_j = \{F, M\}^{n*} \times \big\{ m \in \mathbb{Z}^n : \ 2^{-j-L} m \in Q^2_{lr} \text{ for some } l < j, \text{ some } r \big\} \tag{4.96}$$

be the *main index set* and

$$S^{\Omega,2}_j = \{F, M\}^n \times \big\{ m \in \mathbb{Z}^n : \ 2^{-j-L} m \in Q^2_{jr} \text{ for some } r \big\} \setminus S^{\Omega,1}_j \tag{4.97}$$

** **Added in proof:** Part (i) remains valid for all p, q, s with (4.94).

be the *residual index set*. Since $l \in \mathbb{N}_0$ one has $S_0^{\Omega,1} = \emptyset$. Recall that the numbers l in the scaling factors 2^{-l} of adjacent cubes Q^0_{lr} differ by at most 1. It may happen that an element of $S_j^{\Omega,1}$ with $j \in \mathbb{N}$ and, say, $l = j-1$, belongs also to the first set on the right-hand side of (4.97). Then it is taken out what is indicated by $\setminus S_j^{\Omega,1}$. The cardinal number of $S_j^{\Omega,2}$ is $\sim 2^{j(n-1)}$. We allocate now the wavelets (and scaled scaling functions) to the above index-sets. Recall that $\Psi^{j,l}_{G,m} = \Psi^j_{G,m}$ in (4.60) does not depend on l as far as scaling is concerned. Indexed by

$$S^\Omega = S^{\Omega,1} \cup S^{\Omega,2}, \quad S^{\Omega,1} = \bigcup_{j=0}^\infty S_j^{\Omega,1}, \quad S^{\Omega,2} = \bigcup_{j=0}^\infty S_j^{\Omega,2} \tag{4.98}$$

we get the *main wavelet system*

$$\Psi^{1,\Omega} = \left\{ \Psi^j_{G,m} : \ (j,G,m) \in S^{\Omega,1} \right\} \tag{4.99}$$

and the *residual wavelet system*

$$\Psi^{2,\Omega} = \left\{ \Psi^j_{G,m} : \ (j,G,m) \in S^{\Omega,2} \right\}. \tag{4.100}$$

We always assume that $k \in \mathbb{N}$ is (4.56) is sufficiently large, for example, in agreement with (4.69),

$$k > \max\left(s, \frac{5n}{2} - s \right) \quad \text{if} \quad 1 < p < \infty, \quad 1 < q < \infty, \quad s > 0, \tag{4.101}$$

and that L in (4.71), (4.72) is fixed afterwards. Any element of the main wavelet system $\Psi^{1,\Omega}$ is orthogonal to any element of the residual wavelet system $\Psi^{2,\Omega}$. Let

$$L_2(\Omega) = L_2^{(1)}(\Omega) \oplus L_2^{(2)}(\Omega) \tag{4.102}$$

be the corresponding orthogonal decomposition. Then $\Psi^{1,\Omega}$ is an orthonormal basis of $L_2^{(1)}(\Omega)$. The pairwise scalar products of the elements of $\Psi^{2,\Omega}$ generate a band-limited matrix caused by the mild overlapping of the supports of elements belonging to adjacent cubes. In any case, $\Psi^{2,\Omega}$ is locally finite and the cardinal number of elements of $\Psi^{2,\Omega}$ with supports intersecting

$$\{x \in \Omega : \ \text{dist}\ (x, \partial\Omega) \geq 2^{-j}\}$$

is $\sim 2^{(n-1)j}$. As we shall see $\Psi^{1,\Omega} \cup \Psi^{2,\Omega}$ is a *frame* and almost a basis in $L_2(\Omega)$ and also in other spaces in the understanding of Theorem 4.22 below.

Let p, q, s as in (4.101) (and also k and L as indicated). We apply to each term $\varrho_{lr} f$ of

$$f = \sum_{l=0}^\infty \sum_{r=1}^{M^l} \varrho_{lr} f, \qquad f \in \widetilde{F}^s_{pq}(\Omega), \tag{4.103}$$

the wavelet expansion (4.62), (4.63). Then we get

$$f = \sum_{(j,G,m)\in S^\Omega} \lambda^{j,G}_m(f)\, 2^{-jn/2}\, \Psi^j_{G,m}. \tag{4.104}$$

As for the coefficients $\lambda^{j,G}_m(f)$ one has to check which terms in (4.62), (4.63) with $\varrho_{lr} f$ in place of f contribute to this coefficient. If $(j,G,m) \in S^{\Omega,1}$ then this is the case for the full resolution of unity (4.90), (4.91) and one gets

$$\begin{aligned} \lambda^{j,G}_m(f) &= \sum_{l,r} \lambda^{j,G,l}_m(\varrho_{lr} f) = 2^{jn/2} \sum_{l,r} \int_{\mathbb{R}^n} \varrho_{lr}(x)\, f(x)\ \Psi^{j,l}_{G,m}(x)\, \mathrm{d}x \\ &= 2^{jn/2} \int_{\mathbb{R}^n} f(x)\, \Psi^j_{G,m}(x)\, \mathrm{d}x, \qquad (j,G,m) \in S^{\Omega,1}, \end{aligned} \tag{4.105}$$

(as it should be having in mind the above orthogonality and almost-basis-property). As for the residual terms $(j,G,m) \in S^{\Omega,2}$, not all relevant functions ϱ_{lr} may contribute and one gets

$$\lambda^{j,G}_m(f) = 2^{jn/2} \int_{\mathbb{R}^n} \varrho^j_m(x)\, f(x)\, \Psi^j_{G,m}(x)\, \mathrm{d}x, \qquad (j,G,m) \in S^{\Omega,2}, \tag{4.106}$$

where ϱ^j_m are C^∞ functions with

$$\operatorname{supp} \varrho^j_m \subset Q_{jr}, \qquad |D^\gamma \varrho^j_m(x)| \le c_\gamma\, 2^{j|\gamma|}, \quad \gamma \in \mathbb{N}^n_0, \tag{4.107}$$

where $r = r(m)$ has the same meaning as in (4.97). Recall that we always have the obvious counterpart of (4.71), (4.72). We need the counterpart of the sequence spaces (4.65)–(4.67). Let now

$$\lambda = \{\lambda^{j,G}_m \in \mathbb{C} :\ (j,G,m) \in S^\Omega\}. \tag{4.108}$$

Then $b^{s,\Omega}_{pq}$ is the collection of all sequences (4.108) quasi-normed by

$$\|\lambda\, | b^{s,\Omega}_{pq}\| = \left(\sum_{j=0}^\infty 2^{j(s-n/p)q} \left(\sum_{(G,m)\in S^\Omega_j} |\lambda^{j,G}_m|^p \right)^{q/p} \right)^{1/q} \tag{4.109}$$

with $S^\Omega_j = S^{\Omega,1}_j \cup S^{\Omega,2}_j$ according (4.96), (4.97), and $f^{s,\Omega}_{pq}$ is the collection of all sequences (4.108) quasi-normed by

$$\|\lambda\, | f^{s,\Omega}_{pq}\| = \left\| \left(\sum_{(j,G,m)\in S^\Omega} 2^{jsq}\, \big|\lambda^{j,G}_m \chi_{jm}(\cdot)\big|^q \right)^{1/q} |L_p(\Omega) \right\|. \tag{4.110}$$

Here $0 < p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$ with the usual modifications in (4.109), (4.110) if $p = \infty$ and/or $q = \infty$. Furthermore χ_{jm} is the characteristic function of a cube centred at $2^{-j}m$ with side-length 2^{-j-M} for some $M \in \mathbb{N}$ and sides parallel to the axes of coordinates. (One may think that M is chosen such that these cubes have a distance to $\partial\Omega$ of at least $c2^{-j}$ for some $c > 0$ and all $j \in \mathbb{N}_0$, but this is unimportant). Let $\widetilde{F}^s_{pq}(\Omega)$ and $\widetilde{B}^s_{pq}(\Omega)$ be the spaces according to Definition 4.18 and Remark 4.19. If $p > 1$ and $s > 0$ then, quite obviously,

$$\widetilde{F}^s_{pq}(\Omega) \hookrightarrow L_p(\Omega) \qquad \text{and} \qquad \widetilde{B}^s_{pq}(\Omega) \hookrightarrow L_p(\Omega). \tag{4.111}$$

Theorem 4.22. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ according to Definition* 4.3 *and let*

$$1 < p < \infty,\ 1 < q < \infty,\ s > 0,\ \max\left(s, \frac{5n}{2} - s\right) < k \in \mathbb{N} \tag{4.112}$$

in (4.56).

(i) *Then $f \in L_p(\Omega)$ is an element of $\widetilde{F}^s_{pq}(\Omega)$ if, and only if, it can be represented by*

$$f = \sum_{(j,G,m)\in S^\Omega} \lambda^{j,G}_m\, 2^{-jn/2}\, \Psi^j_{G,m} \tag{4.113}$$

with $\lambda \in f^{s,\Omega}_{pq}$ (unconditional convergence being in $\widetilde{F}^s_{pq}(\Omega)$). Furthermore, $f \in \widetilde{F}^s_{pq}(\Omega)$ can be represented by (4.104) *with* (4.105), (4.106) *and*

$$\|\lambda(f)\,|f^{s,\Omega}_{pq}\| \sim \|f\,|\widetilde{F}^s_{pq}(\Omega)\| = \|f\,|F^s_{pq}(\mathbb{R}^n)\| \tag{4.114}$$

(*equivalent norms*).

(ii) *Then $f \in L_p(\Omega)$ is an element of $\widetilde{B}^s_{pq}(\Omega)$ if, and only if, it can be represented by* (4.113) *with $\lambda \in b^{s,\Omega}_{pq}$ (unconditional convergence being in $\widetilde{B}^s_{pq}(\Omega)$). Furthermore, $f \in \widetilde{B}^s_{pq}(\Omega)$ can be represented by* (4.104) *with* (4.105), (4.106) *and*

$$\|\lambda(f)\,|b^{s,\Omega}_{pq}\| \sim \|f\,|\widetilde{B}^s_{pq}(\Omega)\| = \|f\,|B^s_{pq}(\mathbb{R}^n)\| \tag{4.115}$$

(*equivalent norms*).

Proof. Step 1. This is the counterpart for bounded Lipschitz domains Ω of corresponding assertions for $\mathbb{R}^n$ according to Theorem 3.5 and its modification in Proposition 4.15. Many technical details are the same as before and will not be repeated here. As there, after the correct normalisation, (4.113) is an atomic decomposition and

$$\|f\,|\widetilde{F}^s_{pq}(\Omega)\| \le c\,\|\lambda\,|f^{s,\Omega}_{pq}\| \tag{4.116}$$

for some $c > 0$ which is independent of λ. Conversely if $f \in \widetilde{F}^s_{pq}(\Omega)$ and if $\lambda(f)$ as in (4.105), (4.106) then it follows from the crucial Propositions 4.17(ii) and 4.20(i) that

$$\|\lambda(f)\,|f^{s,\Omega}_{pq}\| \le c\,\|f\,|F^s_{pq}(\mathbb{R}^n)\| = c\,\|f\,|\widetilde{F}^s_{pq}(\Omega)\|. \tag{4.117}$$

Now (4.116), (4.117) prove part (i) of the theorem.

Step 2. Let $1 < p < \infty$, $1 < q < \infty$, $0 < s_0 < s_1 < \infty$ and $s = (1-\theta)s_0 + \theta s_1$. Then one has the real interpolation formula

$$\Big(\widetilde{B}^{s_0}_{pp}(\Omega), \widetilde{B}^{s_1}_{pp}(\Omega)\Big)_{\theta,q} = \widetilde{B}^s_{pq}(\Omega). \tag{4.118}$$

This is a special case of [Tri02a, Theorem 3.5, pp. 496/497]. Although the outcome is the expected one it is by no means obvious. Its proof is based on duality arguments which will not be repeated here. But it may explain the restrictions for p and q in (4.112). The counterpart for the related sequence spaces is given by

$$\big(b^{s_0,\Omega}_{pp}, \, b^{s_1,\Omega}_{pp}\big)_{\theta,q} = b^{s,\Omega}_{pq}. \tag{4.119}$$

As for the proof of (4.119) we first remark that the $\mathbb{R}^n$-counterpart of (4.119) follows from isomorphic maps of sequence spaces of type b^s_{pq} according to Theorem 3.5 (or Proposition 4.15) onto corresponding spaces $B^s_{pq}(\mathbb{R}^n)$ and related well-known interpolation formulas. Sequence spaces of type $b^{s,\Omega}_{pq}$ can be interpreted as complemented subspaces of their $\mathbb{R}^n$-counterparts b^s_{pq}. Then one gets (4.119) from its $\mathbb{R}^n$-counterpart and the method of retraction-coretraction, [Triα, 1.17.1, p. 118], or, more simply by the same arguments as in the proof of Theorem 1.110, especially by a suitable modification of (1.365). Recall that $b^{s,\Omega}_{pp} = f^{s,\Omega}_{pp}$ and $\widetilde{B}^s_{pp}(\Omega) = \widetilde{F}^s_{pp}(\Omega)$. Then part (ii) of the theorem follows from part (i) by the same type of reasoning as just indicated. □

Remark 4.23. There are two reasons for the somewhat disturbing restriction of p, q in (4.112). First we have (4.93) for Lipschitz domains only under the assumption (4.92) in contrast to (4.94) for bounded C^∞ domains. Secondly, (4.118) is known only for the indicated restrictions for p and q (for Lipschitz domains and C^∞ domains). Nevertheless it might be reasonable to collect those assertions which can be obtained from the above propositions but which are not covered by Theorem 4.22. Recall that

$$\widetilde{\mathcal{C}}^s(\Omega) = \widetilde{B}^s_{\infty\infty}(\Omega) = \widetilde{F}^s_{\infty\infty}(\Omega) \tag{4.120}$$

collects all f with $\operatorname{supp} f \subset \overline{\Omega}$ belonging to the Hölder-Zygmund spaces $\mathcal{C}^s(\mathbb{R}^n)$.

Corollary 4.24.

(i) *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ according to Definition* 4.3 *and let $s > 0$. Then $f \in L_\infty(\Omega)$ is an element of $\widetilde{\mathcal{C}}^s(\Omega)$ if, and only if, it can be represented by* (4.113) *with $\lambda \in b^{s,\Omega}_{\infty\infty}$ (unconditional convergence being in $L_\infty(\Omega)$). Furthermore, $f \in \widetilde{\mathcal{C}}^s(\Omega)$ can be represented by* (4.104) *with* (4.105), (4.106) *and*

$$\sup_{(j,G,m)} 2^{js} \, |\lambda^{j,G}_m(f)| \sim \| f \, | \widetilde{\mathcal{C}}^s(\Omega)\| = \| f \, | \mathcal{C}^s(\mathbb{R}^n)\| \tag{4.121}$$

(*equivalent norms*).

(ii) *Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$. Then part* (i) *of Theorem* 4.22 *remains valid for all*

$$0 < p < \infty, \quad 0 < q < \infty, \quad s > \sigma_{pq} \quad \textit{and} \quad f \in L_{\bar{p}}(\Omega) \tag{4.122}$$

with $\bar{p} = \max(p, 1)$.

Proof. This follows from the proof of Theorem 4.22 and the related assertions in Propositions 4.17, 4.20 where k in Proposition 4.15 must be chosen appropriately. □

Remark 4.25. Hence (4.104) with (4.105), (4.106) is a frame representation in the understanding of Remark 3.23 for all spaces covered by Theorem 4.22 and Corollary 4.24. It is almost a basis if $p < \infty$, $q < \infty$. The somewhat disturbing summation over $(j, G, m) \in S^{\Omega,2}$ in (4.104) gives a function belonging locally to $C^k(\Omega)$. In the one-dimensional case one has also a slight improvement for the global behavior (reverse embedding) which will be detailed in Remark 4.31 below. One may call such a frame a *para-basis*, indicating that its main part is an orthonormal system and that its residual part can be neglected (with respect to local and global smoothness, and localisation). The desirable extension of part (ii) of the above corollary to B-spaces is not so clear. It depends on the question of whether the interpolation formula (4.118) can be extended appropriately.

4.2.5 Wavelets in domains: general smoothness

One may ask to what extent representations of type (4.113) apply to other values of p, q, s. If $s < 0$ then one is in a surprisingly good position. The proof of Theorem 3.5 for wavelet representations in $\mathbb{R}^n$ and its preparations in Sections 3.1.2, 3.1.3 are based on atomic representations and local means combined with some duality arguments and maximal inequalities. This will not be repeated here. Again we only discuss those specific points coming in when switching from $\mathbb{R}^n$ to bounded Lipschitz domains Ω in $\mathbb{R}^n$. If $s < 0$ then Section 1.11.6 and the references given there suggest asking for intrinsic wavelet characterisations of type (4.113) for the full spaces $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$. Let now

$$0 < p < \infty, \quad 0 < q < \infty, \quad s < 0 \tag{4.123}$$

and

$$\frac{2n}{\min(p,q)} + \frac{n}{2} - s < k \in \mathbb{N} \tag{4.124}$$

as in (4.69). Let $\lambda \in f^{s,\Omega}_{pq}$ quasi-normed according to (4.110) and

$$f = \sum_{(j,G,m) \in S^\Omega} \lambda^{j,G}_m \, 2^{-jn/2} \, \Psi^j_{G,m} = \sum_{S^{\Omega,1}} + \sum_{S^{\Omega,2}} = f_1 + f_2, \tag{4.125}$$

splitting the sum into the main wavelet system and the residual wavelet system based on (4.98)–(4.100). The terms $\Psi^j_{G,m} \in \Psi^{1,\Omega}$ in (4.99) are atoms in $F^s_{pq}(\mathbb{R}^n)$ as described in Theorem 1.19(ii) and, hence, also in $F^s_{pq}(\Omega)$. This is not necessarily the case for those terms $\Psi^j_{G,m} \in \Psi^{2,\Omega}$ in (4.100) referring to the scaling functions (they do not have the required moment conditions). But

$$\text{diam}\,(\text{supp}\,\Psi^j_{G,m}) \sim \text{dist}\,(\text{supp}\,\Psi^j_{G,m}, \partial\Omega) \sim 2^{-j} \tag{4.126}$$

if $(j, G, m) \in S^{\Omega,2}$. Then $\Psi^j_{G,m}$ are boundary atoms (where moment conditions are no longer needed). The corresponding theory has been described in [ET96, Sections 2.5.2, 2.5.3] with a reference to [TrW96] for details. In particular one can extend a function $\Psi^j_{G,m} \in \Psi^{2,\Omega}$ outside of Ω to a function $\widetilde{\Psi}^j_{G,m}$ being an atom in $F^s_{pq}(\mathbb{R}^n)$ (after correct normalisation) with all required moment conditions and

$$\widetilde{\Psi}^j_{G,m}|\Omega = \Psi^j_{G,m} \quad \text{and} \quad \text{diam}\,(\text{supp}\,\widetilde{\Psi}^j_{G,m}) \sim 2^{-j}. \tag{4.127}$$

Extending (4.125) in this way one gets an atomic decomposition for some $g \in F^s_{pq}(\mathbb{R}^n)$ with the same coefficients and $g|\Omega = f$. Hence, (4.125) can be considered as an atomic decomposition in Ω and

$$\|f\,|F^s_{pq}(\Omega)\| \le \|g\,|F^s_{pq}(\mathbb{R}^n)\| \le c\,\|\lambda\,|f^{s,\Omega}_{pq}\|. \tag{4.128}$$

The last estimate is justified by Theorem 1.19 and Proposition 1.33. The question arises whether these representations characterise the spaces $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$, complementing Theorem 4.22. As before, $D'(\Omega)$ stands for the set of all distributions in Ω.

Theorem 4.26. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ according to Definition 4.3 and let*

$$0 < p < \infty, \quad 0 < q < \infty, \quad s < 0. \tag{4.129}$$

(i) *Let $k \in \mathbb{N}$ be as in (4.124). Then $f \in D'(\Omega)$ is an element of $F^s_{pq}(\Omega)$ if, and only if, it can be represented by (4.113) with $\lambda \in f^{s,\Omega}_{pq}$ (unconditional convergence being in $F^s_{pq}(\Omega)$). Furthermore, $f \in F^s_{pq}(\Omega)$ can be represented by (4.104) with (4.105), (4.106) and*

$$\|\lambda(f)\,|f^{s,\Omega}_{pq}\| \sim \|f\,|F^s_{pq}(\Omega)\| \tag{4.130}$$

(equivalent quasi-norms).

(ii) *Let $\frac{2n}{p} + \frac{n}{2} - s < k \in \mathbb{N}$ (as in (4.68)). Then $f \in D'(\Omega)$ is an element of $B^s_{pq}(\Omega)$ if, and only if, it can be represented by (4.113) with $\lambda \in b^{s,\Omega}_{pq}$ (unconditional convergence being in $B^s_{pq}(\Omega)$). Furthermore, $f \in B^s_{pq}(\Omega)$ can be represented by (4.104) with (4.105), (4.106) and*

$$\|\lambda(f)\,|b^{s,\Omega}_{pq}\| \sim \|f\,|B^s_{pq}(\Omega)\| \tag{4.131}$$

(equivalent quasi-norms).

Proof. By the above considerations we have (4.128) and a similar assertion for the B-spaces. If $f \in F^s_{pq}(\Omega)$ then (4.105) with the kernels $\Psi^j_{G,m}$ and (4.106) with the kernels $\varrho^j_m \Psi^j_{G,m}$ are local means. By Theorem 1.10 and Corollary 1.12 one does not need for $s < 0$ any moment conditions for these kernels. In particular one gets by the same comments as after (4.128) that

$$\|\lambda(f)\,|f^{s,\Omega}_{pq}\| \le c\,\|f\,|F^s_{pq}(\Omega)\|, \qquad f \in F^s_{pq}(\Omega), \tag{4.132}$$

and a B-counterpart. After these preparations one can prove the theorem as follows. First we assume that $f \in D(\Omega) = C^\infty_0(\Omega)$. Then one gets by Theorem 4.22 that f can be represented by (4.104) with (4.105), (4.106) and one has by (4.128), (4.132) that

$$\|f\,|F^s_{pq}(\Omega)\| \sim \|\lambda(f)\,|f^{s,\Omega}_{pq}\|, \qquad f \in D(\Omega), \tag{4.133}$$

and a B-counterpart. On the other hand, the restriction $S(\mathbb{R}^n)|\Omega$ of $S(\mathbb{R}^n)$ to Ω is dense in all spaces considered. Any function $f \in S(\mathbb{R}^n)|\Omega$ can be approximated in $L_{\bar{p}}(\Omega)$ with $\bar{p} = \max(p,1)$ by functions belonging to $D(\Omega)$. Since

$$L_{\bar{p}}(\Omega) \hookrightarrow B^s_{pq}(\Omega) \quad \text{and} \quad L_{\bar{p}}(\Omega) \hookrightarrow F^s_{pq}(\Omega) \tag{4.134}$$

it follows that $D(\Omega)$ is dense in all spaces considered. Now (4.104), (4.133) can be extended by completion to all elements of the above spaces. □

Remark 4.27. The above theorem is based on the favorable situation about (boundary) atoms, local means and the density of $D(\Omega)$ in the spaces with (4.129). This cannot be extended to spaces with $s \ge 0$. But one can combine Theorems 4.22 and 4.26 using the interpolation properties according to Section 1.11.8 and the comments on subspaces in Section 1.11.6. In particular we rely on the notation $\bar{A}^s_{pq}(\Omega)$ with $A = B$ and $A = F$ as introduced in (1.343), (1.344).

Corollary 4.28. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ according to Definition 4.3. Let $K > 0$ and*

$$1 < p < \infty,\ 1 < q < \infty,\ -K < s < K,\ \frac{5n}{2} + K < k \in \mathbb{N}. \tag{4.135}$$

(i) *Then $f \in D'(\Omega)$ is an element of $\bar{F}^s_{pq}(\Omega)$ if, and only if, it can be represented by (4.113) with $\lambda \in f^{s,\Omega}_{pq}$ (unconditional convergence being in $\bar{F}^s_{pq}(\Omega)$). Furthermore $f \in \bar{F}^s_{pq}(\Omega)$ can be represented by (4.104) with (4.105), (4.106) and*

$$\|\lambda(f)\,|f^{s,\Omega}_{pq}\| \sim \|f\,|\bar{F}^s_{pq}(\Omega)\| \tag{4.136}$$

(*equivalent norms*).

(ii) *Then $f \in D'(\Omega)$ is an element of $\bar{B}^s_{pq}(\Omega)$ if, and only if, it can be represented by (4.113) with $\lambda \in b^{s,\Omega}_{pq}$ (unconditional convergence being in $\bar{B}^s_{pq}(\Omega)$). Furthermore, $f \in \bar{B}^s_{pq}(\Omega)$ can be represented by (4.104) with (4.105), (4.106) and*

$$\|\lambda(f)\,|b^{s,\Omega}_{pq}\| \sim \|f\,|\bar{B}^s_{pq}(\Omega)\| \tag{4.137}$$

(*equivalent norms*).

Proof. The cases $s > 0$ and $s < 0$ are covered by Theorems 4.22 and 4.26. In case of $s = 0$ we rely on (1.344) and the complex interpolation formula

$$\left[F^s_{pq}(\Omega)\,,\,F^{-s}_{pq}(\Omega)\right]_{1/2} = F^0_{pq}(\Omega), \qquad 0 < s < 1/p, \tag{4.138}$$

covered by Corollary 1.111(ii). By the same argument as in Step 2 of the proof of Theorem 4.22 it follows that there is a counterpart for the related sequence spaces. Then one obtains part (i) by the same arguments. Similarly for the B-spaces. □

Corollary 4.29. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ according to Definition* 4.3 *and let*

$$0 < p < \infty, \quad 0 < q < \infty, \quad -\infty < s < \min\left(\frac{1}{p}, 1\right). \tag{4.139}$$

If $k \in \mathbb{N}$ is chosen sufficiently large then Corollary 4.28(i) *remains valid with $F^s_{pq}(\Omega)$ in place of $\bar{F}^s_{pq}(\Omega)$. Similarly, Corollary* 4.28(ii) *remains valid with $B^s_{pq}(\Omega)$ in place of $\bar{B}^s_{pq}(\Omega)$.*

Proof. By (1.340) we have

$$F^s_{pq}(\Omega) = \widetilde{F}^s_{pq}(\Omega) \qquad \text{and} \qquad B^s_{pq}(\Omega) = \widetilde{B}^s_{pq}(\Omega) \tag{4.140}$$

if

$$1 < p < \infty, \quad 1 < q < \infty, \quad \frac{1}{p} - 1 < s < \frac{1}{p}. \tag{4.141}$$

Any B-space or F-space with (4.139) can be obtained by complex or real interpolation according to Corollary 1.111 and Proposition 1.114 of spaces covered by (4.140), (4.141) on the one hand and spaces covered by Theorem 4.26 on the other hand. Now the above corollary follows by the same type of arguments as above. □

Remark 4.30. Recall that

$$h_p(\mathbb{R}^n) = F^0_{p,2}(\mathbb{R}^n), \qquad 0 < p < \infty, \tag{4.142}$$

are the well-known (inhomogeneous) *Hardy spaces* with $h_p(\mathbb{R}^n) = L_p(\mathbb{R}^n)$ if $1 < p < \infty$, [Triβ, Theorem 2.5.8/1, p. 92]. Hence one gets by the above corollary intrinsic wavelet characterisations of all spaces $h_p(\Omega)$, where again Ω is a bounded Lipschitz domain in $\mathbb{R}^n$.

Remark 4.31. Let f be an element of one of the B^s_{pq}-spaces in bounded Lipschitz domains covered by one of the Theorems 4.22, 4.26 or Corollaries 4.24, 4.28, 4.29. Let f_2 be the residual part of the wavelet expansion as indicated in (4.125) collecting the scaled scaling functions and the related first wavelets. Then f_2 belongs to $C^k(\omega)$ for any domain ω with $\overline{\omega} \subset \Omega$ and the question arises whether one has even a global improvement

$$f_2 \in B^\sigma_{\widetilde{p}q}(\Omega), \quad 0 < \widetilde{p} \le \infty, \quad -\infty < \sigma = s - \frac{n}{p} + \frac{n}{\widetilde{p}} < k, \tag{4.143}$$

including equivalent quasi-norms (direct and reverse embedding) where $\widetilde{p} < p$ is of interest. By the structure of $b^{s,\Omega}_{pq}$ in (4.109) and atomic arguments (including boundary atoms used above) one gets such an improvement in the one-dimensional case when Ω consists of finitely many intervals. Then the interior ℓ_p-quasi-norm in (4.109) is equivalent to a corresponding $\ell_{\widetilde{p}}$-quasi-norm. Similarly let f be an element of one of the F^s_{pq}-spaces in one of the indicated theorems or corollaries. Then again f_2 belongs to $C^k(\omega)$ for any domain ω with $\overline{\omega} \subset \Omega$ and for $0 < \widetilde{p} < \infty$ and σ as in (4.143) with $n = 1$,

$$f_2 \in F^{\sigma}_{\widetilde{p}\widetilde{q}}(\Omega), \qquad 0 < \widetilde{q} \le \infty, \tag{4.144}$$

including equivalent quasi-norms. This follows from the peculiarities of f_2 and of $f^{s,\Omega}_{pq}$ in (4.110) and (boundary) atomic arguments.

4.3 Sampling numbers

4.3.1 Definitions

Let $A^s_{pq}(\Omega)$ be the spaces introduced in Definition 4.1 where Ω is a bounded Lipschitz domain in $\mathbb{R}^n$ according to Definition 4.3. We are interested in compact embeddings

$$\mathrm{id}: \quad G_1(\Omega) \hookrightarrow G_2(\Omega) \tag{4.145}$$

with the source spaces

$$G_1(\Omega) = A^{s_1}_{p_1 q_1}(\Omega) \tag{4.146}$$

where

$$0 < p_1 \le \infty,\ 0 < q_1 \le \infty,\ s_1 > n/p_1, \quad (p_1 < \infty \text{ in the } F\text{-case}), \tag{4.147}$$

and the target spaces either

$$G_2(\Omega) = L_r(\Omega) \quad \text{with} \quad 0 < r \le \infty \tag{4.148}$$

or

$$G_2(\Omega) = A^{s_2}_{p_2 q_2}(\Omega) \tag{4.149}$$

with

$$0 < p_2 \le \infty,\ 0 < q_2 \le \infty, \quad (p_2 < \infty \text{ in the } F\text{-case}), \tag{4.150}$$

and

$$n\left(\frac{1}{p_2} - 1\right)_+ < s_2 < s_1 - n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ . \tag{4.151}$$

The left-hand side of (4.151) coincides with σ_{p_2} given by (2.6). Then it follows by Proposition 4.6 that

$$\mathrm{id}: \quad A^{s_1}_{p_1 q_1}(\Omega) \hookrightarrow C(\bar{\Omega}) \tag{4.152}$$

with (4.147) and

$$\mathrm{id}: \quad A^{s_2}_{p_2 q_2}(\Omega) \hookrightarrow L_1(\Omega) \tag{4.153}$$

with (4.150), (4.151). In particular if in addition $r \geq 1$ in (4.148) then all spaces $G_1(\Omega)$ and $G_2(\Omega)$ involved are spaces of regular distributions, continuously embedded in $L_1(\Omega)$. If $0 < r < 1$ then $L_r(\Omega)$ cannot be interpreted as a subspace of the space of distributions $D'(\Omega)$. But in this case we have (4.152), and (4.145) makes sense as an embedding between spaces of measurable functions. We will not stress this point in the sequel. In all cases the embedding (4.145) is compact. This is covered by Proposition 4.6 and an additional application of Hölder's inequality in case of (4.148) with $r < 1$.

Next we describe the basic ingredients of *sampling methods* for the compact embedding (4.145) with the indicated specifications for $G_1(\Omega)$ and $G_2(\Omega)$. Let $\{x^j\}_{j=1}^k \subset \Omega$. By (4.152) the *information map*

$$N_k: \quad G_1(\Omega) \mapsto \mathbb{C}^k, \qquad k \in \mathbb{N}, \tag{4.154}$$

given by

$$N_k f = \big(f(x^1), \ldots, f(x^k)\big), \qquad f \in G_1(\Omega), \tag{4.155}$$

makes sense. Obviously, $\mathbb{C}^k$ is the collection of all k-tuples of complex numbers. Let

$$S_k = \Phi_k \circ N_k \qquad \text{where} \quad \Phi_k: \quad \mathbb{C}^k \mapsto G_2(\Omega) \tag{4.156}$$

is an arbitrary map (also called *method* or *algorithm*). Hence,

$$S_k f = \Phi_k \big(f(x^1), \ldots, f(x^k)\big) \in G_2(\Omega), \qquad f \in G_1(\Omega). \tag{4.157}$$

One wishes to recover a given continuous function $f \in G_1(\Omega)$ in $G_2(\Omega)$ by asking for optimally scattered sampling points $\{x^j\}_{j=1}^k$ and optimally chosen methods Φ_k. This results in the following definition of sampling numbers where again we rely on [NoT04].

Definition 4.32. *Let Ω be a bounded Lipschitz domain according to Definition* 4.3. *Let $G_1(\Omega)$ be given by* (4.146), (4.147) *and let $G_2(\Omega)$ be either the spaces in* (4.148) *or the spaces in* (4.149)–(4.151). *Let $k \in \mathbb{N}$ and let* id *be the embedding* (4.145).

(i) *Then*

$$g_k(\mathrm{id}) = \inf \left[\sup \{\|f - S_k f \,|G_2(\Omega)\| : \ \|f\,|G_1(\Omega)\| \leq 1\}\right] \tag{4.158}$$

is the kth *sampling number, where the infimum is taken over all k-tuples $\{x^j\}_{j=1}^k \subset \Omega$ and all maps $S_k = \Phi_k \circ N_k$ according to* (4.154)–(4.157).

(ii) *The linear sampling numbers $g_k^{\mathrm{lin}}(\mathrm{id})$ are given by* (4.158) *where the infimum is taken over all k-tuples $\{x^j\}_{j=1}^k \subset \Omega$ and all linear maps $S_k = \Phi_k \circ N_k$ with*

$$S_k f = \sum_{j=1}^k f(x^j)\, h_j, \qquad h_j \in G_2(\Omega), \quad f \in G_1(\Omega). \tag{4.159}$$

Remark 4.33. In Section 4.4.1 we compare (linear) sampling numbers with other distinguished numbers for compact embeddings such as approximation numbers and entropy numbers. At this moment we only remark as an immediate consequence of the above definition and the definition of approximation numbers $a_k(\mathrm{id})$ according to (1.283) that

$$g_k(\mathrm{id}) \le g_k^{\mathrm{lin}}(\mathrm{id}) \quad \text{and} \quad a_{k+1}(\mathrm{id}) \le g_k^{\mathrm{lin}}(\mathrm{id}), \quad k \in \mathbb{N}. \tag{4.160}$$

There is a huge literature about optimal recovery and sampling. We give some references later on in Remark 4.42.

4.3.2 Basic properties

Again let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ according to Definition 4.3. Let $G_1(\Omega)$ and $G_2(\Omega)$ be as in Definition 4.32. Let $\Gamma = \{x^j\}_{j=1}^l \subset \Omega$. Then

$$\mathrm{id}^\Gamma : \quad G_1^\Gamma(\Omega) = \left\{ f \in G_1(\Omega) : \sum_{j=1}^{l} |f(x^j)| = 0 \right\} \hookrightarrow G_2(\Omega) \tag{4.161}$$

is a compact operator and its quasi-norm $\|\mathrm{id}^\Gamma\|$ has the usual meaning.

Proposition 4.34. *Let Ω be the above bounded Lipschitz domain in $\mathbb{R}^n$. Let $G_1(\Omega)$, $G_2(\Omega)$ and* id *be as in Definition* 4.32 *and let* id^Γ *be given by* (4.161)*. Then*

$$g_k(\mathrm{id}) \sim \inf \left\{ \|\mathrm{id}^\Gamma\| : \ \mathrm{card}\, \Gamma \le k \right\}, \qquad k \in \mathbb{N}, \tag{4.162}$$

where the equivalence constants are independent of k.

Proof. Step 1. Let $k \in \mathbb{N}$. By definition

$$\|\mathrm{id}^\Gamma\| = \sup \left\{ \|f \,|G_2(\Omega)\| : \ \|f\,|G_1(\Omega)\| \le 1, \ f(x^j) = 0 \right\} \tag{4.163}$$

with $\Gamma = \{x^j\}_{j=1}^l \subset \Omega$ and $l \le k$. We denote the right-hand side of (4.162) by $g_k^0(\mathrm{id})$ and prove in this step that

$$g_k^0(\mathrm{id}) \preceq g_k(\mathrm{id}), \qquad k \in \mathbb{N}. \tag{4.164}$$

Let f^0 be the identically vanishing function in Ω and let S_k^ε for $\varepsilon > 0$ be a map approximating $g_k(\mathrm{id})$ in (4.158) up to ε. In particular,

$$\|S_k^\varepsilon f^0 \,|G_2(\Omega)\| \le g_k(\mathrm{id}) + \varepsilon. \tag{4.165}$$

Furthermore,

$$g_k^0(\mathrm{id}) \le \sup \|f\,|G_2(\Omega)\| = \sup \|f - S_k^\varepsilon f + S_k^\varepsilon f^0 \,|G_2(\Omega)\| \tag{4.166}$$

where the supremum is taken over all $f \in G_1(\Omega)$ with $\|f\,|G_1(\Omega)\| \le 1$ and $f(x^j) = 0$. Enlarging the right-hand side of (4.166) by taking the supremum over the whole unit ball in $G_1(\Omega)$ one gets by the above assumption and (4.165) that

$$g_k^0(\mathrm{id}) \preceq g_k(\mathrm{id}) + \varepsilon, \qquad k \in \mathbb{N}, \tag{4.167}$$

uniformly in k and ε. This proves (4.164)

Step 2. We prove the converse of (4.164). Let $\Gamma = \{x^j\}_{j=1}^k \subset \Omega$ be k pairwise different points. We interpret the information map (4.154), (4.155) as a trace operator tr_Γ,

$$\mathrm{tr}_\Gamma = N_k : \quad G_1(\Omega) \mapsto \mathbb{C}^k, \qquad k \in \mathbb{N}. \tag{4.168}$$

It generates a quasi-norm in $\mathbb{C}^k$,

$$\|\{c_j\}\|_\Gamma = \inf \left\{ \|h\,|G_1(\Omega)\| : \ h(x^j) = c_j \right\}. \tag{4.169}$$

We choose for Φ_k in (4.156) a corresponding (non-linear) bounded extension operator ext_Γ,

$$\Phi_k = \mathrm{ext}_\Gamma : \quad \mathbb{C}^k \mapsto G_1(\Omega) \hookrightarrow G_2(\Omega) \tag{4.170}$$

and put

$$S_k = \mathrm{ext}_\Gamma \circ \mathrm{tr}_\Gamma = \Phi_k \circ N_k. \tag{4.171}$$

In particular, S_k is a (non-linear) bounded operator in $G_1(\Omega)$. For given $\varepsilon > 0$ we choose Γ such that

$$\|h\,|G_2(\Omega)\| \le g_k^0(\mathrm{id}) + \varepsilon \quad \text{if} \quad \|h\,|G_1(\Omega)\| \le 1, \quad h(x^j) = 0, \tag{4.172}$$

for $j = 1, \ldots, k$. Then one has for

$$f \in G_1(\Omega) \quad \text{with} \quad \|f\,|G_1(\Omega)\| \le 1 \quad \text{and} \quad h = f - S_k f \tag{4.173}$$

that $\|h\,|G_1(\Omega)\| \preceq 1$ with $h(x^j) = 0$ and hence

$$\|f - S_k f\,|G_2(\Omega)\| = \|h\,|G_2(\Omega)\| \preceq g_k^0(\mathrm{id}) + \varepsilon. \tag{4.174}$$

One gets finally the converse of (4.164). □

Remark 4.35. In case of Banach spaces assertions of type (4.162) are known. They can be strengthened by

$$g_k^0(\mathrm{id}) \le g_k(\mathrm{id}) \le 2 g_k^0(\mathrm{id}), \qquad k \in \mathbb{N}. \tag{4.175}$$

We refer to [TWW88], pp. 45 and 58. This applies to the above spaces $G_1(\Omega)$ and $G_2(\Omega)$ if all p, q, r involved are larger than or equal to 1.

4.3.3 Main assertions

Again we assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$ and that $A^s_{pq}(\Omega)$ are the spaces as introduced in Definition 4.1 with the special case of the Hölder-Zygmund spaces

$$\mathcal{C}^s(\Omega) = B^s_{\infty\infty}(\Omega), \qquad s > 0, \tag{4.176}$$

considered in some detail in Section 1.11.10. Recall that $a_+ = \max(a, 0)$ if $a \in \mathbb{R}$.

Proposition 4.36. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ according to Definition* 4.3. *Let $g^{\mathrm{lin}}_k(\mathrm{id})$ be the linear sampling numbers as introduced in Definition* 4.32(ii) *where the embedding* id *is given by* (4.145)–(4.148).

(i) *Let*

$$0 < p < \infty, \quad 0 < q \le \infty, \quad s > n/p \quad \text{and} \quad 0 < r \le \infty. \tag{4.177}$$

Then

$$g^{\mathrm{lin}}_k \big(\mathrm{id} :\ F^s_{pq}(\Omega) \hookrightarrow L_r(\Omega)\big) \preceq k^{-\frac{s}{n} + (\frac{1}{p} - \frac{1}{r})_+}, \qquad k \in \mathbb{N}. \tag{4.178}$$

(ii) *Let $s > 0$ and $0 < r \le \infty$. Then*

$$g^{\mathrm{lin}}_k \left(\mathrm{id} :\ \mathcal{C}^s(\Omega) \hookrightarrow L_r(\Omega)\right) \preceq k^{-s/n}, \qquad k \in \mathbb{N}. \tag{4.179}$$

Proof. We prove (4.178). The proof of (4.179) is the same. As a consequence of

$$F^s_{pq_1}(\Omega) \hookrightarrow F^s_{pq_2}(\Omega) \quad \text{if} \quad q_1 \le q_2, \tag{4.180}$$

and Hölder's inequality it follows that we may assume $q \ge p$ and $r \ge p$. Let $\tau > 0$ and let $\{x^j\}^k_{j=1} \subset \Omega$ be points having pairwise distance of at least τ such that for some $d > 0$ the balls B^j centred at x^j and of radius $d\tau$ cover Ω. Here d is a positive constant which is independent of τ. We may assume $k \sim \tau^{-n}$ where the equivalence constants are independent of τ. Let $f \in F^s_{pq}(\Omega)$ and $\widetilde{f} \in F^s_{pq}(\mathbb{R}^n)$ with $\widetilde{f}|\Omega = f$ and

$$\|\widetilde{f}\,|F^s_{pq}(\mathbb{R}^n)\| \le 2\,\|f\,|F^s_{pq}(\Omega)\|. \tag{4.181}$$

Let

$$S_k f = \sum^k_{j=1} f(x^j)\, h_j, \qquad h_j \in L_\infty(\Omega) \text{ real}, \tag{4.182}$$

be the polynomial reproducing specification of (4.159) constructed in [Wen01]. Let $\mathcal{P}^M(\Omega)$ be as above in connection with Theorem 4.10. By Remark 4.11 we may assume that Ω is connected. According to [Wen01] there is a number $\tau_0 > 0$ such that one finds for all τ with $0 < \tau \le \tau_0$ and $M \in \mathbb{N}$ a mapping (4.182) with

$$(S_k P)(x) = P(x) \qquad \text{where} \quad P \in \mathcal{P}^M(\Omega), \quad x \in \Omega, \tag{4.183}$$

and

$$\sum^k_{j=1} |h_j(x)| \le 2, \quad x \in \Omega \quad \text{with} \quad \operatorname{supp} h_j \subset bB^j \cap \Omega. \tag{4.184}$$

Here bB^j is a ball concentric with the above ball B^j and of radius $bd\tau$, where $b > 1$ is a suitably chosen number which is independent of τ and M. Let $\Omega_j = B^j \cap \Omega$ and $\tilde{\Omega}_j = aB^j \cap \Omega$ for some $a > 1$ specified later on. Let $r < \infty$. Then one gets for S_k with (4.182)–(4.184) and $P_j \in \mathcal{P}^M(\Omega)$ that

$$\begin{aligned} &\|f - S_k f \,|L_r(\Omega)\|^r \\ &\le \sum_{j=1}^k \|f - P_j + S_k P_j - S_k f \,|L_r(\Omega_j)\|^r \\ &\le c\,\tau^n \sum_{j=1}^k \left(\sup_{x\in\Omega_j} |f(x) - P_j(x)|^r + \sup_{x\in\tilde{\Omega}_j} |f(x) - P_j(x)|^r \right), \end{aligned} \tag{4.185}$$

where the first term on the right-hand side comes from $f - P_j$ and the second term from (4.184) assuming that a is chosen sufficiently large (but independent of τ). Hence

$$\|f - S_k f \,|L_r(\Omega)\|^r \le c\tau^n \sum_{j=1}^k \sup_{x\in aB^j} |\tilde{f}(x) - P_j(x)|^r. \tag{4.186}$$

We use Corollary 4.13(ii) with aB^j, having radius $\lambda = ad\tau$, in place of ω_τ. Since $q \ge p$, (4.52) reduces to (4.50). We choose $u = 1$ and write simply d_t^M in place of the counterpart of $d_{t,u}^{M,\tau}$. Let P_j in (4.186) be the optimal polynomials in (4.53). Using $q \ge p$ and $r \ge p$ one obtains by (4.186), (4.53),

$$\begin{aligned} &\|f - S_k f \,|L_r(\Omega)\|^r \\ &\le c_1\, \tau^{(s-n/p+n/r)r} \sum_{j=1}^k \left(\int_{aB^j} \left(\int_0^\lambda t^{-sq} \left(d_t^M \tilde{f}\right)(x)^q \,\frac{\mathrm{d}t}{t}\right)^{p/q} \mathrm{d}x \right)^{r/p} \\ &\le c_1\, \tau^{(s-n/p+n/r)r} \left(\sum_{j=1}^k \int_{aB^j} \left(\int_0^\lambda t^{-sq} \left(d_t^M \tilde{f}\right)(x)^q \,\frac{\mathrm{d}t}{t}\right)^{p/q} \mathrm{d}x \right)^{r/p} \\ &\le c_2\, \tau^{(s-n/p+n/r)r} \left(\int_{\mathbb{R}^n} \left(\int_0^1 t^{-sq} \left(d_t^M \tilde{f}\right)(x)^q \,\frac{\mathrm{d}t}{t}\right)^{p/q} \mathrm{d}x \right)^{r/p}. \end{aligned} \tag{4.187}$$

Now it follows by Theorem 1.116(iii) and (4.181) that

$$\begin{aligned} \|f - S_k f \,|L_r(\Omega)\| &\le c_1 \tau^{s-n/p+n/r}\, \|\tilde{f} \,|F^s_{pq}(\mathbb{R}^n)\| \\ &\le c_2 \tau^{s-n/p+n/r}\, \|f \,|F^s_{pq}(\Omega)\|. \end{aligned} \tag{4.188}$$

If $r = \infty$ one has to modify in the usual way. Now (4.178) follows from (4.188) and $\tau^{-n} \sim k$. $\square$

Theorem 4.37. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ according to Definition* 4.3. *Let*

$$0 < p \leq \infty, \quad 0 < q \leq \infty, \quad s > n/p \quad \textit{and} \quad 0 < r \leq \infty \tag{4.189}$$

with $p < \infty$ for the F-spaces. Let

$$\mathrm{id}: \quad G_1(\Omega) = A^s_{pq}(\Omega) \hookrightarrow L_r(\Omega) = G_2(\Omega) \tag{4.190}$$

be the compact embedding according to (4.145)–(4.148) *and the explanations given there. Let $g_k(\mathrm{id})$ and $g^{\mathrm{lin}}_k(\mathrm{id})$ be the corresponding sampling numbers as introduced in Definition* 4.32. *Then*

$$g_k(\mathrm{id}) \sim g^{\mathrm{lin}}_k(\mathrm{id}) \sim k^{-\frac{s}{n}+(\frac{1}{p}-\frac{1}{r})_+}, \qquad k \in \mathbb{N}. \tag{4.191}$$

Proof. Step 1. First we extend Proposition 4.36 by real interpolation to the B-spaces. Let $p < \infty$,

$$0 < \theta < 1, \quad p \leq q_0 \leq \infty, \quad p \leq q_1 \leq \infty, \quad 0 < q \leq \infty, \tag{4.192}$$

and

$$n/p < s = (1-\theta)s_0 + \theta s_1 \quad \text{with} \quad s_0 \neq s_1. \tag{4.193}$$

We choose s_0 and s_1 near s such that we can apply (4.188) to $F^{s_0}_{p_0q_0}(\Omega)$ and $F^{s_1}_{p_1q_1}(\Omega)$ with the same S_k, interpreted as a linear bounded operator. Using the real interpolation (1.369) one gets for $r \geq p$,

$$\|f - S_k f \,|L_r(\Omega)\| \leq c\, \tau^{s-n/p+n/r}\, \|f \,|B^s_{pq}(\Omega)\|. \tag{4.194}$$

If $p = \infty$ then one can rely on (4.179) instead of (4.178) and the real interpolation formula (1.368) with $p = q_0 = q_1 = \infty$. Hence by the same arguments as in the proof of Proposition 4.36 we have for all embeddings (4.190),

$$g_k(\mathrm{id}) \leq g^{\mathrm{lin}}_k(\mathrm{id}) \preceq k^{-\frac{s}{n}+(\frac{1}{p}-\frac{1}{r})_+}, \qquad k \in \mathbb{N}. \tag{4.195}$$

Step 2. We prove the converse of (4.195),

$$k^{-\frac{s}{n}+(\frac{1}{p}-\frac{1}{r})_+} \preceq g_k(\mathrm{id}), \qquad k \in \mathbb{N}. \tag{4.196}$$

By (1.299) it is sufficient to deal with the B-spaces. Let $L \in \mathbb{N}$ and $c > 0$ be two suitably chosen fixed numbers which will be specified later on in dependence on Ω. Let $k = 2^{ln}$ with $l \in \mathbb{N}$. For any set $\{x^j\}^k_{j=1} \subset \Omega$ there are lattice points

$$y^j = 2^{-l-L} m^j \in \Omega \quad \text{with} \quad m^j \in \mathbb{Z}^n, \qquad 1 \leq j \leq k, \tag{4.197}$$

with $m^{j_0} \neq m^{j_1}$ if $j_0 \neq j_1$,

$$|y^j - x^w| \geq c2^{-l} \quad \text{and} \quad \mathrm{dist}\,(y^j, \partial\Omega) \geq c2^{-l} \tag{4.198}$$

for $1 \le j, w \le k$. Let ψ be a compactly supported sufficiently smooth Daubechies wavelet, say of type (3.3) with $m = 0$, such that one can apply Theorem 3.5 to $B^s_{pq}(\mathbb{R}^n)$. Let

$$f_k(x) = \sum_{j=1}^{k} a_j \psi(2^{l+L} x - m^j), \qquad a_j \in \mathbb{C}. \tag{4.199}$$

If $L \in \mathbb{N}$ and $c > 0$ are chosen appropriately then one gets for $k = 2^{ln}$ with $l \in \mathbb{N}$,

$$\operatorname{supp} f_k \subset \Omega, \qquad f_k(x^w) = 0 \quad \text{for} \quad w = 1, \ldots, k, \tag{4.200}$$

and by Theorem 3.5 uniformly in $l \in \mathbb{N}$ that

$$\| f_k \,|B^s_{pq}(\Omega)\| \sim 2^{l(s-n/p)} \left(\sum_{j=1}^{k} |a_j|^p \right)^{1/p} \tag{4.201}$$

with the usual modification if $p = \infty$. Furthermore one may assume that the terms in (4.199) have disjoint supports. Then

$$\| f_k \,|L_r(\Omega)\| \sim 2^{-\frac{ln}{r}} \left(\sum_{j=1}^{k} |a_j|^r \right)^{1/r} \tag{4.202}$$

with the usual modification if $r = \infty$. Let $r \le p$, $A = B$ in (4.190) and $\Gamma = \{x^j\}_{j=1}^k$ in (4.161). We choose $a_j = 1$ in (4.199)–(4.202) and get

$$\|\mathrm{id}^\Gamma\| \ge c\, 2^{-ls} = k^{-s/n}, \qquad k = 2^{ln}. \tag{4.203}$$

Then it follows by Proposition 4.34 that

$$k^{-\frac{s}{n} + (\frac{1}{p} - \frac{1}{r})_+} = k^{-s/n} \preceq g_k(\mathrm{id}), \qquad k \in \mathbb{N}. \tag{4.204}$$

If $r > p$ then we choose $a_1 = 1$ and $a_j = 0$ if $j \ge 2$ in (4.199). Applying again the above arguments one gets (4.196) also in this case. □

Corollary 4.38. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ according to Definition* 4.3 *and let $C^l(\bar{\Omega})$, $W^l_1(\Omega)$ and $W^l_\infty(\Omega)$ with $l \in \mathbb{N}$ be the spaces introduced in* (4.16) *and* (4.18), (4.19). *Let $0 < r \le \infty$ and*

$$\mathrm{id}_{l,\infty}: \quad C^l(\bar{\Omega}) \hookrightarrow L_r(\Omega) \quad \text{with} \quad l \in \mathbb{N}, \tag{4.205}$$

$$\mathrm{id}^*_{l,\infty}: \quad W^l_\infty(\Omega) \hookrightarrow L_r(\Omega) \quad \text{with} \quad l \in \mathbb{N}, \tag{4.206}$$

$$\mathrm{id}_{l,1}: \quad W^l_1(\Omega) \hookrightarrow L_r(\Omega) \quad \text{with} \quad n < l \in \mathbb{N}. \tag{4.207}$$

Then for $k \in \mathbb{N}$,

$$g_k(\mathrm{id}_{l,\infty}) \sim g_k(\mathrm{id}^*_{l,\infty}) \sim g_k^{\mathrm{lin}}(\mathrm{id}_{l,\infty}) \sim g_k^{\mathrm{lin}}(\mathrm{id}^*_{l,\infty}) \sim k^{-l/n} \tag{4.208}$$

and

$$g_k(\mathrm{id}_{l,1}) \sim g_k^{\mathrm{lin}}(\mathrm{id}_{l,1}) \sim k^{-\frac{l}{n} + (1 - \frac{1}{r})_+}. \tag{4.209}$$

Proof. Since the equivalences in (4.191) are independent of q the above assertions follow from Theorem 4.37 and Proposition 4.5. □

Remark 4.39. Proposition 4.36, Theorem 4.37 and Corollary 4.38 coincide with Proposition 22, Theorem 24 and Corollary 26 in [NoT04]. This applies also to the proof of Proposition 4.36 and Step 1 of the proof of Theorem 4.37. In Step 2 of the latter proof we relied on corresponding arguments in [Tri05a]. This will be of some use for us in what follows when dealing with the second case of interest, which is the embedding (4.145) with the source space (4.146), (4.147) and the target space (4.149)–(4.151). Recall that we always have (4.152), (4.153).

Theorem 4.40. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ according to Definition 4.3. Let*

$$\mathrm{id}: \quad A^{s_1}_{p_1q_1}(\Omega) \hookrightarrow A^{s_2}_{p_2q_2}(\Omega) \tag{4.210}$$

with

$$0 < p_1 \le \infty, \quad 0 < q_1 \le \infty, \quad s_1 > n/p_1, \quad (p_1 < \infty \text{ in the } F\text{-case}), \tag{4.211}$$

and

$$\begin{cases} 0 < p_2 \le \infty, \quad 0 < q_2 \le \infty, \quad (p_2 < \infty \text{ in the } F\text{-case}), \\ n(\frac{1}{p_2} - 1)_+ < s_2 < s_1 - n(\frac{1}{p_1} - \frac{1}{p_2})_+, \end{cases} \tag{4.212}$$

be the compact embedding according to (4.145)–(4.147) , (4.149)–(4.151) *and the explanations given there. Let $g_k(\mathrm{id})$ be the corresponding sampling numbers as introduced in Definition* 4.32. *Then*

$$g_k(\mathrm{id}) \sim k^{-\frac{s_1-s_2}{n} + (\frac{1}{p_1} - \frac{1}{p_2})_+}, \qquad k \in \mathbb{N}, \tag{4.213}$$

where the equivalence constants are independent of k.

Proof. Step 1. By (1.299) it is sufficient to deal with the B-spaces. First let $p_1 = p_2 = p$ and $1 < p \le \infty$. Then we have by (4.191) that

$$g_k(\mathrm{id}: \ G_1 \hookrightarrow L_p) \sim k^{-s_1/n}, \qquad k \in \mathbb{N}, \tag{4.214}$$

with $G_1 = B^{s_1}_{pp}$, where we now omit Ω in the notation. Using (1.369) we get

$$(G_1, L_p)_{\theta,q_2} = (B^{s_1}_{pp}, L_p)_{\theta,q_2} = B^{s_2}_{pq_2} = G_2 \quad \text{with} \quad s_2 = (1-\theta)s_1, \tag{4.215}$$

where $0 < \theta < 1$. For given k we choose Γ in (4.161) and Proposition 4.34 optimal. Then it follows from (4.214), (4.215) and the interpolation property that

$$\begin{aligned} \|\mathrm{id}^\Gamma: \ G^\Gamma_1 \hookrightarrow G_2\| &\le \|\mathrm{id}^\Gamma: \ G^\Gamma_1 \hookrightarrow G_1\|^{1-\theta} \cdot \|\mathrm{id}^\Gamma: G_1 \hookrightarrow L_p\|^\theta \\ &\le c'\, g^\theta_k(\mathrm{id}: \ G_1 \hookrightarrow L_p) \\ &\le c''\, k^{-\frac{s_1-s_2}{n}}, \qquad k \in \mathbb{N}. \end{aligned} \tag{4.216}$$

Hence,

$$g_k(\mathrm{id} : G_1 \hookrightarrow G_2) \le c\, k^{-\frac{s_1-s_2}{n}}, \qquad k \in \mathbb{N}, \tag{4.217}$$

so far under the restriction $1 < p = p_1 = p_2 \le \infty$. Let $0 < p = p_1 = p_2 \le 1$. Then one can interpret G_1 and G_2 as subspaces of $L_p(\Omega)$ as we discussed briefly after (4.153). Then (4.215) with (4.212) remains valid. We refer to [DeS93], Theorem 6.3, and the explanations given in [Triϵ], pp. 373/374, formula (25.101). Then one gets (4.217) for all admitted spaces G_1 and G_2 with $0 < p_1 = p_2 \le \infty$.

Step 2. Let $p_1 < p_2 \le \infty$ and let

$$G_0 = G_0(\Omega) = B^{s_0}_{p_2 q_1}(\Omega) \quad \text{with} \quad s_0 - n/p_2 = s_1 - n/p_1 > 0. \tag{4.218}$$

As in (1.301) and the references given there one has the continuous embedding

$$\mathrm{id} : \quad G_1 = B^{s_1}_{p_1 q_1}(\Omega) \hookrightarrow B^{s_0}_{p_2 q_1}(\Omega) = G_0. \tag{4.219}$$

Let Γ be as in (4.161) and Proposition 4.34. Since $s_0 - n/p_2 > 0$ the decomposition

$$\mathrm{id}^\Gamma : \quad G_1^\Gamma \hookrightarrow G_2 = (\mathrm{id}^\Gamma :\ G_0^\Gamma \hookrightarrow G_2) \circ (\mathrm{id}^\Gamma :\ G_1^\Gamma \hookrightarrow G_0^\Gamma) \tag{4.220}$$

makes sense. We apply Step 1 to the first factor on the right-hand side. Then it follows by (4.218) and optimally chosen Γ that

$$\|\mathrm{id}^\Gamma :\ G_1^\Gamma \hookrightarrow G_2\| \le c\, k^{-\frac{s_0-s_2}{n}} = c\, k^{-\frac{s_1-s_2}{n} + \frac{1}{p_1} - \frac{1}{p_2}}, \qquad k \in \mathbb{N}. \tag{4.221}$$

By Proposition 4.34 this proves

$$g_k(\mathrm{id} :\ G_1 \hookrightarrow G_2) \le c\, k^{-\frac{s_1-s_2}{n} + (\frac{1}{p_1} - \frac{1}{p_2})_+}, \qquad k \in \mathbb{N}, \tag{4.222}$$

if $p_1 \le p_2$. If $p_2 < p_1$ then one gets (4.222) from the above cases and the continuous embedding

$$\mathrm{id} : \quad B^{s_2}_{p_1 q_2}(\Omega) \hookrightarrow B^{s_2}_{p_2 q_2}(\Omega) \tag{4.223}$$

which will be justified in Remark 4.41 below.

Step 3. The proof of the converse of (4.222) is the same as in Step 2 of the proof of Theorem 4.37. Let f_k be given by (4.199) with a suitably chosen Daubechies wavelet. Then we have (4.201) with $B^{s_1}_{p_1 q_1}$ and $B^{s_2}_{p_2 q_2}$ in place of B^s_{pq}. Afterwards one gets the converse of (4.222) in the same way as there. □

Remark 4.41. Let K be a ball in $\mathbb{R}^n$ and let $s \in \mathbb{R}$, $0 < q \le \infty$, $0 < p_2 < p_1 \le \infty$. If

$$f \in B^s_{p_1 q}(\mathbb{R}^n) \quad \text{with} \quad \mathrm{supp}\, f \subset K, \tag{4.224}$$

then $f \in B^s_{p_2 q}(\mathbb{R}^n)$ and

$$\|f\,|B^s_{p_2 q}(\mathbb{R}^n)\| \le c_K\, \|f\,|B^s_{p_1 q}(\mathbb{R}^n)\| \tag{4.225}$$

where c_K is independent of f with (4.224). This follows from Theorem 1.10 and Hölder's inequality. Now (4.223) is a consequence of this observation and Definition 4.1. In contrast to Theorem 4.37 we have no equivalence assertion of type (4.213) for $g_k^{\mathrm{lin}}(\mathrm{id})$. The above theorem coincides essentially with Theorem 13 in [Tri05a].

Remark 4.42. Problems of sampling and optimal recovery of functions have been studied by many authors, preferably for functions on cubes $\Omega = [0,1]^n$ or periodic functions on the torus and with $C^k(\bar{\Omega})$, Hölder-Zygmund spaces $\mathcal{C}^s(\Omega)$, classical Sobolev spaces $W_p^k(\Omega)$ or classical Besov spaces as source spaces. We refer to [Cia78], [Nov88], [Hei94], [Kud93], [Kud95], [Kud98], [Tem93]. All spaces considered there are Banach spaces (in contrast to the above approach). Corresponding results for bounded Lipschitz domains may be found in [Wen01] (which we used in the above proofs) and [NWW04]. Instead of isotropic spaces one may use spaces with dominating mixed smoothness or anisotropic spaces. We refer to [Tem93] and [BNR99]. In the present Section 4.3 we followed largely [NoT04], [Tri05a]. In [NoT04] one finds further discussions and references especially in connection with other methods such as randomized algorithms and algorithms for quantum computers.

4.4 Complements

4.4.1 Relations to other numbers measuring compactness

We wish to compare the sampling numbers in (4.191) for the compact embedding (4.190) with corresponding approximation numbers and entropy numbers. Recall that the second part of this book beginning with Chapter 2 should be readable independently. This forces us to repeat what is meant by approximation numbers and entropy numbers.

Definition 4.43. *Let A, B be complex quasi-Banach spaces and let $T \in L(A,B)$ be compact. Let U_A be the unit ball in A.*

(i) *Then for all $k \in \mathbb{N}$ the* kth *entropy number $e_k(T)$ of T is defined as the infimum of all $\varepsilon > 0$ such that*

$$T(U_A) \subset \bigcup_{j=1}^{2^{k-1}} (b_j + \varepsilon U_B) \text{ for some } b_1, \ldots, b_{2^{k-1}} \in B. \tag{4.226}$$

(ii) *Then for all $k \in \mathbb{N}$ the* kth *approximation number $a_k(T)$ of T is defined by*

$$a_k(T) = \inf \{ \|T - L\| : \ L \in L(A,B), \ \operatorname{rank} L < k \}, \tag{4.227}$$

where $\operatorname{rank} L$ *is the dimension of the range of L.*

Remark 4.44. This coincides essentially with Definition 1.87. One may consult Section 1.10 for discussions, explanations and properties of these numbers and also for corresponding references. If $T = \mathrm{id}$ is given by (4.190) then one may ask what can be said about the corresponding numbers $a_k(\mathrm{id})$ and $e_k(\mathrm{id})$. We rely here on Theorem 1.107. This forces us to restrict r in (4.190) to $r \geq 1$, although the corresponding assertions should be true for all $0 < r \leq \infty$.

Theorem 4.45. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ according to Definition 4.3. Let*

$$0 < p \le \infty, \quad 0 < q \le \infty, \quad s > n/p \quad \textit{and} \quad 1 \le r \le \infty \tag{4.228}$$

with $p < \infty$ for the F-spaces. Let

$$\mathrm{id}: \quad G_1(\Omega) = A^s_{pq}(\Omega) \hookrightarrow L_r(\Omega) = G_2(\Omega) \tag{4.229}$$

be the compact embedding according to (4.145)–(4.148) (with $r \ge 1$). Let $g_k(\mathrm{id})$, $g^{\mathrm{lin}}_k(\mathrm{id})$, $a_k(\mathrm{id})$, $e_k(\mathrm{id})$ be the numbers as introduced in Definitions 4.32 and 4.43. Then

$$k^{-s/n} \sim e_k(\mathrm{id}) \preceq a_k(\mathrm{id}) \preceq g_k(\mathrm{id}) \sim g^{\mathrm{lin}}_k(\mathrm{id}) \sim k^{-\frac{s}{n}+(\frac{1}{p}-\frac{1}{r})_+} \tag{4.230}$$

where $k \in \mathbb{N}$. Furthermore,

$$e_k(\mathrm{id}) \sim a_k(\mathrm{id}) \quad \textit{if, and only if,} \quad r \le p, \tag{4.231}$$

and

$$a_k(\mathrm{id}) \sim g_k(\mathrm{id}) \ \textit{if, and only if,} \quad \begin{cases} \textit{either} & 0 < p \le r \le 2, \\ \textit{or} & 2 \le p \le r \le \infty, \\ \textit{or} & 1 \le r \le p \le \infty. \end{cases} \tag{4.232}$$

Proof. Step 1. The first equivalence in (4.230) is covered by Theorem 1.97, the embedding (1.299) and the restriction of (4.24), (4.25) to Ω as far as the limiting cases $r = 1$ and $r = \infty$ are concerned. By the same argument one gets the other equivalences in (4.230) as a consequence of (4.191). Hence it remains to prove the inequalities in (4.230) and the equivalences (4.231), (4.232). Let p, r be as on the right-hand side of (4.232). Then the inequalities in (4.230) and the if-parts in (4.231), (4.232) follow from (1.348) in Theorem 1.107 and Theorem 1.97. We divide the remaining cases into four subcases,

$$0 < p < 2 < r < \infty, \qquad s > n \max(1 - 1/r, 1/p), \tag{4.233}$$

$$0 < p < 2 < r < \infty, \qquad s < n \max(1 - 1/r, 1/p), \tag{4.234}$$

and the two limiting cases

$$0 < p < 2 < r < \infty, \qquad s = n \max(1 - 1/r, 1/p), \tag{4.235}$$

$$0 < p < 2, \qquad r = \infty. \tag{4.236}$$

Step 2. In case of (4.233) we have

$$\lambda = \frac{s}{n} - \max\left(\frac{1}{2} - \frac{1}{r}, \frac{1}{p} - \frac{1}{2}\right) > \frac{1}{2} \tag{4.237}$$

and

$$\frac{s}{n} - \frac{1}{p} + \frac{1}{r} < \lambda < \frac{s}{n}. \tag{4.238}$$

Then (1.349) proves the inequalities in (4.230) and that there are no equivalences.

Step 3. We deal with (4.234) and put $\frac{1}{p} + \frac{1}{p'} = 1$ if $1 \le p \le \infty$ and $p' = \infty$ if $p < 1$. Using $s > n/p$ the second part of (4.234) reduces to

$$\frac{n}{p} < s < n\,(1 - \frac{1}{r}) < n. \tag{4.239}$$

In particular, $p > 1$ and $2 < p' < r$. If λ is given by the left-hand side of (4.237) then we have now $\lambda < 1/2$. Then it follows by (1.350) that

$$a_k(\mathrm{id}) \sim k^{-(\frac{s}{n} - \frac{1}{p} + \frac{1}{r})p'/2}, \qquad k \in \mathbb{N}. \tag{4.240}$$

This proves $a_k(\mathrm{id}) \preceq g_k(\mathrm{id})$ and excludes equivalences. By (4.239) and $p < 2$, hence $2s > n$, we get

$$s + \frac{n}{r} < n = \frac{n}{p} + \frac{n}{p'} < \frac{n}{p} + \frac{2s}{p'} \tag{4.241}$$

and

$$(s - \frac{n}{p} + \frac{n}{r}) \cdot p'/2 < s. \tag{4.242}$$

This proves $e_k(\mathrm{id}) \preceq a_k(\mathrm{id})$ and excludes equivalences.

Step 4. We deal with the limiting case (4.235). Then (4.239) and the exponent in (4.240) must be modified by

$$\frac{n}{p} < s = n\,(1 - \frac{1}{r}) < n \quad \text{and} \quad \mu = (\frac{s}{n} - \frac{1}{p} + \frac{1}{r}) \cdot p'/2 = \frac{1}{2}\,. \tag{4.243}$$

Furthermore, if again λ is the number on the left-hand side of (4.237) then we have now $\lambda = 1/2$. Although the case $\lambda = 1/2$ is not covered by Theorem 1.107 it follows by this theorem that for any $\varepsilon > 0$,

$$k^{-\frac{1}{2} - \varepsilon} \preceq a_k(\mathrm{id}) \preceq k^{-\frac{1}{2} + \varepsilon}, \qquad k \in \mathbb{N}. \tag{4.244}$$

Furthermore,

$$\frac{s}{n} = 1 - \frac{1}{r} > \frac{1}{2} \quad \text{and} \quad \frac{s}{n} + \frac{1}{r} - \frac{1}{p} = \frac{1}{p'} < \frac{1}{2} \tag{4.245}$$

are the two relevant exponents for the entropy numbers in (1.306) and the sampling numbers in (4.230). This proves (4.230) also in this limiting case and excludes equivalences. By the same arguments one can incorporate also the limiting case (4.236). □

Remark 4.46. We followed [NoT04] where we formulated Theorem 4.45 without proof. We add a few comments. In case of approximation numbers $a_{k+1}(\mathrm{id})$ one allows all methods

$$S_k f = \sum_{j=1}^{k} L_j(f)\, h_j, \qquad h_j \in L_r(\Omega), \tag{4.246}$$

where $L_j : A^s_{pq}(\Omega) \mapsto \mathbb{C}$ are arbitrary linear continuous functionals. If one admits only $L_j(f) = f(x^j)$ according to (4.159) then one gets the linear sampling numbers $g^{\text{lin}}_k(\text{id})$. Now (4.160) and the corresponding inequalities in (4.230) are obvious. By (4.230), (4.232) it follows that one gets a better rate of convergence for $a_k(\text{id})$ than for $g^{\text{lin}}_k(\text{id})$ if, and only if, $p < 2 < r$. The restriction $r \geq 1$ in the above theorem compared with $r > 0$ in Theorem 4.37 comes from the application of Theorem 1.107. If the latter theorem can be extended at least to the spaces covered by Theorem 4.37 then one can extend also the above theorem to the target spaces $L_r(\Omega)$ with $0 < r \leq \infty$. A corresponding assertion for the entropy numbers of the embedding (4.189), (4.190) may be found in [NoT04], Corollary 28, with the expected outcome,

$$e_k(\text{id}) \sim k^{-s/n}, \qquad k \in \mathbb{N}. \tag{4.247}$$

By the above method it might be possible to extend some of the assertions of the above theorem to the embeddings considered in Theorem 4.40.

4.4.2 Embedding constants

Definition 4.32 of the sampling numbers $g_k(\text{id})$ requires (4.152) for the source spaces $A^{s_1}_{p_1 q_1}(\Omega)$ which is ensured if p_1, q_1, s_1 are restricted by (4.147). This suggests to have a closer look at the behavior of the embedding constants of

$$\text{id} : \quad A^s_{pq}(\Omega) \hookrightarrow C(\bar{\Omega}) \tag{4.248}$$

if $0 < s - n/p \to 0$. Proposition 4.6(ii) gives a first impression that can be expected if one reaches the critical line $s = n/p$ from above. When it comes to embedding constants one has to fix the quasi-norms. We assume that the spaces $A^s_{pq}(\mathbb{R}^n)$ with $A = B$ or $A = F$ are quasi-normed according to Definition 2.1 with a fixed function φ_0 in (2.8). In case of domains Ω we fix the corresponding quasi-norms afterwards according to Definition 4.1. Let for

$$0 < p \leq \infty, \quad 0 < q \leq \infty, \quad s \in \mathbb{R}, \quad \varepsilon = s - n/p \tag{4.249}$$

(with $p < \infty$ for the F-spaces) and $0 < \delta < 1$,

$$M_\delta(B) = \big\{ (p,q,s) : \ 0 < \varepsilon < 1, \ \delta < s < \delta^{-1} \big\} \tag{4.250}$$

and

$$M_\delta(F) = \big\{ (p,q,s) : \ p < \delta^{-1}, \ q > \delta, \ 0 < \varepsilon < 1, \ \delta < s < \delta^{-1} \big\}. \tag{4.251}$$

For fixed δ the atomic representations used below result in quasi-norms which are uniformly equivalent to the above fixed quasi-norms. This follows from the proof of atomic representations. But we will not stress this point in the sequel. Recall that $\frac{1}{p} + \frac{1}{p'} = 1$ if $1 \leq p \leq \infty$ and $p' = \infty$ if $0 < p < 1$.

Theorem 4.47. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ according to Definition 4.3. For $0<\delta<1$ let*

$$\mathrm{id}(B^s_{pq}): \quad B^s_{pq}(\Omega) \hookrightarrow C(\bar{\Omega}), \quad (p,q,s) \in M_\delta(B) \tag{4.252}$$

and

$$\mathrm{id}(F^s_{pq}): \quad F^s_{pq}(\Omega) \hookrightarrow C(\bar{\Omega}), \quad (p,q,s) \in M_\delta(F). \tag{4.253}$$

Then

$$\|\mathrm{id}(B^s_{pq})\| \sim \varepsilon^{-1/q'} \quad \textit{and} \quad \|\mathrm{id}(F^s_{pq})\| \sim \varepsilon^{-1/p'}, \tag{4.254}$$

where the equivalence constants are independent of p,q,s.

Proof. Step 1. First we deal with the B-spaces assuming $1<q\le\infty$, hence $1\le q'<\infty$. Let $f\in B^s_{pq}(\Omega)$ and $g\in B^s_{pq}(\mathbb{R}^n)$ with

$$\|g\,|B^s_{pq}(\mathbb{R}^n)\| \le 2\,\|f\,|B^s_{pq}(\Omega)\|, \qquad g|\Omega = f, \tag{4.255}$$

and an optimal atomic decomposition

$$g(x) = \sum_{j=0}^\infty \sum_{m\in\mathbb{Z}^n} \lambda_{j,m}\, a_{j,m}(x), \qquad x\in\mathbb{R}^n, \tag{4.256}$$

according to Theorem 1.19. Let $0\in\Omega$. Then

$$f(0) = \sum_{j=0}^\infty \lambda_{j,0}\, a_{j,0}(0) + + \tag{4.257}$$

where $++$ indicates a few terms of the same type (controlled overlapping based on the supports of $a_{j,m}$). If in addition $\varepsilon q'\ge 1$ then we get

$$\begin{aligned} |f(0)| &\le c\sum_{j=0}^\infty 2^{-j\varepsilon}\,|\lambda_{j,0}| + + \\ &\le c\left(\sum_{j=0}^\infty |\lambda_{j,0}|^q\right)^{1/q} \cdot \left(\sum_{j=0}^\infty 2^{-j\varepsilon q'}\right)^{1/q'} + + \\ &\le c\,\|f\,|B^s_{pq}(\Omega)\| \end{aligned} \tag{4.258}$$

with the usual modification if $q=\infty$, where c is independent of ε. If $\varepsilon q'<1$ then one has

$$|f(0)| \le c\,\|f\,|B^s_{pq}(\Omega)\| \left(\frac{1}{1-2^{-\varepsilon q'}}\right)^{1/q'} \le c'\,\varepsilon^{-1/q'}\,\|f\,|B^s_{pq}(\Omega)\| \tag{4.259}$$

where c' is independent of ε. If $0 < q \le 1$ then one gets (4.259) with $q' = \infty$ (in good agreement with Proposition 4.6(ii)). Now one obtains that

$$\|\mathrm{id}(B^s_{pq})\| \le c\varepsilon^{-1/q'}. \tag{4.260}$$

Next we deal with the F-case. Let (4.256) be an optimal decomposition of the F-counterpart of (4.255) and $1 < p < \infty$. Then it follows from the arguments in Example 1.35 that

$$|f(0)| \le c \left(\sum_{j=0}^{\infty} |\lambda_{j,0}|^p \right)^{1/p} \varepsilon^{-1/p'} + + \le c'\, \varepsilon^{-1/p'} \, \|f \,|F^s_{pq}(\Omega)\|. \tag{4.261}$$

If $0 < p \le 1$ then one gets (4.261) with $p' = \infty$. Both together give the F-counterpart of (4.260).

Step 2. We prove the estimate from below for the B-spaces assuming $q > 1$. Let $0 \le \psi \in D(\mathbb{R}^n)$ with $\psi(x) > 0$ if, and only if, $|x| < \eta$ (with a sufficiently small $\eta > 0$) and $\psi(0) = 1$. Again let $0 \in \Omega$. Then

$$f(x) = \sum_{j=0}^{k-1} 2^{-j\varepsilon}\, \psi(2^j x) \in D(\Omega) \tag{4.262}$$

is an atomic decomposition in $B^s_{pq}(\mathbb{R}^n)$ according to Theorem 1.19. For given $0 < \varepsilon < 1$ we choose $k \in \mathbb{N}$ such that $k\varepsilon \sim 1$. Then it follows from the assumed estimate

$$\|f \,|C(\bar{\Omega})\| \le c(B^s_{pq})\, \|f \,|B^s_{pq}(\Omega)\| \le c(B^s_{pq})\, \|f \,|B^s_{pq}(\mathbb{R}^n)\| \tag{4.263}$$

that

$$k \sim \varepsilon^{-1} \sim \frac{1 - 2^{-k\varepsilon}}{1 - 2^{-\varepsilon}} = \sum_{j=0}^{k-1} 2^{-j\varepsilon} \le c(B^s_{pq})\, k^{1/q} \tag{4.264}$$

and hence

$$c(B^s_{pq}) \ge c\, k^{1/q'} \sim \varepsilon^{-1/q'}. \tag{4.265}$$

If $0 < q \le 1$ then one may choose $k = 1$ in (4.262). This proves the converse of (4.260) and hence the first assertion in (4.254). In the F-case we assume $0 < p < \infty$ and, in addition, $q \ge 1$. Then one gets from Theorem 1.19 and Example 1.35 the F-counterpart of the above estimate with p' in place of q'. If $q < 1$ then (4.262) might be no longer an atomic decomposition (maybe some moment conditions are needed). But how to circumvent this minor difficulty may be found in [Triϵ], Step 5 of the proof of Theorem 13.2, pp. 209/210. □

Remark 4.48. We followed [Tri05a]. If $0 < q \le 1$ then $\|\mathrm{id}(B^s_{pq})\|$ in (4.254) is uniformly bounded. Similarly for the F-spaces if $0 < p \le 1$. This is in good agreement with Proposition 4.6(ii). In [Tri05a] one finds further results in this direction especially if one prescribes the values of f at points $\{x^j\}_{j=1}^k \subset \Omega$, hence $f(x^j) = a_j \in \mathbb{C}$.

Chapter 5

Anisotropic Function Spaces

5.1 Definitions and basic properties

5.1.1 Introduction

In the preceding Chapters 2–4 we dealt in detail with some recent aspects of the theory of isotropic (inhomogeneous unweighted) spaces B^s_{pq} and F^s_{pq} on $\mathbb{R}^n$ and on domains in $\mathbb{R}^n$. Also the self-contained survey in Chapter 1 covers only isotropic spaces on $\mathbb{R}^n$, domains in $\mathbb{R}^n$, and rough structures such as fractals. On the other hand, the Russian school in particular considered anisotropic spaces on $\mathbb{R}^n$ and on (adapted) domains, together with isotropic spaces, from the very beginning of the theory of function spaces as a self-contained branch of functional analysis in the late 1950s and early 1960s. One might think of the *classical anisotropic Sobolev spaces*

$$W^{\bar{k}}_p(\mathbb{R}^n) = \Big\{ f \in L_p(\mathbb{R}^n) : \ \|f \,|W^{\bar{k}}_p(\mathbb{R}^n)\| < \infty \Big\}, \tag{5.1}$$

where

$$1 < p < \infty, \quad \bar{k} = (k_1, \ldots, k_n), \quad k_l \in \mathbb{N}, \tag{5.2}$$

and

$$\|f \,|W^{\bar{k}}_p(\mathbb{R}^n)\| = \|f \,|L_p(\mathbb{R}^n)\| + \sum_{l=1}^n \left\| \frac{\partial^{k_l} f}{\partial x_l^{k_l}} \,|L_p(\mathbb{R}^n) \right\| \tag{5.3}$$

and the *classical anisotropic Besov spaces*

$$B^{\bar{s}}_{pq}(\mathbb{R}^n) = \big\{ f \in L_p(\mathbb{R}^n) : \ \|f \,|B^{\bar{s}}_{pq}(\mathbb{R}^n)\|_{\bar{M}} < \infty \big\} \tag{5.4}$$

where

$$1 < p < \infty, \ 1 \le q \le \infty, \ \bar{s} = (s_1, \ldots, s_n), \ 0 < s_l < M_l \in \mathbb{N}, \tag{5.5}$$

and

$$\begin{aligned} &\| f \,|B^{\bar{s}}_{pq}(\mathbb{R}^n)\|_{\bar{M}} \\ &= \| f \,|L_p(\mathbb{R}^n)\| + \sum_{l=1}^{n} \left(\int_0^1 t^{-s_l q} \left\| \Delta^{M_l}_{t,l} f \,|L_p(\mathbb{R}^n)\right\|^q \frac{\mathrm{d}t}{t} \right)^{1/q} \end{aligned} \tag{5.6}$$

(usual modification if $q = \infty$) with $\bar{M} = (M_1, \ldots, M_n)$. Here

$$\left(\Delta^m_{t,l} f\right)(x) = \left(\Delta^m_h f\right)(x) \quad \text{with} \quad h = te_l, \quad t \in \mathbb{R}, \tag{5.7}$$

$l = 1, \ldots, n$, are the iterated differences according to (4.32) in the direction of the lth coordinate where $e_l = (0, \ldots, 0, 1, 0, \ldots, 0)$ is the corresponding unit vector with 1 at place l. If

$$k_1 = \cdots = k_n = k \in \mathbb{N} \quad \text{then} \quad W^{\bar{k}}_p(\mathbb{R}^n) = W^k_p(\mathbb{R}^n) \tag{5.8}$$

are the classical (isotropic) Sobolev spaces according to (1.3), (1.4), and if

$$s_1 = \cdots = s_n = s > 0 \quad \text{then} \quad B^{\bar{s}}_{pq}(\mathbb{R}^n) = B^s_{pq}(\mathbb{R}^n) \tag{5.9}$$

are the classical (isotropic) Besov spaces according to (1.13). The theory of the classical anisotropic Besov spaces and of the (fractional) anisotropic Sobolev spaces (also denoted as anisotropic Bessel-potential spaces) was developed systematically in the 1960s and 1970s by the Russian school and may be found in the books [Nik77] (first edition 1969), [BIN75] and in the surveys [BKLN88], [KuN88]. We rely here on the direct anisotropic generalisation of the Fourier analytical Definition 2.1 of the isotropic spaces. The theory of the corresponding isotropic spaces as developed in the 1970s and early 1980s was based on maximal inequalities and (more or less sophisticated) Fourier multipliers and may be found in [Triβ]. A full anisotropic counterpart of these isotropic techniques was established from the 1980s up to the middle of the 1990s in several papers which will be mentioned later on, but there is no comprehensive presentation in a book. We have a similar situation as far as more recent developments are concerned such as atomic and quarkonial expansions and local means. All this has been available for a few years (references will be given below) and the theory of the anisotropic spaces reached in many respects the same degree of sophistication as the corresponding assertions for isotropic spaces. But we cannot present this theory in detail here. We collect in this Section 5.1 without proofs (but with detailed references) some basic assertions and those specific topics which are needed later on, in particular atomic expansions and local means. Based on these ingredients we prove in Section 5.2 wavelet isomorphisms for the anisotropic spaces considered in generalisation of the Theorems 3.5 and 3.12. As a somewhat surprising application we describe in Section 5.3 a transference method allowing us to transfer some results for isotropic spaces directly to anisotropic spaces without any additional efforts.

5.1.2 Definitions

We call

$$\alpha = (\alpha_1, \ldots, \alpha_n) \quad \text{with} \quad 0 < \alpha_1 \le \cdots \le \alpha_n < \infty, \quad \sum_{j=1}^n \alpha_j = n, \tag{5.10}$$

an *anisotropy* in $\mathbb{R}^n$. For $t > 0$ and $r \in \mathbb{R}$ we put

$$t^\alpha x = (t^{\alpha_1} x_1, \ldots, t^{\alpha_n} x_n) \quad \text{and} \quad t^{r\alpha} x = (t^r)^\alpha x, \tag{5.11}$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. We modify the (isotropic) resolution of unity (2.8)–(2.10) as follows. Let $\varphi_0^\alpha \in S(\mathbb{R}^n)$ with

$$\varphi_0^\alpha(x) = 1 \text{ if } \sup_l |x_l| \le 1 \quad \text{and} \quad \varphi_0^\alpha(y) = 0 \text{ if } \sup_l 2^{-\alpha_l} |y_l| \ge 1, \tag{5.12}$$

where again $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. Let

$$\varphi_k^\alpha(x) = \varphi_0^\alpha \left(2^{-k\alpha} x\right) - \varphi_0^\alpha \left(2^{-(k-1)\alpha} x\right), \quad k \in \mathbb{N}, \quad x \in \mathbb{R}^n. \tag{5.13}$$

Then

$$\sum_{j=0}^\infty \varphi_j^\alpha(x) = 1 \quad \text{for all } x \in \mathbb{R}^n, \tag{5.14}$$

is an anisotropic resolution of unity with

$$\text{supp } \varphi_0^\alpha \subset R_1^\alpha \quad \text{and} \quad \text{supp } \varphi_k^\alpha \subset R_{k+1}^\alpha \backslash R_{k-1}^\alpha, \quad k \in \mathbb{N}, \tag{5.15}$$

where R_j^α are rectangles

$$R_j^\alpha = \left\{ x \in \mathbb{R}^n : \ |x_l| \le 2^{\alpha_l j} \right\}, \qquad j \in \mathbb{N}_0. \tag{5.16}$$

Definition 2.1 can now be extended to anisotropic spaces as follows.

Definition 5.1. *Let α be an anisotropy according to* (5.10) *and let $\varphi^\alpha = \{\varphi_j^\alpha\}_{j=0}^\infty$ with* (5.12)–(5.16) *be a corresponding anisotropic resolution of unity.*

(i) *Let*

$$0 < p \le \infty, \quad 0 < q \le \infty, \quad s \in \mathbb{R}, \tag{5.17}$$

and

$$\|f \,|B_{pq}^{s,\alpha}(\mathbb{R}^n)\|_{\varphi^\alpha} = \left(\sum_{j=0}^\infty 2^{jsq} \left\| (\varphi_j^\alpha \hat{f})^\vee \,|L_p(\mathbb{R}^n) \right\|^q \right)^{1/q} \tag{5.18}$$

(*with the usual modification if $q = \infty$*). *Then*

$$B_{pq}^{s,\alpha}(\mathbb{R}^n) = \left\{ f \in S'(\mathbb{R}^n) : \ \|f \,|B_{pq}^{s,\alpha}(\mathbb{R}^n)\|_{\varphi^\alpha} < \infty \right\}. \tag{5.19}$$

(ii) *Let*

$$0 < p < \infty, \quad 0 < q \le \infty, \quad s \in \mathbb{R}, \tag{5.20}$$

and

$$\| f \, | F^{s,\alpha}_{pq}(\mathbb{R}^n) \|_{\varphi^\alpha} = \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} \left| (\varphi^\alpha_j \hat{f})^\vee(\cdot) \right|^q \right)^{1/q} | L_p(\mathbb{R}^n) \right\| \tag{5.21}$$

(*with the usual modification if* $q = \infty$). *Then*

$$F^{s,\alpha}_{pq}(\mathbb{R}^n) = \left\{ f \in S'(\mathbb{R}^n) : \ \| f \, | F^{s,\alpha}_{pq}(\mathbb{R}^n) \|_{\varphi^\alpha} < \infty \right\}. \tag{5.22}$$

Remark 5.2. As for basic notation we refer to Sections 2.1.2, 2.1.3. If $\alpha = (1, \ldots, 1)$ then the above definition is only a minor technical modification of Definition 2.1. In particular,

$$B^{s,\alpha}_{pq}(\mathbb{R}^n) = B^s_{pq}(\mathbb{R}^n) \quad \text{and} \quad F^{s,\alpha}_{pq}(\mathbb{R}^n) = F^s_{pq}(\mathbb{R}^n) \tag{5.23}$$

for $\alpha = (1, \ldots, 1)$ and all admitted parameters s, p, q.

Theorem 5.3. *Let* α *be an anisotropy according to* (5.10) *and let* $\varphi^\alpha = \{\varphi^\alpha_j\}_{j=0}^\infty$ *with* (5.12)–(5.16) *be a corresponding anisotropic resolution of unity.*

(i) *Let* p, q, s *be given by* (5.17). *Then* $B^{s,\alpha}_{pq}(\mathbb{R}^n)$ *according to* (5.18), (5.19) *is a quasi-Banach space* (*Banach space if* $p \ge 1$, $q \ge 1$) *and it is independent of* φ^α (*equivalent quasi-norms*).

(ii) *Let* p, q, s *be given by* (5.20). *Then* $F^{s,\alpha}_{pq}(\mathbb{R}^n)$ *according to* (5.21), (5.22) *is a quasi-Banach space* (*Banach space if* $p \ge 1$, $q \ge 1$) *and it is independent of* φ^α (*equivalent quasi-norms*).

Remark 5.4. The classical theory of anisotropic function spaces of Sobolev type and Besov type, as outlined in our Introduction 5.1.1, was mainly developed by the Russian school and is covered by the references to the Russian literature given there. In particular, $\bar{s}$ in (5.5) and s, α in Definition 5.1, Theorem 5.3, are related by

$$\frac{1}{s} = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{s_j} \quad \text{and} \quad \alpha_k = \frac{s}{s_k} \text{ where } k = 1, \ldots, n. \tag{5.24}$$

It explains why s is called the *mean smoothness* (of $\bar{s}$). This notation applies also to (5.24) with $s < 0$. An approach to anisotropic Sobolev spaces and Besov spaces of type (5.1)–(5.3) and (5.4)–(5.6) from the point of view of (abstract) semi-groups of operators and interpolation theory had been given in [ScT76], [Scm77] in generalisation of corresponding considerations in [Triα]. (The somewhat curious timing is caused by the fact that [Triα] was mainly written in 1973 but needed five years to be published due to some unpleasant circumstances.) The above Fourier-analytical Definition 5.1 goes back to [Tri77], Section 2.5.2 and [StT79]. We refer in this context also to [ST87], Chapter 4, and to the short description in [Triβ], Section 10.1, where one finds also further references reflecting the state of art in the middle of the 1980s. The corresponding theory of the isotropic spaces in the 1970s and 1980s was mainly based on more or less sophisticated maximal functions and Fourier multipliers. We refer to [Pee76], [Tri77], [Triβ] and the relevant parts of [Triγ], in particular the historically-oriented survey given in Chapter 1 of the latter book. There is now a full counterpart of this part of the theory for anisotropic spaces. In addition to [Tri77], [StT79] we refer to [Yam86], [See89], [Din95], [Joh95]. But unfortunately there is no comprehensive treatment of this part of the theory in a book or any other unified way. In particular the above theorem is a consequence of these results.

5.1.3 Concrete spaces

We described in Section 1.2 some concrete isotropic spaces in $\mathbb{R}^n$ of Sobolev-Besov type. We look now at their anisotropic generalisations. Again let α be an anisotropy according to (5.10). The counterpart of the lift I_σ in (1.5) with $\sigma \in \mathbb{R}$ is now given by

$$I^\alpha_\sigma : \quad f \mapsto \left(\left[\sum_{r=1}^n (1+\xi_r^2)^{1/2\alpha_r} \right]^\sigma \widehat{f} \right)^\vee , \qquad f \in S'(\mathbb{R}^n). \tag{5.25}$$

Quite obviously, I^α_σ maps $S(\mathbb{R}^n)$ onto itself and $S'(\mathbb{R}^n)$ onto itself. Furthermore by

$$\sum_{r=1}^n (1+\xi_r^2)^{1/2\alpha_r} \sim 2^k \quad \text{if} \quad \xi \in R^\alpha_{k+1} \backslash R^\alpha_{k-1}, \quad k \in \mathbb{N}, \tag{5.26}$$

it follows that I^α_σ is well adapted to (5.15), (5.16). By analogy to (1.7) we denote

$$H^{\bar{s}}_p(\mathbb{R}^n) = I^\alpha_{-s} L_p(\mathbb{R}^n), \qquad 1<p<\infty, \quad s \in \mathbb{R}, \tag{5.27}$$

with

$$\bar{s} = (s_1, \ldots, s_n), \qquad s_r = s/\alpha_r, \quad r = 1, \ldots, n, \tag{5.28}$$

as *anisotropic Sobolev spaces* (whereas the notation *classical anisotropic Sobolev spaces* is reserved for the spaces in (5.1)–(5.3)).

Theorem 5.5. *Let α be an anisotropy in $\mathbb{R}^n$ according to* (5.10)*, let $\bar{s}$ be given by* (5.28) *with the mean smoothness $s \in \mathbb{R}$, and let $1 < p < \infty$. Let $F^{s,\alpha}_{p,2}(\mathbb{R}^n)$ and $H^{\bar{s}}_p(\mathbb{R}^n)$ be the spaces as introduced in Definition* 5.1 *and in* (5.27), (5.25)*. Then*

$$H^{\bar{s}}_p(\mathbb{R}^n) = F^{s,\alpha}_{p,2}(\mathbb{R}^n) \tag{5.29}$$

and, in particular,

$$L_p(\mathbb{R}^n) = F^{0,\alpha}_{p,2}(\mathbb{R}^n). \tag{5.30}$$

If, in addition, $s_r = k_r \in \mathbb{N}$ *and* $\bar{k} = (k_1, \ldots, k_n)$ *then*

$$H_p^{\bar{s}}(\mathbb{R}^n) = W_p^{\bar{k}}(\mathbb{R}^n) \tag{5.31}$$

are the classical anisotropic Sobolev spaces according to (5.1)–(5.3).

Remark 5.6. As mentioned at the end of Remark 5.4 there is a full counterpart of all the technical instruments needed to prove Theorems 5.3 and 5.5. In particular, (5.30) is an anisotropic Paley-Littlewood theorem. A proof may be found in [Tri77], Section 2.5.2, p. 106, with references to related anisotropic multiplier theorems and to [StT79], Section 4.4, for further details. This covers also (5.29) and, as a consequence, (5.31). A proof of (5.29) has also been given in [Yam86], Proposition 3.11, p. 157. As mentioned very briefly in Remark 1.1 in the isotropic case the Paley-Littlewood assertions (5.30) can be extended to the (inhomogeneous) *Hardy spaces* $h_p(\mathbb{R}^n)$, hence

$$h_p(\mathbb{R}^n) = F^0_{p,2}(\mathbb{R}^n), \qquad 0 < p < \infty, \tag{5.32}$$

[Triβ], Theorem 2.5.8/1, p. 92. Corresponding anisotropic Hardy spaces $h^\alpha_p(\mathbb{R}^n)$ have been introduced in [CaTo75], [CaTo77]. We do not go into detail, but there is a full counterpart of (5.32),

$$h^\alpha_p(\mathbb{R}^n) = F^{0,\alpha}_{p,2}(\mathbb{R}^n), \qquad 0 < p < \infty, \tag{5.33}$$

[Tri80], [Triβ], Section 10.1, Remark 2, p. 270. Obviously, (5.27) must be understood as an isomorphic map of $L_p(\mathbb{R}^n)$ onto $H^{\bar{s}}_p(\mathbb{R}^n)$. Conversely I^α_s maps $H^{\bar{s}}_p(\mathbb{R}^n)$ onto $L_p(\mathbb{R}^n)$ and one has

$$\|f \,|H^{\bar{s}}_p(\mathbb{R}^n)\| \sim \left\| \left(\left[\sum_{r=1}^n (1+\xi_r^2)^{1/2\alpha_r} \right]^s \widehat{f} \right)^\vee |L_p(\mathbb{R}^n) \right\|. \tag{5.34}$$

If, in addition, $s > 0$, then it follows from anisotropic Fourier multiplier assertions covered by the above literature and by (5.28), that

$$\begin{aligned} \|f \, H^{\bar{s}}_p(\mathbb{R}^n)\| &\sim \sum_{r=1}^n \left\| \left[(1+\xi_r^2)^{s_r/2} \widehat{f} \right]^\vee |L_p(\mathbb{R}^n) \right\| \\ &= \sum_{r=1}^n \|f \,|H^{s_r}_{p,x_r}(\mathbb{R}^n)\|, \qquad s_r > 0, \ 1 < p < \infty, \end{aligned} \tag{5.35}$$

where the latter is the preferred notation in the Russian literature, [Nik77]. In particular, $H^{\bar{s}}_p(\mathbb{R}^n)$ is the intersection

$$H^{\bar{s}}_p(\mathbb{R}^n) = \bigcap_{r=1}^n H^{s_r}_{p,x_r}(\mathbb{R}^n), \qquad 1 < p < \infty, \quad s_r > 0, \tag{5.36}$$

of the corresponding Sobolev spaces on $\mathbb{R}$ in the rth direction of coordinates, L_p-extended to the other $n-1$ directions. By (5.31) and (one-dimensional) Fourier multipliers, (5.1), (5.3) is a special case,

$$W_p^{\bar{k}}(\mathbb{R}^n) = \bigcap_{r=1}^{n} W_{p,x_r}^{k_r}(\mathbb{R}^n), \qquad 1 < p < \infty, \quad k_r \in \mathbb{N}. \tag{5.37}$$

Remark 5.7. We describe the B-counterpart of (5.29). Let $p, q, \bar{s}$ be restricted by (5.5) and let the corresponding mean smoothness s and the anisotropy $\alpha = (\alpha_1, \ldots, \alpha_n)$ be given by (5.24). Then

$$B_{pq}^{\bar{s}}(\mathbb{R}^n) = B_{pq}^{s,\alpha}(\mathbb{R}^n). \tag{5.38}$$

In particular these spaces can be normed by (5.6) (equivalent norms) and we have in obvious notation

$$B_{pq}^{\bar{s}}(\mathbb{R}^n) = \bigcap_{r=1}^{n} B_{pq,x_r}^{s_r}(\mathbb{R}^n). \tag{5.39}$$

This is the counterpart of (5.36), (5.37).

Isotropic spaces are anisotropic spaces (curiously enough from a linguistic point of view). Then we have for the classical Sobolev spaces the norm (5.3), but also (1.4), and for the classical Besov spaces the norm (5.6), but also (1.13) and its extension according to Theorem 1.116. There is the following counterpart for the corresponding anisotropic spaces. Recall that

$$\sigma_p = n\left(\frac{1}{p} - 1\right)_+ \qquad \text{where} \quad 0 < p \le \infty, \tag{5.40}$$

and that the differences Δ_h^M and the directional differences $\Delta_{t,l}^M$ in $\mathbb{R}^n$ are given by (1.11), (4.32) and (5.7). Furthermore if α is an anisotropy according to (5.10) and $h = (h_1, \ldots, h_n) \in \mathbb{R}^n$ then we put

$$|h|_\alpha = \sum_{l=1}^{n} |h_l|^{1/\alpha_l} \qquad \text{(anisotropic distance)}. \tag{5.41}$$

Theorem 5.8.

(i) *Let*

$$1 < p < \infty \qquad \textit{and} \quad \bar{k} = (k_1, \ldots, k_n) \quad \textit{with} \quad k_r \in \mathbb{N}. \tag{5.42}$$

Let

$$\|f \,|W_p^{\bar{k}}(\mathbb{R}^n)\|^* = \sum \|D^\gamma f \,|L_p(\mathbb{R}^n)\| \tag{5.43}$$

where the sum is taken over all

$$\gamma \in \mathbb{N}_0^n \qquad \textit{with} \quad \sum_{r=1}^{n} \gamma_r / k_r \le 1. \tag{5.44}$$

Then

$$W^{\bar{k}}_p(\mathbb{R}^n) = \Big\{ f \in L_p(\mathbb{R}^n) : \ \|f \,|W^{\bar{k}}_p(\mathbb{R}^n)\|^* < \infty \Big\} \tag{5.45}$$

(*equivalent norms*).

(ii) *Let* α *be an anisotropy according to* (5.10) *and let*

$$0 < p \le \infty, \qquad 0 < q \le \infty, \qquad \sigma_p < s < \infty, \tag{5.46}$$

where σ_p *is given by* (5.40). *Let* $s_l = s/\alpha_l$ *as in* (5.24) *and let* $s_l < M_l \in \mathbb{N}$. *Then*

$$\|f \,|L_p(\mathbb{R}^n)\| + \sum_{l=1}^n \Big(\int_0^1 t^{-s_l q} \, \|\Delta^{M_l}_{t,l} f \,|L_p(\mathbb{R}^n)\|^q \, \frac{\mathrm{d}t}{t} \Big)^{1/q} \tag{5.47}$$

and

$$\|f \,|L_p(\mathbb{R}^n)\| + \sum_{l=1}^n \Big(\int_0^1 t^{-s_l q} \sup_{0<\tau\le t} \|\Delta^{M_l}_{\tau,l} f \,|L_p(\mathbb{R}^n)\|^q \, \frac{\mathrm{d}t}{t} \Big)^{1/q} \tag{5.48}$$

are equivalent quasi-norms in $B^{s,\alpha}_{pq}(\mathbb{R}^n)$ (*usual modification if* $q = \infty$).

(iii) *Let* α, p, q, s *and* s_l *be as in part* (ii). *Let* $M \in \mathbb{N}$ *be sufficiently large. Then*

$$\|f \,|L_p(\mathbb{R}^n)\| + \Bigg(\int\limits_{|h|_\alpha \le 1} |h|_\alpha^{-sq} \, \|\Delta^M_h f \,|L_p(\mathbb{R}^n)\|^q \, \frac{\mathrm{d}h}{|h|^n_\alpha} \Bigg)^{1/q} \tag{5.49}$$

and

$$\|f \,|L_p(\mathbb{R}^n)\| + \Big(\int_0^1 t^{-sq} \sup_{0<|h|_\alpha\le t} \|\Delta^M_h f \,|L_p(\mathbb{R}^n)\|^q \, \frac{\mathrm{d}t}{t} \Big)^{1/q} \tag{5.50}$$

are equivalent quasi-norms in $B^{s,\alpha}_{pq}(\mathbb{R}^n)$ (*usual modification if* $q = \infty$).

Remark 5.9. Part (i) is a classical assertion as it may be found in [Nik77], Chapter 9, again based on anisotropic Fourier multiplier assertions. The isotropic specialisations of the parts (ii) and (iii) are well known. They are corner-stones of the theory of Besov spaces. We mentioned some assertions of this type in (1.13) (classical spaces) and in Theorem 1.116. Detailed proofs for the isotropic case may be found in [Triβ], Theorem 2.5.12, p. 110, and Theorem 2.5.13, p. 115, and in particular in [Triγ], Theorem 2.6.1, p. 140. The proof in [Triγ] is based on general characterisations of isotropic B-spaces and F-spaces with differences as a special application. These general characterisations have been extended in [Far00] and [Dac03] from isotropic spaces to anisotropic spaces. Then one gets again the above parts (ii) and (iii) as an application of these general characterisations to differences. The above formulations may be found in [Dac03]. A first result about equivalent norms of type (5.49) with $M = 1$ in $B^{\bar{s}}_{pp}(\mathbb{R}^n)$ with $1 \le p < \infty$, $0 < s_l < 1$, were obtained in [LoT79]. We refer in this context also to [BIN75], especially §18.

Remark 5.10. In generalisation of (1.10) and (2.21) we denote

$$\mathcal{C}^{s,\alpha}(\mathbb{R}^n) = B^{s,\alpha}_{\infty\infty}(\mathbb{R}^n), \qquad s \in \mathbb{R}, \tag{5.51}$$

as anisotropic Hölder-Zygmund spaces, where α is an anisotropy according to (5.10). If $s > 0$ and $s/\alpha_l = s_l < M_l \in \mathbb{N}$ then $\mathcal{C}^{s,\alpha}(\mathbb{R}^n)$ can be equivalently normed by

$$\|f \,|L_\infty(\mathbb{R}^n)\| + \sum_{l=1}^{n} \sup t^{-s_l} \left| (\Delta^{M_l}_{t,l} f)(x) \right| \tag{5.52}$$

where the supremum is taken over $0 < t < 1$ and $x \in \mathbb{R}^n$. Furthermore if $M \in \mathbb{N}$ is sufficiently large then

$$\|f \,|L_\infty(\mathbb{R}^n)\| + \sup |h|_\alpha^{-s} \left| (\Delta^M_h f)(x) \right| \tag{5.53}$$

is also an equivalent norm where the supremum is taken over $h \in \mathbb{R}^n$ with $0 < |h|_\alpha \le 1$ and $x \in \mathbb{R}^n$. These are special cases of the above Theorem 5.8.

5.1.4 New developments

As outlined in Remark 5.4 the theory of the isotropic spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ according to Definition 2.1 up to the early 1990s, characterised by the extensive use of maximal inequalities and Fourier multipliers, as presented in [Triβ] and in some parts of [Triγ], has a full anisotropic counterpart. The corresponding references may be found at the end of Remark 5.4. At least as far as our own interests are concerned the theory of the above isotropic spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ for the full range of the parameters s, p, q has been based in the 1990s (and up to now) on the following two developments having, in turn, numerous applications, for example to fractal analysis.

- (i) Characterisations of $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ in terms of general means as described in Section 1.3, especially in Theorem 1.7.
- (ii) Characterisations of these spaces by (more or less) elementary building blocks such as atoms (and molecules), wavelet bases, wavelet frames, and quarks.

One may ask whether these elaborated theories can be extended to anisotropic spaces according to Definition 5.1. The characterisation of the isotropic spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ in terms of general means as described in Section 1.3 is a polished version reflecting the situation at the end of the 1990s. It has its origin in corresponding characterisations and equivalent quasi-norms for the spaces $F^s_{pq}(\mathbb{R}^n)$ in [Triγ], Section 2.4, and for the spaces $B^s_{pq}(\mathbb{R}^n)$ in [Triγ], Section 2.5. At this latter level there is a fully elaborated counterpart for the anisotropic spaces according to Definition 5.1. We refer to [Far00] and [Dac03]. An outstanding example of these general means are the local means. In Section 1.4 we collected some main assertions of this powerful instrument for the isotropic spaces. A corresponding theory for anisotropic spaces will be described in Section 5.1.6 below. As for (smooth) atoms the situation is quite similar. The anisotropic generalisation of Section 1.5, and especially of Theorem 1.19 is the subject of Section 5.1.5. It is due to [Far00]. In this paper one finds also some quarkonial expansions in generalisation of Section 1.6. A few further comments and references may be found in Remark 1.48. But this will not be the subject of this book. The main aim of this Chapter 5 is the description of wavelet isomorphisms and wavelet bases in anisotropic spaces in generalisation of Sections 1.7 and 3.1. This will be done in Section 5.2. For this purpose we need anisotropic atoms (and molecules) and anisotropic local means, and this is just the reason for having a closer look at these ingredients below. Finally the last but not least building blocks for isotropic spaces considered above are the wavelet frames according to Section 1.8 and in greater detail in Section 3.2. An anisotropic generalisation of this theory may be found in [HaTa05], [Tam06]. Altogether there is a good chance to extend the recent theory of building blocks for (inhomogeneous) isotropic spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ to corresponding anisotropic spaces. Some notions have been defined (atoms, quarks, wavelet frames) and something is subject of what follows (wavelet bases). Building blocks in isotropic function spaces might be considered as a topic for its own sake. But even more interesting are the numerous applications such as to pseudodifferential operators, numerics, signal analysis and local smoothness theory, time-frequency analysis, fractal analysis and a related spectral theory of fractal operators, entropy methods, spaces on quasi-metric spaces, to mention only a few. Some of this is the main subject of this book. We refer in particular to Chapters 4, 7–8. Full counterparts for anisotropic spaces of the diverse ingredients pave the way to corresponding anisotropic theories.

5.1.5 Atoms

We describe the anisotropic counterpart of smooth atoms according to Section 1.5.1 and Definition 2.5. We follow [Far00]. As for basic notation we refer to Sections 2.1.2, 2.1.3, where we need now some anisotropic modifications related to the anisotropy α given by (5.10). Let $Q^\alpha_{\nu m}$ be the rectangle in $\mathbb{R}^n$ with sides parallel to the axes of coordinates, centred at

$$2^{-\nu\alpha}m = (2^{-\nu\alpha_1}m_1, \dots, 2^{-\nu\alpha_n}m_n), \quad \text{where } m \in \mathbb{Z}^n \text{ and } \nu \in \mathbb{N}_0, \tag{5.54}$$

and with side lengths $2^{-(\nu-1)\alpha_1}, \dots, 2^{-(\nu-1)\alpha_n}$. We used the abbreviation (5.11). If Q is a rectangle in $\mathbb{R}^n$ and $r > 0$ then rQ is the rectangle in $\mathbb{R}^n$ concentric with Q and with side lengths r times the side lengths of Q. In the isotropic case $\alpha = (1, \dots, 1)$ we have $Q^\alpha_{\nu m} = Q_{\nu m}$ where $Q_{\nu m}$ are the cubes according to Section 2.1.2. In particular, Q^α_{0m} are rectangles of side lengths $2^{\alpha_1}, \dots, 2^{\alpha_n}$ centred at $m \in \mathbb{Z}^n$. If $x \in \mathbb{R}^n$, $\gamma \in \mathbb{N}^n_0$ and α as in (5.10) then

$$\alpha\gamma = \gamma\alpha = \sum_{j=1}^n \gamma_j \alpha_j \quad \text{and} \quad x^\gamma = x_1^{\gamma_1} \cdots x_n^{\gamma_n}. \tag{5.55}$$

Definition 5.11. *Let α be an anisotropy according to* (5.10).

(i) *Let $K \ge 0$ and $c \ge 1$. A continuous function $a : \mathbb{R}^n \to \mathbb{C}$ for which there exist all derivatives $D^\gamma a$ if $\gamma\alpha \le K$ is called a* 1*-atom (more precisely 1^α_K-atom) if*

$$\operatorname{supp} a \subset c\, Q^\alpha_{0m} \qquad \textit{for some} \quad m \in \mathbb{Z}^n, \tag{5.56}$$

and

$$|D^\gamma a(x)| \le 1 \qquad \textit{for} \quad \gamma\alpha \le K. \tag{5.57}$$

(ii) *Let $s \in \mathbb{R}$, $0 < p \le \infty$, $K \ge 0$, $L \ge 0$ and $c \ge 1$. A continuous function $a : \mathbb{R}^n \to \mathbb{C}$ for which there exist all derivatives $D^\gamma a$ if $\alpha\gamma \le K$ is called an (s,p)-atom (more precisely an anisotropic $(s,p)^\alpha_{K,L}$-atom) if*

$$\operatorname{supp} a \subset c\, Q^\alpha_{\nu m} \qquad \textit{for some} \quad \nu \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n, \tag{5.58}$$

$$|D^\gamma a(x)| \le 2^{-\nu(s-n/p-\gamma\alpha)} \qquad \textit{for} \quad \gamma\alpha \le K \tag{5.59}$$

and

$$\int_{\mathbb{R}^n} x^\beta\, a(x)\, \mathrm{d}x = 0 \quad \textit{for} \quad \beta \in \mathbb{N}^n_0 \quad \textit{with} \quad \beta\alpha < L. \tag{5.60}$$

Remark 5.12. This is the anisotropic extension of Definition 1.15. As there (5.60) is empty if $L = 0$. Otherwise we have the obvious counterpart of (1.61). Since $|Q^\alpha_{\nu m}| = 2^{-(\nu-1)n}$ the p-normalised characteristic function

$$\chi^{(p),\alpha}_{\nu m}(x) = 2^{(\nu-1)n/p} \text{ if } x \in Q^\alpha_{\nu m} \quad \text{and} \quad \chi^{(p),\alpha}_{\nu m}(x) = 0 \text{ if } x \notin Q^\alpha_{\nu m} \tag{5.61}$$

generalises (1.62), where again $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$.

Definition 5.13. *Let α be an anisotropy according to* (5.10) *and let $0 < p \le \infty$, $0 < q \le \infty$,*

$$\lambda = \{\lambda_{\nu m} \in \mathbb{C} : \ \nu \in \mathbb{N}_0,\ m \in \mathbb{Z}^n\}, \tag{5.62}$$

$$\|\lambda \,|b_{pq}\| = \left(\sum_{\nu=0}^\infty \Big(\sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}|^p \Big)^{q/p} \right)^{1/q} \tag{5.63}$$

and

$$\|\lambda \,|f^\alpha_{pq}\| = \left\| \left(\sum_{\nu=0}^\infty \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}\, \chi^{(p),\alpha}_{\nu m}(\cdot)|^q \right)^{1/q} |L_p(\mathbb{R}^n) \right\| \tag{5.64}$$

(with the usual modification if $p = \infty$ and/or $q = \infty$). Then

$$b_{pq} = \{\lambda : \ \|\lambda\,|b_{pq}\| < \infty\} \tag{5.65}$$

and

$$f^\alpha_{pq} = \{\lambda : \ \|\lambda\,|f^\alpha_{pq}\| < \infty\}. \tag{5.66}$$

Remark 5.14. This is the direct counterpart of Definition 1.17. If $p = q$ then $f^\alpha_{pq} = b_{pq}$ for all anisotropies. Recall that

$$\sigma_p = n\left(\frac{1}{p} - 1\right)_+ \quad \text{and} \quad \sigma_{pq} = n\left(\frac{1}{\min(p,q)} - 1\right)_+ . \tag{5.67}$$

Furthermore we indicate the location and the size of a 1^α_K-atom or an $(s,p)^\alpha_{K,L}$-atom according to Definition 5.11 with ν, m as there by writing $a^\alpha_{\nu m}$ in place of a.

Theorem 5.15. *Let α be an anisotropy according to* (5.10).

(i) *Let $0 < p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$. Let both K with*

$$K \ge \begin{cases} 0 & \text{if } s < 0, \\ s + \alpha_n & \text{if } s \ge 0, \end{cases} \tag{5.68}$$

and L with

$$L \ge 0 \qquad \text{and} \qquad L > \sigma_p - s, \tag{5.69}$$

be fixed. Then $f \in S'(\mathbb{R}^n)$ belongs to $B^{s,\alpha}_{pq}(\mathbb{R}^n)$ if, and only if, it can be represented as

$$f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a^\alpha_{\nu m}, \quad \text{unconditional convergence being in } S'(\mathbb{R}^n), \tag{5.70}$$

where for fixed $c \ge 1$, $a^\alpha_{\nu m}$ are 1^α_K-atoms ($\nu = 0$) or $(s,p)^\alpha_{K,L}$-atoms ($\nu \in \mathbb{N}$) according to Definition 5.11 *and Remark* 5.14*, and $\lambda \in b_{pq}$. Furthermore,*

$$\|f \,|B^{s,\alpha}_{pq}(\mathbb{R}^n)\| \sim \inf \|\lambda \,|b_{pq}\| \tag{5.71}$$

are equivalent quasi-norms where the infimum is taken over all admissible representations (5.70).

(ii) *Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$. Let K be as in* (5.68) *and let L with*

$$L \ge 0 \qquad \text{and} \qquad L > \sigma_{pq} - s, \tag{5.72}$$

be fixed. Then $f \in S'(\mathbb{R}^n)$ belongs to $F^{s,\alpha}_{pq}(\mathbb{R}^n)$ if, and only if, it can be represented by (5.70)*, where now for fixed $c \ge 1$, $a^\alpha_{\nu m}$ are 1^α_K-atoms ($\nu = 0$) or $(s,p)^\alpha_{K,L}$-atoms ($\nu \in \mathbb{N}$) with* (5.68), (5.72) *and $\lambda \in f^\alpha_{pq}$. Furthermore,*

$$\|f \,|F^{s,\alpha}_{pq}(\mathbb{R}^n)\| \sim \inf \|\lambda \,|f^\alpha_{pq}\| \tag{5.73}$$

are equivalent quasi-norms where the infimum is taken over all admissible representations (5.70).

Remark 5.16. This is the anisotropic generalisation of Theorem 1.19. As for technical explanations concerning the convergence of (5.70) we refer to Remark 1.20. The above theorem coincides essentially with [Far00], Theorem 3.3. Compared with the isotropic Theorem 1.19 the term α_n in (5.68) looks a little bit curious. Maybe one can overcome this (otherwise unimportant) technical point by asking for an anisotropic counterpart of the theory of (isotropic) non-smooth atoms as developed in Section 2.2. More recent atomic and molecular representation theorems may be found in [Kyr04], [BoH05], [Bow05].

5.1.6 Local means

Atoms are one of the two corner-stones for wavelet isomorphisms and wavelet bases in anisotropic function spaces $B^{s,\alpha}_{pq}(\mathbb{R}^n)$ and $F^{s,\alpha}_{pq}(\mathbb{R}^n)$ in generalisation of Section 3.1 where we developed the corresponding theory of the isotropic spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$. The second corner-stone comprises local means asking for an anisotropic version of Section 1.4. Again we follow [Far00].

Let k be a C^∞ function in $\mathbb{R}^n$ with

$$\operatorname{supp} k \subset B = \{y : \ |y| < 1\} \tag{5.74}$$

and let α be an anisotropy according to (5.10). Then the anisotropic version of the local means (1.41) is given by

$$\begin{aligned} k^\alpha(t,f)(x) &= \int_{\mathbb{R}^n} k(y)\, f(x + t^\alpha y)\, \mathrm{d}y \\ &= t^{-n} \int_{\mathbb{R}^n} k\left(t^{-\alpha}(y-x)\right)\, f(y)\, \mathrm{d}y \end{aligned} \tag{5.75}$$

where $f \in S'(\mathbb{R}^n)$. We used (5.10), (5.11). We put $k(1,f) = k^\alpha(1,f)$. Recall that σ_p and σ_{pq} are given by (5.67).

Theorem 5.17. *Let α be an anisotropy in $\mathbb{R}^n$ according to* (5.10). *Let k_0 and k be two C^∞ functions in $\mathbb{R}^n$ with*

$$\operatorname{supp} k_0 \subset B, \qquad \operatorname{supp} k \subset B, \tag{5.76}$$

$k_0^\vee(0) \neq 0$, *and for some $\varepsilon > 0$ and $\varkappa > 0$,*

$$k^\vee(\xi) \neq 0 \text{ if } 0 < |\xi| \le \varepsilon \quad \text{and} \quad (D^\beta k^\vee)(0) = 0 \text{ if } \beta\alpha < \varkappa. \tag{5.77}$$

(i) *Let $0 < p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$ and $\varkappa > \max(s, \sigma_p) + \sigma_p$. Then*

$$\|k_0(1,f)|L_p(\mathbb{R}^n)\| + \left(\sum_{j=1}^\infty 2^{jsq}\, \|k^\alpha(2^{-j},f)\,|L_p(\mathbb{R}^n)\|^q \right)^{1/q} \tag{5.78}$$

(*usual modification if $q = \infty$*) *is an equivalent quasi-norm in $B^{s,\alpha}_{pq}(\mathbb{R}^n)$.*

(ii) *Let* $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$ *and* $\varkappa > \max(s, \sigma_p) + n/\min(p,q)$. *Then*

$$\|k_0(1,f)\,|L_p(\mathbb{R}^n)\| + \left\| \left(\sum_{j=1}^{\infty} 2^{jsq} \left| k^{\alpha}(2^{-j},f)(\cdot)\right|^q \right)^{1/q} |L_p(\mathbb{R}^n)\right\| \tag{5.79}$$

(*usual modification if* $q = \infty$) *is an equivalent quasi-norm in* $F^{s,\alpha}_{pq}(\mathbb{R}^n)$.

Remark 5.18. This theorem coincides essentially with [Far00], Theorem 4.9. The corresponding isotropic Theorem 1.10 is in a more final shape. The conditions for the kernels k_0 and k given there are more natural. Furthermore, (1.47) and (1.48) are characterisations. The above formulations are nearer to the original isotropic assertions in [Triγ], Theorems 2.4.6, pp. 122/123, and 2.5.3, p. 138. Some improvements came later. Corresponding references may be found in Remark 1.11. On the other hand all technicalities about local means and related maximal functions which we used in the isotropic case in connection with Proposition 3.3, Theorem 3.5 and their proofs have full anisotropic counterparts and are covered by [Far00] and [Dac03]. In other words, the two major technical ingredients for an anisotropic extension of the wavelet isomorphisms according to Theorem 3.5, atomic representations and local means, are covered by Theorems 5.15 and 5.17 and the indicated complements. This applies in particular to local means (5.75) where the kernel k has only a limited smoothness. In this connection we used the duality (3.17) which justifies (3.45). We add now a comment on this point.

5.1.7 A comment on dual pairings

Again let α be an anisotropy in $\mathbb{R}^n$ according to (5.10). If $0 < p < \infty$ then $S(\mathbb{R}^n)$ is dense in $B^{s,\alpha}_{pp}(\mathbb{R}^n)$ and it makes sense to interpret the dual $B^{s,\alpha}_{pp}(\mathbb{R}^n)'$ of $B^{s,\alpha}_{pp}(\mathbb{R}^n)$ as a subset of $S'(\mathbb{R}^n)$. Recall that σ_p is given by (5.67). Then

$$B^{s,\alpha}_{pp}(\mathbb{R}^n)' = B^{-s+\sigma_p,\alpha}_{p'p'}(\mathbb{R}^n) \tag{5.80}$$

where p' is given by (3.18). This is the anisotropic generalisation of (3.17). If $1 < p < \infty$ and, hence, $\sigma_p = 0$ then (5.80) is covered by [ST87], Section 4.2.4, p. 201. Otherwise one can follow the proofs of the corresponding assertions in [Triβ], Sections 2.11.2, 2.11.3, pp. 178–182. But we have no direct reference of (5.80) if $p \le 1$. On the other hand we used duality in the isotropic case only to justify (3.15) with kernels Ψ^G of limited smoothness (Daubechies wavelet kernels) and as a consequence in (3.45). But one can argue more directly, especially when one does not care about best possible smoothness exponents. Let

$$s \in \mathbb{R},\ 0 < p \le \infty, \qquad \text{and} \qquad L \ge 0,\ L > \frac{n}{p} - s, \tag{5.81}$$

and let Ψ be a continuous function with a compact support in the unit ball having all classical continuous derivatives of order

$$D^\beta \Psi, \quad \frac{\partial}{\partial x_j} D^\beta \Psi \quad \text{where} \quad \beta\alpha < L, \quad j = 1, \ldots, n. \tag{5.82}$$

Let $f \in B^{s,\alpha}_{pq}(\mathbb{R}^n)$ be expanded according to (5.70). We wish to show that

$$(f, \Psi) = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \int_{\mathbb{R}^n} a^\alpha_{\nu m}(x)\, \Psi(x)\, \mathrm{d}x \tag{5.83}$$

converges absolutely in $\mathbb{C}$. According to (5.69) we may assume that the atoms $a^\alpha_{\nu m}$ with $\nu \in \mathbb{N}$ satisfy (5.60) where L is given by the more restrictive assumption (5.81). We expand Ψ at $2^{-\nu\alpha}m$ in a Taylor polynomial and a remainder term $r_{\nu m}$,

$$\begin{aligned} &\int_{\mathbb{R}^n} a^\alpha_{\nu m}(x)\, \Psi(x)\, \mathrm{d}x \\ &= \int_{\mathbb{R}^n} a^\alpha_{\nu m}(x) \left[\sum_{\beta\alpha < L} \frac{D^\beta \Psi(2^{-\nu\alpha}m)}{\beta!} (x - 2^{-\nu\alpha}m)^\beta + r_{\nu m}(x) \right] \mathrm{d}x \\ &= \int_{\mathbb{R}^n} a^\alpha_{\nu m}(x)\, r_{\nu m}(x)\, \mathrm{d}x \end{aligned} \tag{5.84}$$

as a consequence of (5.60) and (5.82). Then it follows by (5.58), (5.59) that

$$\left| \int_{\mathbb{R}^n} a^\alpha_{\nu m}(x)\, \Psi(x)\, \mathrm{d}x \right| \le c\, 2^{-\nu(s-n/p)}\, 2^{-\nu L}\, 2^{-\nu n} \tag{5.85}$$

where the last factor comes from the volume of $Q^\alpha_{\nu m}$. Since Ψ has a compact support in the unit ball one gets

$$\begin{aligned} \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}| \left| \int_{\mathbb{R}^n} a^\alpha_{\nu m}(x)\, \Psi(x)\, \mathrm{d}x \right| &\le c \sum_{\nu=0}^{\infty} \sup_m |\lambda_{\nu m}|\, 2^{-\nu(s+L-s/p)} \\ &\le c' \, \|\lambda \,|\ell_\infty\| \\ &\le c' \, \|\lambda \,|b_{pp}\|, \end{aligned} \tag{5.86}$$

where we used (5.81) and (5.63). Hence by Theorem 5.15, the right-hand side of (5.83) converges absolutely in $\mathbb{C}$ and (f, Ψ) makes sense as a dual pairing. Identifying Ψ in (3.15) with Ψ^G then one gets an elementary justification of (3.15), (3.45), and its anisotropic generalisations considered below.

5.2 Wavelets

5.2.1 Anisotropic multiresolution analysis

It is the main aim of Section 5.2 to extend the wavelet isomorphisms according to Theorems 3.5 and 3.12 from the (inhomogeneous) isotropic spaces $B^s_{pq}(\mathbb{R}^n)$ and

$F^s_{pq}(\mathbb{R}^n)$ to the corresponding anisotropic spaces $B^{s,\alpha}_{pq}(\mathbb{R}^n)$ and $F^{s,\alpha}_{pq}(\mathbb{R}^n)$ according to Definition 5.1 where again α is the anisotropy given by (5.10). The first attempt to replace $2^{j-1}x$ in (3.5) by something like

$$2^{j\alpha}x = \left(2^{j\alpha_1}x_1, \ldots, 2^{j\alpha_n}x_n\right), \qquad x \in \mathbb{R}^n, \tag{5.87}$$

does not work in general. We return to this point in Remark 5.25. Here we stick at the one-dimensional (dyadic) multiresolution analysis as described in Definition 1.49, Remark 1.50, and Proposition 1.51, but we re-organise its (isotropic) n-dimensional version according to Remark 1.52 and Proposition 1.53 in an anisotropic procedure.

Let $\{V_j : j \in \mathbb{N}_0\}$ be a one-dimensional (inhomogeneous) multiresolution analysis in $L_2(\mathbb{R})$ with a scaling function (father wavelet) ψ_F and an (associated) wavelet (mother wavelet) ψ_M as described in Definition 1.49 and Remark 1.50. Let α be an anisotropy according to (5.10). Let $2 \le n \in \mathbb{N}$. We introduce the index sets $I^{j,\alpha}$,

$$I^{j,\alpha} \subset \{F,M\}^n \times \mathbb{N}_0^n, \qquad j \in \mathbb{N}_0, \tag{5.88}$$

where $\{F,M\}^n$ is the collection of all indices $G = (G_1, \ldots, G_n)$ with G_r being either F or M. Let $\{F,M\}^{n*}$ be the collection of all $G \in \{F,M\}^n$ such that at least one component G_r is an M (hence the cardinal number of $\{F,M\}^{n*}$ is $2^n - 1$). Then, by definition, $I^{0,\alpha}$ has only one element (G,k) where $G_1 = \cdots G_n = F$ and $k = 0$. If $j \in \mathbb{N}$ then, by definition, $I^{j,\alpha}$ is the collection of all elements (G,k) with $G \in \{F,M\}^{n*}$ and $k \in \mathbb{N}_0^n$ such that

$$k_r = [(j-1)\alpha_r] \qquad \text{if} \qquad G_r = F \tag{5.89}$$

and

$$[(j-1)\alpha_r] \le k_r < [j\alpha_r] \qquad \text{if} \qquad G_r = M. \tag{5.90}$$

Obviously, $r = 1, \ldots, n$ and $k = (k_1, \ldots, k_n)$. Furthermore, $[c]$ is the largest integer smaller than or equal to $c \in \mathbb{R}$. In the isotropic case where all $\alpha_r = 1$ we have (1.134) with $k_r = j-1$ for all r. Since $\alpha_n \ge 1$, in all anisotropic cases $I^{j,\alpha}$ with $j \in \mathbb{N}$ has at least one element and at most

$$(2^n - 1)\prod_{r=1}^n \left(1 + [j\alpha_r] - [(j-1)\alpha_r]\right) \le (2^n-1)\prod_{r=1}^n (2+\alpha_r) \tag{5.91}$$

elements. Next we describe the anisotropic counterpart of (1.135) and (1.138). Let V_j^α with $j \in \mathbb{N}_0$ be the subspace of $L_2(\mathbb{R}^n)$ spanned by the orthonormal basis

$$\Psi^{j,\alpha}_m(x) = \prod_{r=1}^n 2^{[j\alpha_r]/2}\, \psi_F\left(2^{[j\alpha_r]}x_r - m_r\right), \quad m \in \mathbb{Z}^n. \tag{5.92}$$

As in the n-dimensional isotropic case one has

$$V_j^\alpha \subset V_{j+1}^\alpha, \quad j \in \mathbb{N}_0, \qquad L_2(\mathbb{R}^n) = \operatorname{span} \bigcup_{j=0}^\infty V_j^\alpha \tag{5.93}$$

and the orthogonal decomposition

$$L_2(\mathbb{R}^n) = V_0^\alpha \oplus \bigoplus_{j=0}^{\infty} W_j^\alpha \tag{5.94}$$

with

$$V_{j+1}^\alpha = V_j^\alpha \oplus W_j^\alpha, \qquad j \in \mathbb{N}_0. \tag{5.95}$$

Recall that $|k| = k_1 + \cdots + k_n$ if $(k_1, \ldots, k_n) = k \in \mathbb{N}_0^n$.

Proposition 5.19. *Let α be an anisotropy according to (5.10) and let $I^{j,\alpha}$ be the above index-sets. Let ψ_F and ψ_M be the above functions. Then*

$$\Psi_m^{j,(G,k),\alpha}(x) = 2^{|k|/2} \prod_{r=1}^{n} \psi_{G_r}\left(2^{k_r} x_r - m_r\right) \tag{5.96}$$

with $j \in \mathbb{N}_0$, $(G,k) \in I^{j,\alpha}$ and $m \in \mathbb{Z}^n$, is an orthonormal basis in $L_2(\mathbb{R}^n)$.

Proof. This assertion follows by the same arguments as in the proof of Proposition 1.53. □

Remark 5.20. We are mainly interested in the compactly supported Daubechies wavelets ψ_M and ψ_F according to Theorem 1.61(ii). Then (5.96) is the anisotropic generalisation of (3.5) and there is a number $c > 0$ such that

$$\operatorname{supp} \Psi_m^{j,(G,k),\alpha} \subset \left\{ x \in \mathbb{R}^n : \ |x_r - 2^{-k_r} m_r| \le c\, 2^{-k_r} \right\}. \tag{5.97}$$

Since $(G,k) \in I^{j,\alpha}$ we have in particular (5.90). If $j \in \mathbb{N}_0$ and $(G,k) \in I^{j,\alpha}$ one gets for $m \in \mathbb{Z}^n$ a tiling of $\mathbb{R}^n$ which may serve as an admitted substitute of the canonical anisotropic tiling of $\mathbb{R}^n$ with the rectangles Q_{jm}^α as introduced at the beginning of Section 5.1.5. This applies in particular to Definition 5.11 and Theorem 5.15. Next we discuss a counterpart of this comment in connection with anisotropic local means as described in Section 5.1.6 and Theorem 5.17. In the isotropic case we needed local means in Step 2 of the proof of Theorem 3.5 in connection with estimates from above. This was based on Proposition 3.3. But for the corresponding proof and the underlying references one can replace $\varphi_j(x) = \varphi(2^{-j+1}x)$ in connection with (3.24), (3.25) by $\varphi_j(x) = \varphi(2^{\varepsilon_j} 2^{-j} x)$ with $|\varepsilon_j| \le c$ for some $c > 0$ which is independent of $j \in \mathbb{N}$. This means that one can replace $\Psi^G(2^{j-1}\xi)$ there by $\Psi^G(2^{j-\varepsilon_j}\xi)$. There is a full anisotropic counterpart of all these observations, [Far00]. In particular in connection with Theorem 5.17 (denoting temporarily the kernels by K in place of k to avoid any confusion with the above use of $k \in \mathbb{N}_0^n$) and the anisotropic counterpart of Proposition 3.3 one can replace in the local means $K^\alpha(t,f)$ in (5.75) with $t = 2^{-j}$ the corresponding kernels

$$K(2^{j\alpha} y) \qquad \text{by} \qquad K\left(2^{k_1} y_1, \ldots, 2^{k_n} y_n\right) \tag{5.98}$$

with $k = (k_1, \ldots, k_n)$ restricted by (5.90).

5.2.2 Main assertions

After the preparations in the preceding subsection one can carry over the wavelet isomorphisms for isotropic spaces from Theorem 3.5 (Daubechies wavelets) and Theorem 3.12 (Meyer wavelets) to the corresponding anisotropic spaces. First we introduce the anisotropic extension of Definition 3.1. We adapt the anisotropic rectangles $Q^\alpha_{\nu m}$ used in Definition 5.11 to (5.97), (5.98). This is unimportant, but otherwise one must discuss some index-shifting of $m \in \mathbb{Z}^n \to m' \in \mathbb{Z}^n$ to relate the rectangles in (5.97) to some $Q^\alpha_{\nu m'}$. Let

$$Q^k_m = \big\{ x \in \mathbb{R}^n : \ |x_r - 2^{-k_r} m_r| \le 2^{-k_r+1}, \ r = 1, \ldots, n \big\} \tag{5.99}$$

where $k \in \mathbb{N}^n_0$ and $m \in \mathbb{Z}^n$. Let χ^k_m be the characteristic function of Q^k_m.

Definition 5.21. *Let α be an anisotropy according to* (5.10) *and let $I^{j,\alpha}$ be the index-sets as introduced in* (5.88)–(5.90). *Let $0 < p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$,*

$$\lambda = \Big\{ \lambda^{j,(G,k)}_m \in \mathbb{C} : \ j \in \mathbb{N}_0, \ (G,k) \in I^{j,\alpha}, \ m \in \mathbb{Z}^n \Big\}, \tag{5.100}$$

$$\|\lambda \,|b^{s,\alpha}_{pq}\| = \left(\sum_{j=0}^\infty \sum_{(G,k) \in I^{j,\alpha}} 2^{j(s-n/p)q} \left(\sum_{m \in \mathbb{Z}^n} |\lambda^{j,(G,k)}_m|^p \right)^{q/p} \right)^{1/q} \tag{5.101}$$

and

$$\|\lambda \,|f^{s,\alpha}_{pq}\| = \left\| \left(\sum_{j,(G,k),m} 2^{jsq} \left| \lambda^{j,(G,k)}_m \, \chi^k_m(\cdot) \right|^q \right)^{1/q} |L_p(\mathbb{R}^n) \right\| \tag{5.102}$$

(*with the usual modification if $p = \infty$ and/or $q = \infty$*). *Then*

$$b^{s,\alpha}_{pq} = \big\{ \lambda : \ \|\lambda \,|b^{s,\alpha}_{pq}\| < \infty \big\} \tag{5.103}$$

and

$$f^{s,\alpha}_{pq} = \big\{ \lambda : \ \|\lambda \,|f^{s,\alpha}_{pq}\| < \infty \big\}. \tag{5.104}$$

Remark 5.22. Obviously, the summation in (5.102) is the same as in (5.100). Both $b^{s,\alpha}_{pq}$ and $f^{s,\alpha}_{pq}$ are quasi-Banach spaces. In the isotropic case $\alpha = 1$ they coincide with the corresponding spaces in Definition 3.1.

After these preparations one can now extend Theorem 3.5 to anisotropic spaces. In particular we assume that ψ_F and ψ_M are the compactly supported Daubechies wavelets according to Theorem 1.61(ii). The counterpart of (3.33) is now given by

$$\sum_{j,(G,k),m} \lambda^{j,(G,k)}_m \, 2^{-|k|/2} \, \Psi^{j,(G,k),\alpha}_m, \qquad \lambda \in b^{s,\alpha}_{pq}. \tag{5.105}$$

One can repeat the considerations at the beginning of Section 3.1.3. In particular, if ψ_F and ψ_M are sufficiently smooth then $\Psi_m^{j,(G,k),\alpha}$ are anisotropic atoms in $B^{s,\alpha}_{pq}(\mathbb{R}^n)$ according to Definition 5.11 where we shifted now the normalising factors to the sequence spaces. Here we used (5.97) and corresponding assertions as far as derivatives and moment conditions are concerned. By the same arguments as at the beginning of Section 3.1.3 the series (5.105) converges unconditionally in $S'(\mathbb{R}^n)$ and locally in $B^{\sigma,\alpha}_{pq}(\mathbb{R}^n)$ with $\sigma < s$. This justifies the abbreviation

$$\sum_{j,(G,k),m} = \sum_{j=0}^{\infty} \sum_{(G,k)\in I^{j,\alpha}} \sum_{m\in\mathbb{Z}^n}. \tag{5.106}$$

In Definition 1.56 we said what is meant by an unconditional (Schauder) basis in a quasi-Banach space.

Theorem 5.23. *Let α be an anisotropy according to* (5.10) *and let $\Psi_m^{j,(G,k),\alpha}$ be given by* (5.96) *with the Daubechies wavelets*

$$\psi_F \in C^u(\mathbb{R}) \quad \text{and} \quad \psi_M \in C^u(\mathbb{R}), \qquad u \in \mathbb{N}, \tag{5.107}$$

as in Theorem 1.61(ii)*.*

(i) *Let $0 < p \le \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. Then there is a natural number $u^\alpha(s,p)$ with the following property. Let $u > u^\alpha(s,p)$ in* (5.107)*. Let $f \in S'(\mathbb{R}^n)$. Then $f \in B^{s,\alpha}_{pq}(\mathbb{R}^n)$ if, and only if, it can be represented as*

$$f = \sum_{j,(G,k),m} \lambda_m^{j,(G,k)}\, 2^{-|k|/2}\, \Psi_m^{j,(G,k),\alpha}, \qquad \lambda \in b^{s,\alpha}_{pq}, \tag{5.108}$$

unconditional convergence being in $S'(\mathbb{R}^n)$ and locally in any $B^{\sigma,\alpha}_{pq}(\mathbb{R}^n)$ with $\sigma < s$. The representation (5.108) *is unique,*

$$\lambda_m^{j,(G,k)} = 2^{|k|/2} \left(f, \Psi_m^{j,(G,k),\alpha} \right) \tag{5.109}$$

and

$$I: \ f \mapsto \left\{ 2^{|k|/2} \left(f, \Psi_m^{j,(G,k),\alpha} \right) \right\} \tag{5.110}$$

is an isomorphic map of $B^{s,\alpha}_{pq}(\mathbb{R}^n)$ **onto** *$b^{s,\alpha}_{pq}$. If, in addition, $p < \infty$, $q < \infty$, then $\left\{ \Psi_m^{j,(G,k),\alpha} \right\}$ is an unconditional basis in $B^{s,\alpha}_{pq}(\mathbb{R}^n)$.*

(ii) *Let $0 < p < \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. Then there is a natural number $u^\alpha(s,p,q)$ with the following property. Let $u > u^\alpha(s,p,q)$ in* (5.107)*. Let $f \in S'(\mathbb{R}^n)$. Then $f \in F^{s,\alpha}_{pq}(\mathbb{R}^n)$ if, and only if, it can be represented as*

$$f = \sum_{j,(G,k),m} \lambda_m^{j,(G,k)}\, 2^{-|k|/2}\, \Psi_m^{j,(G,k),\alpha}, \qquad \lambda \in f^{s,\alpha}_{pq}, \tag{5.111}$$

unconditional convergence being in $S'(\mathbb{R}^n)$ *and locally in any* $F^{\sigma,\alpha}_{pq}(\mathbb{R}^n)$ *with* $\sigma < s$. *The representation is unique where* $\lambda^{j,(G,k)}_m$ *is given by* (5.109). *Furthermore, I in* (5.110) *is an isomorphic map of* $F^{s,\alpha}_{pq}(\mathbb{R}^n)$ **onto** $f^{s,\alpha}_{pq}$. *If, in addition,* $q < \infty$ *then* $\left\{\Psi^{j,(G,k),\alpha}_m\right\}$ *is an unconditional basis in* $F^{s,\alpha}_{pq}(\mathbb{R}^n)$.

Proof. After the above preparations the proof of Theorem 3.5 can be carried over from the isotropic case to the above anisotropic one. The atomic representation (5.105) gives the anisotropic counterpart of (3.41). The discussion about local means in Remark 5.20 and the anisotropic extension of Proposition 3.3, which is covered by [Far00] and the references in the proof of Proposition 3.3, ensure the anisotropic counterpart of (3.43), (3.44). Using duality as described in Section 5.1.7 one gets the anisotropic version of (3.45), (3.46), in particular the claimed uniqueness. The rest is the same as there. □

It is not a surprise that one can replace the Daubechies wavelets in the above theorem by Meyer wavelets as we did in the isotropic case in Theorem 3.12, with the following outcome.

Theorem 5.24. *Let* α *be an anisotropy according to* (5.10) *and let* $\Psi^{j,(G,k),\alpha}_m$ *be given by* (5.96) *with the one-dimensional Meyer wavelets* ψ_F *and* ψ_M *according to Theorem* 1.61(i), (3.63), (3.64).

(i) *Let* $0 < p \leq \infty$, $0 < q \leq \infty$ *and* $s \in \mathbb{R}$. *Let* $f \in S'(\mathbb{R}^n)$. *Then* $f \in B^{s,\alpha}_{pq}(\mathbb{R}^n)$ *if, and only if, it can be represented by* (5.108), *unconditional convergence being in* $S'(\mathbb{R}^n)$ *and locally in any* $B^{\sigma,\alpha}_{pq}(\mathbb{R}^n)$ *with* $\sigma < s$. *This representation is unique where* $\lambda^{j,(G,k)}_m$ *is given by* (5.109). *Furthermore, I in* (5.110) *is an isomorphic map of* $B^{s,\alpha}_{pq}(\mathbb{R}^n)$ **onto** $b^{s,\alpha}_{pq}$. *If, in addition,* $p < \infty$, $q < \infty$ *then* $\left\{\Psi^{j,(G,k),\alpha}_m\right\}$ *is an unconditional basis in* $B^{s,\alpha}_{pq}(\mathbb{R}^n)$.

(ii) *Let* $0 < p < \infty$, $0 < q \leq \infty$ *and* $s \in \mathbb{R}$. *Let* $f \in S'(\mathbb{R}^n)$. *Then* $f \in F^{s,\alpha}_{pq}(\mathbb{R}^n)$ *if, and only if, it can be represented by* (5.111), *unconditional convergence being in* $S'(\mathbb{R}^n)$ *and locally in any* $F^{\sigma,\alpha}_{pq}(\mathbb{R}^n)$ *with* $\sigma < s$. *This representation is unique where* $\lambda^{j,(G,k)}_m$ *is given by* (5.109). *Furthermore, I in* (5.110) *is an isomorphic map of* $F^{s,\alpha}_{pq}(\mathbb{R}^n)$ **onto** $f^{s,\alpha}_{pq}$. *If, in addition,* $q < \infty$ *then* $\left\{\Psi^{j,(G,k),\alpha}_m\right\}$ *is an unconditional basis in* $F^{s,\alpha}_{pq}(\mathbb{R}^n)$.

Proof. One has to combine the arguments from the proofs of Theorems 3.12 and 5.23. This applies in particular to the Tauberian condition ensured by (3.63), (3.64). In case of anisotropic molecules one can rely on the anisotropic version of the reduction of molecules to atoms as outlined in Section 3.2.3 and (3.133). Some details may also be found in [HaTa05], Proposition 3.2. This is sufficient for the simple case needed here. But one can also consult [Kyr04], [BoH05], [Bow05] extending atomic and molecular decompositions for isotropic spaces according to [FJW91] to (weighted) anisotropic spaces. □

Remark 5.25. Section 5.2 is an improved and modified version of the relevant parts of [Tri05b]. In particular we developed in this paper the anisotropic multiresolution analysis as presented in Section 5.2.1. Compared with the elegant isotropic version as described in Proposition 1.53 it looks somewhat complicated and one might ask whether there are simpler versions. The first attempt would be to replace the n-dimensional isotropic dilation factors 2^{j-1}in (1.136) by $2^{(j-1)\alpha_r}$ where α is the anisotropy (5.10). This can be reduced to the question of whether there is a (compactly supported, sufficiently smooth) one-dimensional scaling function ψ_F and a related wavelet ψ_M such that for given $1 < b \in \mathbb{R}$,

$$\Big\{ \psi_F(x-m),\ b^{j/2}\psi_M(b^j x - m) :\ j \in \mathbb{N}_0,\, m \in \mathbb{Z} \Big\} \tag{5.112}$$

is an orthonormal basis in $L_2(\mathbb{R})$ in modification of Proposition 1.51 and Theorem 1.61. This problem attracted some attention. We refer to [Aus92] and [Bow03b]. In the latter paper one finds also the corresponding recent literature. According to [Bow03b], Theorem 4.1, for irrational b there is no orthonormal basis in $L_2(\mathbb{R})$ with a compactly supported wavelet ψ_M (this negative assertion remains valid if one admits not only one wavelet ψ_M but finitely many). The case $b = v/w$ with $v \in \mathbb{N}$, $w \in \mathbb{N}$, relatively prime, has been considered in [Aus92]. The situation is better, but not good enough for the above purposes if $v > w > 1$. Then there is no compactly supported ψ_M such that (5.112) is an orthonormal basis in $L_2(\mathbb{R})$, [Aus92], Theorem 1(ii). The situation improves if $b - 1 \in \mathbb{N}$ in (5.112) and if, maybe, not only one but finitely many wavelets are admitted. Transferred to the above anisotropic situation one arrives at the assumption that $2^{\alpha_r} \in \mathbb{N}$ (or at least that 2^{α_r} is rational) where α is an anisotropy according to (5.10). Under these restrictions it is possible to construct some wavelet characterisations for anisotropic function spaces based on (5.112). We refer to [GaT02] for spaces of type $B^{s,\alpha}_{pq}(\mathbb{R}^n)$ with $s > 0$, $1 \le p \le \infty$, $1 \le q \le \infty$, to [GHT04] for some extensions of these spaces to $p < 1$ (where the corresponding B-spaces do not always coincide with the B-spaces according to Definition 5.1), to [Bow03a], Chapter 2, where wavelet bases belonging to $S(\mathbb{R}^n)$ are considered in anisotropic Hardy spaces (5.33) with $0 < p \le 1$, and to [Kyr04] for the full scales of spaces according to Definition 5.1. In any case wavelet characterisations of anisotropic spaces according to Definition 5.1 in terms of isomorphisms or bases via tensor products of one-dimensional L_2-bases of type (5.112) require additional assumptions for the anisotropy α. Our approach works for all anisotropies α, but the outcome is less elegant. On the other hand, constructions of this type have been considered before. Nearest to us is [BeN93], [BeN95] dealing with wavelet bases belonging to $S(\mathbb{R}^n)$ of Meyer type in the spaces from Definition 5.1. Similar ideas have been used in [Hoch02] in connection with anisotropic Besov spaces in cubes in $\mathbb{R}^n$.

5.3 The transference method

5.3.1 The method

Let α be an anisotropy according to (5.10) and let Q^α_{jm} with $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$ be the same rectangle as at the beginning of Section 5.1.5 with the characteristic function χ^α_{jm}. Then for $s \in \mathbb{R}$, $0 < p \le \infty$, $0 < q \le \infty$ the quasi-Banach spaces $\bar{f}^{s,\alpha}_{pq}$ and $\bar{b}^{s,\alpha}_{pq}$ are defined as the collection of all sequences

$$\lambda = \left\{ \lambda^j_m \in \mathbb{C} : \; j \in \mathbb{N}_0, \; m \in \mathbb{Z}^n \right\} \tag{5.113}$$

such that the corresponding quasi-norms

$$\| \lambda \, | \bar{f}^{s,\alpha}_{pq} \| = \left\| \left(\sum_{j=0}^\infty \sum_{m \in \mathbb{Z}^n} 2^{jsq} \, \left| \lambda^j_m \chi^\alpha_{jm}(\cdot) \right|^q \right)^{1/q} | L_p(\mathbb{R}^n) \right\| \tag{5.114}$$

and

$$\begin{aligned} \| \lambda \, | \bar{b}^{s,\alpha}_{pq} \| &= \left(\sum_{j=0}^\infty 2^{j(s-n/p)q} \left(\sum_{m \in \mathbb{Z}^n} |\lambda^j_m|^p \right)^{q/p} \right)^{1/q} \\ &\sim \left(\sum_{j=0}^\infty 2^{jsq} \left\| \sum_{m \in \mathbb{Z}^n} \lambda^j_m \chi^\alpha_{jm} \, | L_p(\mathbb{R}^n) \right\|^q \right)^{1/q} \end{aligned} \tag{5.115}$$

are finite (with the usual modifications if $p = \infty$ and/or $q = \infty$). Compared with Definition 5.13 we incorporated now the weight factors. By Theorems 5.23, 5.24 one can try to shift problems for the spaces $B^{s,\alpha}_{pq}(\mathbb{R}^n)$ and $F^{s,\alpha}_{pq}(\mathbb{R}^n)$ to corresponding problems for the sequence spaces $b^{s,\alpha}_{pq}$ and $f^{s,\alpha}_{pq}$ in Definition 5.21. The above sequence spaces $\bar{b}^{s,\alpha}_{pq}$ and $\bar{f}^{s,\alpha}_{pq}$ are the hard core of these sequence spaces. Recall that the cardinal number of $I^{j,\alpha}$ can be estimated independently of j by (5.91). Furthermore, $\bar{b}^{s,\alpha}_{pq}$ is also independent of α. This makes clear that one can *transfer* problems for anisotropic B-spaces via sequence spaces to corresponding problems for isotropic B-spaces. Below we illuminate this *transference method* by several examples. But first we clarify that there is no immediate counterpart of this observation for the F-spaces.

Proposition 5.26. *Let $s \in \mathbb{R}$, $0 < p \le \infty$, $0 < q \le \infty$. Let α^1 and α^2 be two anisotropies according to* (5.10).

(i) *Then*

$$\bar{b}^{s,\alpha^1}_{pq} = \bar{b}^{s,\alpha^2}_{pq}. \tag{5.116}$$

(ii) *Let $\alpha^1 \ne \alpha^2$. Then*

$$\bar{f}^{s,\alpha^1}_{pq} = \bar{f}^{s,\alpha^2}_{pq} \quad \textit{if, and only if,} \quad p = q. \tag{5.117}$$

Proof. By (5.114), (5.115) we have (5.116) and also (5.117) with $p = q$. To disprove equality in (5.117) if $p \neq q$ we may assume $s = 0$, $\alpha^1 = (1, \ldots, 1)$ and $\alpha^2 = \alpha$ according to (5.10) with $\alpha_1 < 1$. We denote the corresponding spaces by $\bar{f}_{pq}$ (isotropic case) and $\bar{f}^{\alpha}_{pq}$. Let Q^l be a cube in $\mathbb{R}^n$ with the left corner-point $\bar{l} = (l, \ldots, l)$, where $l \in \mathbb{N}$, and of side-length 1. At level $j+1$ with $j \in \mathbb{N}_0$ we divide this cube naturally into 2^{jn} disjoint (open) cubes of side-length $2^{-j-1}2 = 2^{-j}$ and put

$$\lambda^{j+1}_m = \lambda_j \quad \text{if } 2^{-j-1}\, m \text{ is the centre of one of these cubes} \tag{5.118}$$

and $\lambda^{j+1}_m = 0$ otherwise. In particular,

$$l\, 2^{j+1} < m_r < (l+1)\, 2^{j+1}, \qquad r = 1, \ldots, n, \tag{5.119}$$

for $m \in \mathbb{Z}^n$ with (5.118), and

$$\|\lambda \,|\bar{f}_{pq}\| = \left(\sum_{j=0}^{\infty} |\lambda_j|^q \right)^{1/q}. \tag{5.120}$$

The same m as in (5.118), (5.119) result for the anisotropic space $\bar{f}^{\alpha}_{pq}$ in a corresponding subdivision of the rectangle $Q^{\alpha,l,j}$,

$$l\, 2^{j+1}\, 2^{-(j+1)\alpha_r} \le x_r < (l+1)\, 2^{j+1}\, 2^{-(j+1)\alpha_r}, \tag{5.121}$$

where $|Q^{\alpha,l,j}| = 1$. If $1 + 1/l < 2^{1-\alpha_1}$ then

$$(l+1)\, 2^{j(1-\alpha_1)} = (1 + 1/l)\, l\, 2^{j(1-\alpha_1)} < l\, 2^{(j+1)(1-\alpha_1)}. \tag{5.122}$$

In particular, the rectangles $Q^{\alpha,l,j}$ are pairwise disjoint. Hence,

$$\|\lambda \,|\bar{f}^{\alpha}_{pq}\| = \left(\sum_{j=0}^{\infty} |\lambda_j|^p \right)^{1/p}. \tag{5.123}$$

This proves the only-if-part of (5.117). □

Remark 5.27. In particular the transference method for B-spaces which we outlined above cannot be extended immediately to F-spaces as we conjectured in [Tri05b]. But the negative outcome (5.117) might not be the last word. The numbers λ^j_m in (5.113), (5.114) are located at the rectangle Q^{α}_{jm}. Then one gets for different anisotropies the decoupling as used in the above proof. One could try to avoid this effect by a more sophisticated one-to-one relation between rectangles Q^{α}_{jm} for different anisotropies at each level j using that a cube of side-length 1 is intersected by $\sim 2^{jn}$ rectangles Q^{α}_{jm} independently of α. In addition there is an anisotropic counterpart of the useful observation described in Section 1.5.3. But nothing has been done so far.

5.3.2 Embeddings

We illustrate the transference method for B-spaces by three rather different examples.

Theorem 5.28. *Let α be an anisotropy according to* (5.10). *Let $s_1 \in \mathbb{R}$, $s_2 \in \mathbb{R}$ and $p_1, p_2, q_1, q_2 \in (0,\infty]$. Let $B^s_{pq}(\mathbb{R}^n)$ and $B^{s,\alpha}_{pq}(\mathbb{R}^n)$ be the spaces as introduced in the corresponding Definitions* 2.1 *and* 5.1. *Then*

$$B^{s_1,\alpha}_{p_1q_1}(\mathbb{R}^n) \hookrightarrow B^{s_2,\alpha}_{p_2q_2}(\mathbb{R}^n) \quad \textit{if, and only if,} \quad B^{s_1}_{p_1q_1}(\mathbb{R}^n) \hookrightarrow B^{s_2}_{p_2q_2}(\mathbb{R}^n). \tag{5.124}$$

Proof. By Theorem 5.24 the question of the first embedding in (5.124) is equivalent to a corresponding question for related sequence spaces of type $b^{s,\alpha}_{pq}$ and, hence, by Section 5.3.1 for related sequence spaces of type $\bar{b}^{s,\alpha}_{pq}$. But by Proposition 5.26 the latter spaces are independent of α. This proves (5.124). □

Remark 5.29. Hence any embedding theorem for isotropic B-spaces can be carried over to anisotropic spaces. To which extent a corresponding assertion is valid for F-spaces and for sharp limiting embeddings of type (1.299)–(1.303) is not so clear. For classical embeddings one may consult the references given in Section 5.1.1. Sharp limiting embeddings for anisotropic spaces may be found in [Yam86, §3] and [FJS00, Appendix C.3]. An affirmative answer to the problem described in Remark 5.27 would result in an extension of the transference method from B-spaces to F-spaces and mixed embeddings including a transference of the theory of envelopes as described in Section 1.9 from isotropic spaces to anisotropic ones. Quite recently it came out by direct arguments that the envelopes of anisotropic spaces do not depend on the anisotropy α. Hence they are the same as for isotropic spaces, [MNP05].

5.3.3 Entropy numbers

Let $B^{s,\alpha}_{pq}(\mathbb{R}^n)$ be the anisotropic spaces as introduced in Definition 5.1 and let Ω be an arbitrary domain in $\mathbb{R}^n$. Then $B^{s,\alpha}_{pq}(\Omega)$ is the collection of all $f \in D'(\Omega)$ such that there is a $g \in B^{s,\alpha}_{pq}(\mathbb{R}^n)$ with $g|\Omega = f$. Furthermore,

$$\|f\,|B^{s,\alpha}_{pq}(\Omega)\| = \inf \|g\,|B^{s,\alpha}_{pq}(\mathbb{R}^n)\| \tag{5.125}$$

where the infimum is taken over all $g \in B^{s,\alpha}_{pq}(\mathbb{R}^n)$ such that its restriction $g|\Omega$ to Ω coincides in $D'(\Omega)$ with f. This is the extension of the Definitions 1.95 and 4.1 for isotropic B-spaces to anisotropic ones. Of course, $B^{s,\alpha}_{pq}(\Omega)$ are quasi-Banach spaces. For isotropic spaces in bounded domains Ω we have Theorem 1.97. Recall that e_k are the entropy numbers according to Definitions 1.87, 4.43.

Theorem 5.30. *Let Ω be an arbitrary bounded domain in $\mathbb{R}^n$ and let α be an anisotropy according to* (5.10). *Let $p_0, p_1, q_0, q_1 \in (0,\infty]$ and*

$$-\infty < s_1 < s_0 < \infty, \qquad s_0 - n/p_0 > s_1 - n/p_1. \tag{5.126}$$

Then the embedding

$$\mathrm{id}: \quad B^{s_0,\alpha}_{p_0 q_0}(\Omega) \hookrightarrow B^{s_1,\alpha}_{p_1 q_1}(\Omega) \tag{5.127}$$

is compact and

$$e_v(\mathrm{id}) \sim v^{-\frac{s_0-s_1}{n}}, \qquad v \in \mathbb{N}. \tag{5.128}$$

Proof. The isotropic case is covered by Theorem 1.97 with a reference to [Triδ], Theorem 23.2, p. 186. By the same arguments as there it is sufficient to deal with the embedding of distributions

$$f \in B^{s_0,\alpha}_{p_0 q_0}(\mathbb{R}^n) \qquad \text{with} \qquad \mathrm{supp}\, f \subset K \tag{5.129}$$

into $B^{s_1,\alpha}_{p_1 q_1}(\mathbb{R}^n)$ where K is a fixed bounded domain, say, the unit ball. We choose u in (5.107) so large that we can apply Theorem 5.23 both to the source space and to the target space. In particular in (5.108), (5.109) only terms with

$$K \cap \mathrm{supp}\, \Psi^{j,(G,k),\alpha}_m \neq \emptyset \tag{5.130}$$

are of relevance. Recall that $|Q^k_m| \sim 2^{-jn}$ for Q^k_m in (5.99) with $(G,k) \in I^{j,\alpha}$. Hence (5.130) applies only to $M_j \sim 2^{jn}$ terms, where the equivalence constants may depend on α (and K) but not on j. This can be transferred to the sequence spaces $\bar{b}^{s,\alpha}_{pq}$ in (5.115) where the relevant parts are the spaces $\bar{b}^{s,\alpha}_{pq}(K)$, consisting of all (newly numbered) sequences

$$\lambda = \left\{ \lambda^j_\nu \in \mathbb{C} : \ j \in \mathbb{N}_0, \ \nu = 1, \ldots, M_j \right\} \tag{5.131}$$

quasi-normed by

$$\|\lambda \,|\bar{b}^{s,\alpha}_{pq}(K)\| = \left(\sum_{j=0}^{\infty} 2^{j(s-n/p)q} \left(\sum_{\nu=1}^{M_j} |\lambda^j_\nu|^p \right)^{q/p} \right)^{1/q}. \tag{5.132}$$

Hence the embedding (5.127) can be transferred to the sequence side,

$$e_v(\mathrm{id}) \sim e_v(\mathrm{id}_b) \quad \text{with} \quad \mathrm{id}_b : \ \bar{b}^{s_0,\alpha}_{p_0 q_0}(K) \hookrightarrow \bar{b}^{s_1,\alpha}_{p_1 q_1}(K), \tag{5.133}$$

which is independent of α. Then the above theorem follows from Theorem 1.97. □

Remark 5.31. We followed essentially [Tri05b]. Otherwise one may consult Remark 1.98 and the references given there about the substantial history of entropy numbers in isotropic (unweighted) function spaces. As far as anisotropic spaces are concerned we are aware of only two papers, [Boz71] (classical Sobolev spaces) and [Tri75] (fractional Sobolev spaces and classical Besov spaces). However these papers have nothing in common with the recent approach to problems of this type via sequence spaces, but their results are covered by the above theorem. This follows from the elementary embedding

$$B^{s,\alpha}_{p,\min(p,q)}(\mathbb{R}^n) \hookrightarrow F^{s,\alpha}_{pq}(\mathbb{R}^n) \hookrightarrow B^{s,\alpha}_{p,\max(p,q)}(\mathbb{R}^n), \tag{5.134}$$

as a consequence of Definition 5.1, the concrete spaces listed in Theorem 5.5 and the independence of (5.128) of q_0, q_1.

5.3.4 Besov characteristics

Let $f \in S'(\mathbb{R}^n)$ be a compactly supported distribution which is not C^∞. Let α be an anisotropy according to (5.10). Then

$$s_f^\alpha(t) = \sup\left\{s : \, f \in B^{s;\alpha}_{p\infty}(\mathbb{R}^n)\right\}, \quad 0 \le t = 1/p < \infty, \tag{5.135}$$

is the anisotropic generalisation of the (isotropic) Besov characteristics $s_f(t)$ in (1.620). One may ask to which extent properties for $s_f(t)$, as discussed in Section 1.18 and will be considered later on in greater detail in Section 7.2.3, can be transferred to $s_f^\alpha(t)$. Recall that a real function on an interval is called *increasing* if it is *non-decreasing.*

Theorem 5.32. *Let α be an anisotropy according to* (5.10)*.*

(i) *Let $f \in S'(\mathbb{R}^n)$ be not C^∞ and let* supp f *be compact. Then $s_f^\alpha(t)$ according to* (5.135) *is an increasing concave function in the (t,s)-diagram in Figure* 1.17.1 *of slope smaller than or equal to n.*

(ii) *For any real increasing concave function $s(t)$, where $0 \le t < \infty$, of slope smaller than or equal to n there is a compactly supported distribution $f \in S'(\mathbb{R}^n)$ with $s(t) = s_f^\alpha(t)$.*

Proof. One can transfer the corresponding assertion in Theorem 1.199 from the isotropic case to the anisotropic one. We rely now on the universal Meyer representation Theorems 3.12 and 5.24 which apply simultaneously to all corresponding isotropic and anisotropic B-spaces. Let $f \in S'(\mathbb{R}^n)$ be expanded according to (3.36) and Theorem 3.12,

$$f = \sum_{j=0}^\infty \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} \lambda_m^{j,G}\, 2^{-jn/2}\, \Psi_m^{j,G}. \tag{5.136}$$

For fixed $j \in \mathbb{N}_0$ one can map the index-set $G^j \times \mathbb{Z}^n$ one-to-one onto a subset of its anisotropic counterpart $I^{j,\alpha} \times \mathbb{Z}^n$ in Section 5.2.1. Let

$$g = \sum_{j=0}^\infty \sum_{(G,k) \in I^{j,\alpha}} \sum_{m \in \mathbb{Z}^n} \mu_m^{j,(G,k)}\, 2^{-|k|/2}\, \Psi_m^{j,(G,k),\alpha} \tag{5.137}$$

according to (5.108) and Theorem 5.24 (Meyer wavelets), where the coefficients $\mu_m^{j,(G,k)}$ are either zero or coincide for fixed $j \in \mathbb{N}_0$ one-to-one with the coefficients $\lambda_m^{j,G}$ in (5.136). Then (3.7) and its anisotropic counterpart (5.101) are essentially the same. Hence $s_g^\alpha(t) = s_f(t)$. Similarly one can start from the anisotropic side and arrive at the isotropic one. Then one gets both parts of the above theorem with one exception. The Meyer wavelets are not local. Assuming that f in (5.136) has a compact support, say, in the unit ball. Then g in (5.137) need not have a compact support. We outline how this defect can be repaired. First one checks that the coefficients $\lambda_m^{j,G}$ in (5.136) given by (3.37) decay rapidly with respect to j and m if $2^{-j}|m| \to \infty$. The corresponding terms in g generate a function in $S(\mathbb{R}^n)$ which can be omitted. Furthermore in the above indicated one-to-one map of $G^j \times \mathbb{Z}^n$ onto a subset of $I^{j,\alpha} \times \mathbb{Z}^n$ it can be assumed that points inside a ball of radius $R \geq 1$ remain inside a ball of radius cR for some $c > 0$ which is independent of R. But then one can check that for a suitable cut-off function $\chi \in D(\mathbb{R}^n)$ one has $\tilde{g} = (1-\chi)g \in S(\mathbb{R}^n)$. Hence $s^{\alpha}_{\tilde{g}}(t) = s^{\alpha}_{g}(t) = s_f(t)$. We omit the (unimportant) details. In a somewhat different situation we used arguments of this type more carefully in Step 2 of the proof of Theorem 3.37. □

Remark 5.33. Hence one can transfer at least some assertions for (isotropic) Besov characteristics to anisotropic spaces. But this does not immediately apply to other properties based on finite Radon measures. Then a direct approach parallel to the isotropic spaces may result in better assertions. An extension of (5.135) and Theorem 5.32 to not necessarily compactly supported $f \in S'(\mathbb{R}^n)$ has been given recently in [Ved06].

Chapter 6

Weighted Function Spaces

6.1 Definitions and basic properties

6.1.1 Introduction

In the preceding Chapters 2–5 we dealt exclusively with unweighted function spaces in $\mathbb{R}^n$ or on domains in $\mathbb{R}^n$. On the other hand corresponding weighted spaces with a wide range of diverse types of weights have attracted a lot of attention. We are interested here in a rather special case replacing $L_p(\mathbb{R}^n)$ in Definition 2.1 by a weighted L_p-space where the corresponding (smooth) weights are of at most polynomial growth without local singularities. We continue the studies which began in [HaT94a], [HaT94b] and had been presented in detail in [ET96, Ch. 4]. But we repeat all that is needed to make Chapter 6 independently readable. We rely mostly on [HaT05]. Our main result is a wavelet isomorphism for the weighted spaces considered, Section 6.2. As an application we deal in detail with entropy numbers of compact embeddings between some of these weighted function spaces with weights of polynomial type, Section 6.4, preceded by a preparation in Section 6.3 looking at corresponding compact embeddings between related weighted ℓ_p-spaces. This will be complemented in Section 6.5 where we deal with more general weights (than in Section 6.4) and with (weighted and unweighted) radial spaces.

6.1.2 Definitions

We use the same basic notation as in Section 2.1.2. Furthermore, $C^k(\mathbb{R}^n)$ with $k \in \mathbb{N}_0$ is the space as introduced in (4.10), (4.11).

Definition 6.1. *Let $n \in \mathbb{N}$. The class W^n of admissible weight functions is the collection of all positive C^∞ functions w on $\mathbb{R}^n$ with the following properties.*

(i) *For all* $\gamma \in \mathbb{N}_0^n$ *there is a positive constant* c_γ *with*

$$|D^\gamma w(x)| \le c_\gamma w(x) \qquad \textit{for all } x \in \mathbb{R}^n. \tag{6.1}$$

(ii) *There are two constants* $c > 0$ *and* $\alpha \ge 0$ *such that*

$$0 < w(x) \le c\, w(y) \left(1 + |x-y|^2\right)^{\alpha/2} \quad \textit{for all } x \in \mathbb{R}^n,\ y \in \mathbb{R}^n. \tag{6.2}$$

Remark 6.2. Let w be a measurable function in $\mathbb{R}^n$ satisfying (6.2). Let g be a non-negative C^∞ in $\mathbb{R}^n$ with support in the unit ball and, say, $\widehat{g}(0) = 1$. Then

$$w_g(x) = (w \star g)(x) = \int_{\mathbb{R}^n} g(x-y)\, w(y)\, \mathrm{d}y \in W^n. \tag{6.3}$$

Furthermore, $w \sim w_g$. Nevertheless, (6.1) will be of some service for us, although one might consider $w \in W^n$ with (6.1), (6.2) as a smooth representative in the class of equivalent measurable functions with (6.2). If $\lambda > 0$ and $w \in W^n$ then

$$\lambda w \in W^n, \quad w^\lambda \in W^n, \quad w^{-1} \in W^n. \tag{6.4}$$

Furthermore, $w_1 + w_2 \in W^n$ and $w_1 w_2 \in W^n$ if $w_1 \in W^n$, $w_2 \in W^n$. We relied in [HaT94a], [HaT94b] and [ET96] on this class W^n as the underlying weights for B-spaces and F-spaces in the framework of Fourier-analytical definitions. But it is a special case of larger classes of weights and function spaces based on ultra-distributions as considered in [Tri77], [Triβ, Ch. 6,7], and studied in detail in [ST87, Section 5.1]. There one finds also references to the original papers and the relevant literature.

Recall that $L_p(\mathbb{R}^n)$ with $0 < p \le \infty$ is the quasi-Banach space as introduced in Section 2.1.2 and quasi-normed by (2.1). Then $L_p(\mathbb{R}^n, w)$ with $w \in W^n$ and $0 < p \le \infty$ is the usual quasi-Banach space quasi-normed by

$$\|f \,|L_p(\mathbb{R}^n, w)\| = \|wf \,|L_p(\mathbb{R}^n)\|. \tag{6.5}$$

Otherwise we use the same notation as in Section 2.1.3 and in Definition 2.1.

Definition 6.3. *Let* $\varphi = \{\varphi_j\}_{j=0}^\infty$ *be the dyadic resolution of unity in* $\mathbb{R}^n$ *according to* (2.8)–(2.10). *Let* $w \in W^n$ *be as introduced in Definition* 6.1. *Let* $s \in \mathbb{R}$ *and* $0 < q \le \infty$.

(i) *Let* $0 < p \le \infty$. *Then* $B^s_{pq}(\mathbb{R}^n, w)$ *is the collection of all* $f \in S'(\mathbb{R}^n)$ *such that*

$$\|f \,|B^s_{pq}(\mathbb{R}^n, w)\|_\varphi = \left(\sum_{j=0}^\infty 2^{jsq} \|(\varphi_j \widehat{f})^\vee \,|L_p(\mathbb{R}^n, w)\|^q \right)^{1/q} \tag{6.6}$$

(*with the usual modification if* $q = \infty$) *is finite.*

(ii) *Let $0 < p < \infty$. Then $F^s_{pq}(\mathbb{R}^n, w)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that*

$$\|f\,|F^s_{pq}(\mathbb{R}^n,w)\|_\varphi = \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} |(\varphi_j \widehat{f})^\vee(\cdot)|^q \right)^{1/q} |L_p(\mathbb{R}^n, w) \right\| \tag{6.7}$$

(*with the usual modification if $q = \infty$*) *is finite.*

Remark 6.4. If $w = 1$ then we have the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ as introduced in Definition 2.1. There is a huge literature about weighted function spaces. We restrict ourselves to a few references closely related to the above approach. In particular, [ST87, Section 5.1] (based on the above mentioned literature) deals with spaces of the above type $B^s_{pq}(\mathbb{R}^n, w)$ and $F^s_{pq}(\mathbb{R}^n, w)$ in the more general framework of ultra-distributions. Then one can admit weights w where (6.2) is replaced by

$$0 < w(x) \le c w(y) e^{|x-y|^\beta}, \quad \text{with } 0 < \beta < 1 \text{ and } x \in \mathbb{R}^n,\ y \in \mathbb{R}^n. \tag{6.8}$$

It is remarkable that it is even possible to develop a corresponding theory with smooth weights of exponential growth, $w(x) = e^{|x|}$ if $|x| \ge 1$, for example. Then one has to replace the above Fourier-analytical definitions by corresponding definitions in terms of local means. We refer to [Sco98a], [Sco98b], [Sco99]. Spaces of the above type with (non-smooth) Muckenhoupt weights w in place of the above smooth weights have been studied in [BPT96], [BPT97]. The most general approach to these types of spaces, combining (polynomial and exponential) smooth weights with Muckenhoupt weights has been given in [Ry01]. We rely here on [HaT94a], [HaT94b], and [ET96, Ch. 4] as far as basic assertions are concerned. Some more specific references will be given in the Remarks 6.17 and 6.32, 6.34, in connection with wavelet isomorphisms and, in particular, entropy numbers.

6.1.3 Basic properties

We collect a few basic properties available in the literature following mostly [ET96, Ch. 4] and [ST87, Section 5.1].

Theorem 6.5. *Let $w \in W^n$ according to Definition 6.1 and let $B^s_{pq}(\mathbb{R}^n, w)$ and $F^s_{pq}(\mathbb{R}^n, w)$ be the spaces introduced in Definition 6.3 with*

$$s \in \mathbb{R}, \quad 0 < p \le \infty \ (p < \infty \text{ for the } F\text{-spaces}), \quad 0 < q \le \infty. \tag{6.9}$$

(i) *Then $B^s_{pq}(\mathbb{R}^n, w)$ and $F^s_{pq}(\mathbb{R}^n, w)$ are quasi-Banach spaces (Banach spaces if $p \ge 1$, $q \ge 1$). They are independent of φ (equivalent quasi-norms).*

(ii) *The operator $f \mapsto wf$ is an isomorphic mapping from $B^s_{pq}(\mathbb{R}^n, w)$ onto $B^s_{pq}(\mathbb{R}^n)$ and from $F^s_{pq}(\mathbb{R}^n, w)$ onto $F^s_{pq}(\mathbb{R}^n)$. In particular,*

$$\|wf\,|B^s_{pq}(\mathbb{R}^n)\| \sim \|f\,|B^s_{pq}(\mathbb{R}^n, w)\| \tag{6.10}$$

and

$$\|wf\,|F^s_{pq}(\mathbb{R}^n)\| \sim \|f\,|F^s_{pq}(\mathbb{R}^n,w)\| \tag{6.11}$$

(*equivalent quasi-norms*).

(iii) *Let* I_σ *with* $\sigma \in \mathbb{R}$ *be given by* (1.5). *Then*

$$I_\sigma B^s_{pq}(\mathbb{R}^n,w) = B^{s-\sigma}_{pq}(\mathbb{R}^n,w) \quad \textit{and} \quad I_\sigma F^s_{pq}(\mathbb{R}^n,w) = F^{s-\sigma}_{pq}(\mathbb{R}^n,w). \tag{6.12}$$

Remark 6.6. All assertions are covered by [ET96], Section 4.2.2, pp. 156–160, where one finds also references to the original papers. Part (i) justifies our omission of the subscript φ in (6.6), (6.7) in what follows (as we already did in (6.10), (6.11)). The substantial assertion in part (ii) has a little history. The first proof was given in [Fra86b] based on paramultiplications, followed by [ST87], Section 5.1.3, pp. 245–249, using maximal functions and direct arguments. The shortest available proof based on local means may be found in [HaT94a] and [ET96], Section 4.2.2. The lifting property (6.12) extends (1.6) to the above weighted spaces. One can replace I_σ in (6.12) by

$$I_{\sigma,v}: \quad f \mapsto \left(\langle\xi\rangle^\sigma \widehat{vf}\right)^\vee v^{-1}, \qquad \sigma \in \mathbb{R}, \tag{6.13}$$

with $v \in W^n$. We refer again to [ET96], Section 4.2.2, p. 158.

Let $w \in W^n$. Then the embeddings for unweighted spaces as described in the points (i)–(iii) at the end of Section 1.11.1 remain valid if one replaces there (Ω) by $(\mathbb{R}^n,w)$. This follows from (6.10), (6.11). Comparing spaces of the above type with different weights one gets the following assertion.

Theorem 6.7. *Let* $w^1 \in W^n$, $w^2 \in W^n$, $0 < q_1 \le \infty$, $0 < q_2 \le \infty$. *Let*

$$-\infty < s_2 < s_1 < \infty \qquad \textit{and} \qquad 0 < p_1 \le p_2 < \infty. \tag{6.14}$$

(i) *Then*

$$F^{s_1}_{p_1q_1}(\mathbb{R}^n,w^1) \hookrightarrow F^{s_2}_{p_2q_2}(\mathbb{R}^n,w^2) \tag{6.15}$$

if, and only if,

$$s_1 - n/p_1 \ge s_2 - n/p_2 \qquad \textit{and} \qquad \frac{w^2(x)}{w^1(x)} \le c < \infty \tag{6.16}$$

for some $c > 0$ *and all* $x \in \mathbb{R}^n$.

(ii) *Then the embedding* (6.15) *is compact if, and only if,*

$$s_1 - n/p_1 > s_2 - n/p_2 \qquad \textit{and} \qquad \frac{w^2(x)}{w^1(x)} \to 0 \quad \textit{as } |x| \to \infty. \tag{6.17}$$

Remark 6.8. We refer for a proof to [ET96], Section 4.2.3, pp. 160–161. It is an easy consequence of Theorem 6.5 and compactness assertions for embeddings of (unweighted) spaces in bounded domains as stated in Theorem 1.97. As there one can ask what can be said about the degree of compactness expressed in terms of entropy numbers. We return to this point later on in some detail. Quite obviously, combining this theorem and, as indicated above, the embeddings (i)–(iii) in Section 1.11.1 with $(\mathbb{R}^n, w)$ in place of (Ω), one gets corresponding results for the B-spaces. Explicit formulations may be found in [ET96], pp. 161/162.

6.1.4 Special cases

The classical Sobolev spaces, Hölder-Zygmund spaces, and classical Besov spaces in $\mathbb{R}^n$ as described in Section 1.2 have more or less obvious weighted counterparts. The question arises to which extent they fit in the scheme of the spaces as introduced in Definition 6.3. This is largely the case. Even more, some assertions of Theorem 1.116 can be extended from the unweighted spaces $B^s_{pq}(\mathbb{R}^n)$, $F^s_{pq}(\mathbb{R}^n)$ to their weighted generalisations. We restrict ourselves to a brief description following largely [ST87], Section 5.1.4, pp. 249–253.

Let $w \in W^n$ according to Definition 6.1, $m \in \mathbb{N}$ and $1 < p < \infty$. Then $W^m_p(\mathbb{R}^n, w)$ is the collection of all $f \in L_p(\mathbb{R}^n, w)$ such that

$$\|f \,|W^m_p(\mathbb{R}^n, w)\| = \Big(\sum_{|\alpha| \le m} \|D^\alpha f \,|L_p(\mathbb{R}^n, w)\|^p \Big)^{1/p} < \infty, \tag{6.18}$$

in generalisation of (1.4). Put $W^0_p(\mathbb{R}^n, w) = L_p(\mathbb{R}^n, w)$. Furthermore, let $\Delta^M_h f$ be the differences in $\mathbb{R}^n$ according to (1.11) or (4.32), where $M \in \mathbb{N}$ and $h \in \mathbb{R}^n$. We specify the ball means in (1.377) by

$$\big(d^M_t f\big)(x) = t^{-n} \int_{|h| \le t} |\Delta^M_h f(x)| \, \mathrm{d}h, \quad x \in \mathbb{R}^n, \quad t > 0. \tag{6.19}$$

Recall that

$$\sigma_p = n\left(\frac{1}{p} - 1\right)_+ \qquad \text{and} \qquad \sigma_{pq} = n\left(\frac{1}{\min(p,q)} - 1\right)_+ \tag{6.20}$$

where $0 < p \le \infty$, $0 < q \le \infty$.

Theorem 6.9. *Let $w \in W^n$ according to Definition* 6.1 *and let $B^s_{pq}(\mathbb{R}^n, w)$ and $F^s_{pq}(\mathbb{R}^n, w)$ be the spaces as introduced in Definition* 6.3.

(i) *Let $1 < p < \infty$ and $m \in \mathbb{N}_0$. Then*

$$W^m_p(\mathbb{R}^n, w) = F^m_{p,2}(\mathbb{R}^n, w). \tag{6.21}$$

(ii) *Let* $0 < p \leq \infty$, $0 < q \leq \infty$, $\sigma_p < s < M \in \mathbb{N}$. *Then*

$$\|f \,|L_p(\mathbb{R}^n, w)\| + \left(\int\limits_{|h|\leq 1} |h|^{-sq} \, \left\| \Delta_h^M f \,|L_p(\mathbb{R}^n, w) \right\|^q \, \frac{\mathrm{d}h}{|h|^n} \right)^{1/q} \tag{6.22}$$

(*with the usual modification if* $q = \infty$) *is an equivalent quasi-norm in* $B^s_{pq}(\mathbb{R}^n, w)$.

(iii) *Let* $0 < p < \infty$, $0 < q < \infty$, $\sigma_{pq} < s < M \in \mathbb{N}$. *Then*

$$\|f \,|L_p(\mathbb{R}^n, w)\| + \left\| \left(\int_0^1 t^{-sq} \, d_t^M f(\cdot)^q \, \frac{\mathrm{d}t}{t} \right)^{1/q} \, |L_p(\mathbb{R}^n, w) \right\| \tag{6.23}$$

is an equivalent quasi-norm in $F^s_{pq}(\mathbb{R}^n, w)$.

Remark 6.10. This theorem is a special case of [ST87], Section 5.1.4, pp. 249–250, which applies to a larger class of weights in the framework of ultra-distributions. Of course, (6.21) with $m = 0$ is the Paley-Littlewood theorem,

$$L_p(\mathbb{R}^n, w) = F^0_{p,2}(\mathbb{R}^n, w), \qquad w \in W^n, \quad 1 < p < \infty. \tag{6.24}$$

References to the original papers may be found in [Triβ, Ch. 7] and [ST87, Ch. 5]. If $w = 1$ then one has the well-known assertions for the unweighted spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$. We refer to Theorem 1.116, [Triβ, Section 2.5.12, pp. 109/110] and [Triγ, Section 3.5.3, p. 194].

6.2 Wavelets

It is one of the main aims of Chapter 6 to extend the wavelet representations in Theorem 3.5 for the (unweighted) spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ to the corresponding weighted spaces $B^s_{pq}(\mathbb{R}^n, w)$ and $F^s_{pq}(\mathbb{R}^n, w)$ according to Definition 6.3. We use the same notation as in Theorem 3.5. In particular, the functions of the orthonormal basis

$$\Psi^{j,G}_m \qquad \text{with } j \in \mathbb{N}_0, \;\; G \in G^j, \;\; m \in \mathbb{Z}^n, \tag{6.25}$$

in $L_2(\mathbb{R}^n)$ are denoted as *k-wavelets* if they are constructed according to (3.2)–(3.5), based on the one-dimensional Daubechies wavelets with a real compactly supported scaling function $\psi_F \in C^k(\mathbb{R})$ and a real compactly supported associated wavelet $\psi_M \in C^k(\mathbb{R})$. Here $k \in \mathbb{N}$. Further details and references may be found in Section 1.7. We need the counterpart of Definition 3.1. Again let Q_{jm} with $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$ be the dyadic cubes in $\mathbb{R}^n$ as introduced in Section 2.1.2 and let χ_{jm} be the characteristic function of Q_{jm}.

Definition 6.11. *Let $w \in W^n$ according to Definition* 6.1*. Let $0 < p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$,*

$$\lambda = \left\{ \lambda_m^{j,G} \in \mathbb{C} : \ j \in \mathbb{N}_0, \ G \in G^j, \ m \in \mathbb{Z}^n \right\}, \tag{6.26}$$

$$\|\lambda \,|b^s_{pq}(w)\| = \left(\sum_{j=0}^\infty 2^{j(s-n/p)q} \sum_{G \in G^j} \left(\sum_{m \in \mathbb{Z}^n} w(2^{-j}m)^p \, |\lambda_m^{j,G}|^p \right)^{q/p} \right)^{1/q} \tag{6.27}$$

and

$$\|\lambda \,|f^s_{pq}(w)\| = \left\| \left(\sum_{j,G,m} 2^{jsq} \, w(2^{-j}m)^q \, |\lambda_m^{j,G} \chi_{jm}(\cdot)|^q \right)^{1/q} |L_p(\mathbb{R}^n) \right\| \tag{6.28}$$

(*with the usual modification if $p = \infty$ and/or $q = \infty$*)*. Then*

$$b^s_{pq}(w) = \left\{ \lambda : \ \|\lambda \,|b^s_{pq}(w)\| < \infty \right\} \tag{6.29}$$

and

$$f^s_{pq}(w) = \left\{ \lambda : \ \|\lambda \,|f^s_{pq}(w)\| < \infty \right\}. \tag{6.30}$$

Remark 6.12. If $w = 1$ then the above definition coincides with Definition 3.1. In (6.28) the index set (j, G, m) is the same as in (6.27). Obviously, $b^s_{pq}(w)$ and $f^s_{pq}(w)$ are quasi-Banach spaces. Furthermore,

$$\|\lambda \,|f^s_{pq}(w)\| \sim \left\| \left(\sum_{j,G,m} 2^{jsq} \, |\lambda_m^{j,G} \, \chi_{jm}(\cdot)|^q \right)^{1/q} |L_p(\mathbb{R}^n, w) \right\| \tag{6.31}$$

as an immediate consequence of (6.28), (6.2).

One of the main instruments to extend Theorem 3.5 to weighted spaces is the following *localisation principle* for F-spaces.

Proposition 6.13. *Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$ and $w \in W^n$. Let ϱ be a non-negative compactly supported C^∞ function in $\mathbb{R}^n$ such that*

$$\sum_{m \in \mathbb{Z}^n} \varrho_m(x) \sim 1, \quad x \in \mathbb{R}^n, \qquad \textit{where} \quad \varrho_m(x) = \varrho(x-m). \tag{6.32}$$

Then $f \in F^s_{pq}(\mathbb{R}^n, w)$ if, and only if, $\varrho_m f \in F^s_{pq}(\mathbb{R}^n)$ for any $m \in \mathbb{Z}^n$ and the right-hand side of the equivalence

$$\|f \,|F^s_{pq}(\mathbb{R}^n, w)\|^p \sim \sum_{m \in \mathbb{Z}^n} w^p(m) \, \|\varrho_m f \,|F^s_{pq}(\mathbb{R}^n)\|^p \tag{6.33}$$

is finite.

Proof. The unweighted case

$$\|f \,|F^s_{pq}(\mathbb{R}^n)\|^p \sim \sum_{m\in\mathbb{Z}^n} \|\varrho_m f \,|F^s_{pq}(\mathbb{R}^n)\|^p \tag{6.34}$$

follows from [Triγ], Theorem 2.4.7, pp. 124/125, and pointwise multiplier assertions of type

$$\|gf \,|F^s_{pq}(\mathbb{R}^n)\| \le c_l \sup_{|\alpha|\le l, x\in\mathbb{R}^n} |D^\alpha g(x)| \cdot \|f \,|F^s_{pq}(\mathbb{R}^n)\| \tag{6.35}$$

where

$$\mathbb{N} \ni l > \max(s, \sigma_p - s). \tag{6.36}$$

The latter assertion follows from [Triγ], Section 4.2.2. But now (6.33) is a consequence of (6.11), (6.34), and an application of (6.35) to w and w^{-1} with (6.1), (6.2). □

Remark 6.14. The above arguments can also be applied to

$$\mathcal{C}^s(\mathbb{R}^n, w) = B^s_{\infty,\infty}(\mathbb{R}^n, w) \qquad \text{where } s \in \mathbb{R}. \tag{6.37}$$

Then one gets

$$\|f \,|\mathcal{C}^s(\mathbb{R}^n, w)\| \sim \sup_{m\in\mathbb{Z}^n} w(m)\, \|\varrho_m f \,|\mathcal{C}^s(\mathbb{R}^n)\|. \tag{6.38}$$

The counterpart of (3.33) is now given by, say,

$$\sum_{j,G,m} \lambda^{j,G}_m \, 2^{-jn/2}\, \Psi^{j,G}_m, \qquad \lambda \in f^s_{pq}(w). \tag{6.39}$$

As there one has again *unconditional convergence* in $S'(\mathbb{R}^n)$, which justifies the abbreviation (3.34). This follows from the unweighted case and (6.27). It can be extended to the B-spaces (by elementary embeddings). If $p < \infty$, $q < \infty$, in (6.39) and its b-counterpart, then the corresponding series converge unconditionally in $F^s_{pq}(\mathbb{R}^n, w)$ or $B^s_{pq}(\mathbb{R}^n, w)$. Otherwise one has as in Theorem 3.5 at least local convergence, say, in $B^s_{pq}(K)$ for any ball K and $\sigma < s$. But this can now conveniently be reformulated as unconditional convergence in $B^\sigma_{pq}(\mathbb{R}^n, \tilde{w})$ with $\sigma < s$ and $\tilde{w} \in W^n$ such that $\tilde{w}(x)\, w^{-1}(x) \to 0$ if $|x| \to \infty$. (The latter is only needed if $p = \infty$, hence for the spaces $B^s_{\infty,q}(\mathbb{R}^n, w)$, in particular $\mathcal{C}^s(\mathbb{R}^n, w)$ according to (6.37)). This can also be obtained as a by-product of the proof of the following theorem. We add a comment about this question in the remark after the proof.

Theorem 6.15. *Let $w \in W^n$ as in Definition* 6.1 *and let $\Psi^{j,G}_m$ be the k-wavelets according to* (6.25) *and as specified in* (3.1)–(3.5).

(i) *Let $0 < p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$ and*

$$\max\left(s, \frac{2n}{p} + \frac{n}{2} - s\right) < k \in \mathbb{N}. \tag{6.40}$$

Let $f \in S'(\mathbb{R}^n)$. *Then* $f \in B^s_{pq}(\mathbb{R}^n, w)$ *if, and only if, it can be represented as*

$$f = \sum_{j,G,m} \lambda^{j,G}_m \, 2^{-jn/2} \, \Psi^{j,G}_m, \qquad \lambda \in b^s_{pq}(w), \tag{6.41}$$

unconditional convergence being in $S'(\mathbb{R}^n)$ *and in any space* $B^\sigma_{pq}(\mathbb{R}^n, \tilde{w})$ *with* $\sigma < s$ *and* $\tilde{w}(x)\, w^{-1}(x) \to 0$ *if* $|x| \to \infty$. *Furthermore, the representation* (6.41) *is unique,*

$$\lambda^{j,G}_m = 2^{jn/2} \, \left(f, \Psi^{j,G}_m\right), \tag{6.42}$$

and

$$I: \quad f \to \left\{ 2^{jn/2} \, \left(f, \Psi^{j,G}_m\right) \right\} \tag{6.43}$$

is an isomorphic map of $B^s_{pq}(\mathbb{R}^n, w)$ **onto** $b^s_{pq}(w)$. *If, in addition,* $p < \infty$, $q < \infty$, *then* $\left\{\Psi^{j,G}_m\right\}$ *is an unconditional basis in* $B^s_{pq}(\mathbb{R}^n, w)$.

(ii) *Let* $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$ *and*

$$\max\left(s, \frac{2n}{\min(p,q)} + \frac{n}{2} - s\right) < k \in \mathbb{N}. \tag{6.44}$$

Let $f \in S'(\mathbb{R}^n)$. *Then* $f \in F^s_{pq}(\mathbb{R}^n, w)$ *if, and only if, it can be represented as*

$$f = \sum_{j,G,m} \lambda^{j,G}_m \, 2^{-jn/2} \, \Psi^{j,G}_m, \qquad \lambda \in f^s_{pq}(w), \tag{6.45}$$

unconditional convergence being in $S'(\mathbb{R}^n)$ *and in any space* $F^\sigma_{pq}(\mathbb{R}^n, \tilde{w})$ *with* $\sigma < s$ *and* $\tilde{w}(x)\, w^{-1}(x) \to 0$ *if* $|x| \to \infty$. *Furthermore, the representation* (6.45) *is unique with* (6.42), *and* I *in* (6.43) *is an isomorphic map of* $F^s_{pq}(\mathbb{R}^n, w)$ **onto** $f^s_{pq}(w)$. *If, in addition,* $q < \infty$ *then* $\left\{\Psi^{j,G}_m\right\}$ *is an unconditional basis in* $F^s_{pq}(\mathbb{R}^n, w)$.

Proof. Step 1. Let

$$f = \sum_{j,G,m} \lambda^{j,G}_m \, 2^{-jn/2} \, \Psi^{j,G}_m = \sum_{l \in \mathbb{Z}^n} f_l, \qquad \lambda \in f^s_{pq}(w), \tag{6.46}$$

where f_l are partial sums with

$$\operatorname{supp} f_l \subset \{y : \ |y - l| \le a\} \tag{6.47}$$

for some (sufficiently large) $a > 0$. In particular one obtains by Theorem 3.5 that $f_l \in F^s_{pq}(\mathbb{R}^n)$. Using the pointwise multiplier assertion (6.35) and Proposition 6.13 one gets $f \in F^s_{pq}(\mathbb{R}^n, w)$ and

$$\|f \,|F^s_{pq}(\mathbb{R}^n, w)\| \le c \, \|\lambda \,|f^s_{pq}(w)\| \tag{6.48}$$

for some $c > 0$ which is independent of λ.

Step 2. We prove the converse of (6.48). Let $C > 0$. We apply Proposition 6.13 to $f \in F^s_{pq}(\mathbb{R}^n, w)$ where we may assume that $\varrho(x) = 1$ for $|x| \le C$. If C is large then one finds for any $j \in \mathbb{N}_0$, $G \in G^j$ and $m \in \mathbb{Z}^n$ at least one $l \in \mathbb{Z}^n$ such that

$$\left(f, \Psi^{j,G}_m\right) = \left(\varrho_l f, \Psi^{j,G}_m\right). \tag{6.49}$$

Then the converse of (6.48) follows from Proposition 6.13 and Theorem 3.5. The remaining assertions are the same as in the unweighted case. In particular we have (6.42) and the isomorphic map (6.43) of $F^s_{pq}(\mathbb{R}^n, w)$ onto $f^s_{pq}(w)$. As for convergence we refer to the above comments complemented below in Remark 6.16. If $q < \infty$ then we have convergence in $F^s_{pq}(\mathbb{R}^n, w)$ and $\left\{\Psi^{j,G}_m\right\}$ is an unconditional basis. This proves part (ii).

Step 3. We prove part (i). By (6.37), (6.38) the above arguments can be extended to $\mathcal{C}^s(\mathbb{R}^n, w) = B^s_{\infty\infty}(\mathbb{R}^n, w)$ which gives the possibility to incorporate now $p = \infty$. Otherwise we rely on interpolation applied to $B^s_{pp}(\mathbb{R}^n, w) = F^s_{pp}(\mathbb{R}^n, w)$ (if $p < \infty$). According to [Triβ], Theorem 2.4.2, p. 64, we have for $0 < p \le \infty$, $0 < q \le \infty$, $0 < \theta < 1$,

$$-\infty < s_0 < s_1 < \infty \qquad \text{and} \qquad s = (1-\theta)s_0 + \theta s_1, \tag{6.50}$$

the real interpolation

$$\left(B^{s_0}_{pp}(\mathbb{R}^n), B^{s_1}_{pp}(\mathbb{R}^n)\right)_{\theta,q} = B^s_{pq}(\mathbb{R}^n). \tag{6.51}$$

Choosing k in Theorem 3.5 sufficiently large, then we can apply the isomorphic map in (3.38) to all spaces in (6.51) and one gets the sequence counterpart of (6.51),

$$\left(b^{s_0}_{pp}, b^{s_1}_{pp}\right)_{\theta,q} = b^s_{pq}. \tag{6.52}$$

Using the isomorphic map in Theorem 6.5 and its obvious sequence counterpart one obtains

$$\left(B^{s_0}_{pp}(\mathbb{R}^n, w), B^{s_1}_{pp}(\mathbb{R}^n, w)\right)_{\theta,q} = B^s_{pq}(\mathbb{R}^n, w) \tag{6.53}$$

and

$$\left(b^{s_0}_{pp}(w), b^{s_1}_{pp}(w)\right)_{\theta,q} = b^s_{pq}(w). \tag{6.54}$$

Recall that $b^s_{pp}(w) = f^s_{pp}(w)$. Now part (i) of the theorem follows from part (ii) and the indicated interpolations. In particular if $p < \infty$, $q < \infty$ then $\left\{\Psi^{j,G}_m\right\}$ is an unconditional basis in $B^s_{pq}(\mathbb{R}^n, w)$. □

Remark 6.16. We return briefly to the question of convergence. If $p < \infty$, $q < \infty$, then the series in (6.41), (6.45) converge unconditionally in the corresponding spaces. This follows from the isomorphisms and the same properties of the related sequence spaces. Let $p < \infty$, $q = \infty$. Then it follows from, say, (6.41) that we have unconditional convergence in $B^\sigma_{pq}(\mathbb{R}^n, w)$ with $\sigma < s$ and that all partial sums are uniformly bounded in $B^s_{pq}(\mathbb{R}^n, w)$. But this is sufficient to ensure that $f \in B^s_{pq}(\mathbb{R}^n, w)$ as a consequence of (6.10) and the *Fatou property* for unweighted spaces as it may be found in [Triϵ], p. 360. Similarly for $F^s_{p\infty}(\mathbb{R}^n, w)$. In case of $B^s_{\infty q}(\mathbb{R}^n, w)$ one has to compensate in addition the behavior at infinity which can be done, as indicated, by introducing a second smaller weight $\tilde{w}$ with $\tilde{w}(x)\, w^{-1}(x) \to 0$ if $|x| \to \infty$.

Remark 6.17. We followed essentially [HaT05]. Another proof for

$$B^s_{pq}(\mathbb{R}^n, w) \quad \text{with} \quad s > 0, \quad 1 \le p \le \infty, \quad 1 \le q \le \infty,$$

was given in [KLSS03b] extending wavelet bases for the spaces in (1.155) to their weighted counterparts. References for wavelets in unweighted spaces $B^s_{pq}(\mathbb{R}^n)$, $F^s_{pq}(\mathbb{R}^n)$ and in some of their modifications have been given in Remarks 1.63, 1.66, 3.7, 5.25. We always prefer inhomogeneous wavelets. For unweighted (isotropic) spaces one can develop a corresponding theory for homogeneous spaces largely parallel to inhomogeneous wavelets. One may consult the above references. But in case of weighted spaces it apparently makes a big difference whether one deals with inhomogeneous wavelet bases of Daubechies type or Meyer type or with related homogeneous wavelet bases. In [Lem94] the question is treated under which conditions for a positive Borel measure μ on $\mathbb{R}$ the homogeneous Daubechies wavelets are an unconditional Schauder basis in $L_p(\mathbb{R}, \mu)$ with $1 < p < \infty$. It comes out that this is the case if, and only if, $\mu = v\, \mu_L$ where μ_L is the Lebesgue measure and v belongs to the Muckenhoupt class $\mathcal{A}_p$, which restricts the growth of $v(x)$ if $|x| \to \infty$. By Theorem 6.15 the situation is different if one deals with inhomogeneous Daubechies wavelets (as far as the necessity is concerned). An extension of [Lem94] to $\mathbb{R}^n$ and to more general, not necessarily compactly supported, homogeneous wavelet bases in $L_p(\mathbb{R}^n, \mu)$ with $1 < p < \infty$, has been given in [ABM03]. There one finds further references. Matrix-valued generalisations of weighted homogeneous Besov spaces

$$\dot{B}^s_{pq}(\mathbb{R}^n, w), \qquad s \in \mathbb{R}, \quad 1 \le p < \infty, \quad 0 < q \le \infty, \tag{6.55}$$

have been considered in [Rou02], where again w is a Muckenhoupt weight (but also some more general weights are treated). There are obtained isomorphic maps of type (6.43) in terms of homogeneous wavelet bases of Daubechies type and Meyer type. We refer in particular to [Rou02], Corollary 10.3, p. 309. Most recent results may be found in [HaPi05], [Pio06] including interesting connections to fractals.

6.3 A digression: Sequence spaces

6.3.1 Basic spaces

By Theorem 6.15 problems of continuous and compact embeddings between weighted function spaces can be reduced to corresponding assertions between sequence spaces. This Section 6.3 might be considered as a preparation for what follows in Section 6.4. But we hope that the assertions are also of some self-contained interest. In some sense we continue our studies from [Triδ], Chapter II. Recall our use of $\sim$ indicating equivalence according to Remark 1.98.

Definition 6.18. *Let $d > 0$, $\delta \geq 0$, $0 < p \leq \infty$, and $0 < q \leq \infty$. Let*

$$M_j \in \mathbb{N} \quad \textit{such that} \quad M_j \sim 2^{jd} \quad \textit{where} \quad j \in \mathbb{N}_0, \tag{6.56}$$

$$\lambda = \{\lambda_{jr} \in \mathbb{C} : \ j \in \mathbb{N}_0; \ r = 1, \ldots, M_j\}, \tag{6.57}$$

and

$$\left\| \lambda \,| \ell_q \left(2^{j\delta} \ell_p^{M_j} \right) \right\| = \left(\sum_{j=0}^{\infty} 2^{j\delta q} \left(\sum_{r=1}^{M_j} |\lambda_{jr}|^p \right)^{q/p} \right)^{1/q} \tag{6.58}$$

(with the usual modification if $p = \infty$ and/or $q = \infty$). Then

$$\ell_q \left(2^{j\delta} \ell_p^{M_j} \right) = \left\{ \lambda : \ \left\| \lambda \,| \ell_q \left(2^{j\delta} \ell_p^{M_j} \right) \right\| < \infty \right\}. \tag{6.59}$$

Remark 6.19. In case of $\delta = 0$ we write $\ell_q(\ell_p^{M_j})$. It is quite obvious that $\ell_q(2^{j\delta}\ell_p^{M_j})$ is a quasi-Banach space (Banach space if $p \geq 1$, $q \geq 1$). These spaces have been introduced in [Triδ], Section 8. They played in [Triδ] and also in [Triϵ] a decisive role in connection with the spectral theory of fractal elliptic operators. We are interested in compact embeddings between these spaces expressed in terms of entropy numbers. Entropy numbers $e_k(T)$ for continuous embeddings T between quasi-Banach spaces have been introduced in Definitions 1.87 and 4.43.

Theorem 6.20. *Let $d > 0$, $\delta > 0$, and $M_j \in \mathbb{N}$ according to* (6.56)*. Let $0 < p_1 \leq \infty$,*

$$\frac{1}{p_*} = \frac{1}{p_1} + \frac{\delta}{d}, \tag{6.60}$$

$p_* < p_2 \leq \infty$, $0 < q_1 \leq \infty$, $0 < q_2 \leq \infty$. Then

$$\mathrm{id} : \quad \ell_{q_1} \left(2^{j\delta} \ell_{p_1}^{M_j} \right) \hookrightarrow \ell_{q_2} \left(\ell_{p_2}^{M_j} \right) \tag{6.61}$$

is compact and

$$e_k(\mathrm{id}) \sim k^{-\frac{\delta}{d} + \frac{1}{p_2} - \frac{1}{p_1}}, \qquad k \in \mathbb{N}. \tag{6.62}$$

Proof. *Step* 1. For $p_2 \geq p_1$ a proof may be found in [Triδ], Theorem 8.2, pp. 39–41.

Step 2. Let $p_* < p_2 < p_1$. We decompose id as

$$\mathrm{id} = \mathrm{id}_2 \circ \mathrm{id}_1 \tag{6.63}$$

with

$$\mathrm{id}_1 : \quad \ell_{q_1} \left(2^{j\delta} \ell_{p_1}^{M_j} \right) \hookrightarrow \ell_{q_2} \left(2^{j\tilde{\delta}} \ell_{p_1}^{M_j} \right), \qquad \tilde{\delta} = d \left(\frac{1}{p_2} - \frac{1}{p_1} \right), \tag{6.64}$$

and

$$\mathrm{id}_2 : \quad \ell_{q_2} \left(2^{j\tilde{\delta}} \ell_{p_1}^{M_j} \right) \hookrightarrow \ell_{q_2} \left(\ell_{p_2}^{M_j} \right). \tag{6.65}$$

By Hölder's inequality, based on $\frac{1}{p_2} = \frac{1}{p_1} + (\frac{1}{p_2} - \frac{1}{p_1})$, it follows that id_2 is a linear and bounded map. Then one gets by Step 1 and Proposition 1.89,

$$e_k(\mathrm{id}) \le c\, e_k(\mathrm{id}_1) \le c'\, k^{-\frac{1}{d}(\delta - \tilde{\delta})} = c'\, k^{-\frac{\delta}{d} + \frac{1}{p_2} - \frac{1}{p_1}} \tag{6.66}$$

where c and $c'$are independent of $k \in \mathbb{N}$.

Step 3. It remains to prove that the inequalities in (6.66) are equivalences, again under the hypothesis $p_* < p_2 < p_1$. Let, in addition, $p_1 < \infty$. We apply Proposition 1.91(i) to

$$A = \ell_{q_1}\left(2^{j\delta} \ell_{p_1}^{M_j}\right), \quad B_0 = \ell_{q_2}\left(\ell_{p_2}^{M_j}\right), \quad B_1 = \ell_{q_2}\left(\ell_{p_3}^{M_j}\right), \tag{6.67}$$

with $p_2 < p_1 < p_3$. Then Hölder's inequality leads to (1.288) with

$$B_\theta = \ell_{q_2}\left(\ell_{p_1}^{M_j}\right) \qquad \text{where} \quad \frac{1}{p_1} = \frac{1-\theta}{p_2} + \frac{\theta}{p_3}. \tag{6.68}$$

We assume now that for given d, δ, p_1, q_1 (with $p_1 < \infty$) the converse of (6.66) is wrong for some p_2 (with $p_* < p_2 < p_1$) and q_2. Then there is a sequence $k_j \in \mathbb{N}$ with $k_j \to \infty$ if $j \to \infty$ and

$$e_{k_j}(\mathrm{id} : A \hookrightarrow B_0) \cdot k_j^{\frac{\delta}{d} - \frac{1}{p_2} + \frac{1}{p_1}} \to 0 \quad \text{if} \quad j \to \infty. \tag{6.69}$$

Since $(1-\theta)(\frac{1}{p_2} - \frac{1}{p_1}) + \theta(\frac{1}{p_3} - \frac{1}{p_1}) = 0$ it follows by (1.289), (6.69), and the corresponding estimate from above for $A \hookrightarrow B_1$ that

$$e_{2k_j}(\mathrm{id} :\; A \hookrightarrow B_\theta) \cdot k_j^{\delta/d} \to 0 \quad \text{if} \quad j \to \infty. \tag{6.70}$$

But this contradicts (6.62) with $p_2 = p_1$ which is covered by Step 1. If $p_1 = \infty$ then the above arguments do not work. Since we wish to disprove the counterpart of (6.69) we may assume $q_1 < \infty$. Let $d/\delta = p_* < p_2 < p_1 = \infty$. We apply Proposition 1.91(ii) to

$$A_0 = \ell_{q_1}\left(2^{j\delta} \ell_\infty^{M_j}\right), \quad A_1 = \ell_{q_1}\left(2^{j\delta} \ell_{q_1\theta}^{M_j}\right), \quad B = \ell_{q_2}\left(\ell_{p_2}^{M_j}\right) \tag{6.71}$$

where $0 < \theta < 1$. Then $\mathrm{id} : A_1 \hookrightarrow B$ is also covered by the conditions of the theorem. We may assume $q_1\theta = 1$. By [Triα], Theorem 1.18.3/2, p. 127, we have

$$\left(\ell_\infty^M, \ell_1^M\right)_{\theta, q_1} = \ell_{q_1}^M, \qquad q_1\theta = 1, \tag{6.72}$$

and, hence,

$$A = (A_0, A_1)_{\theta, q_1} = \ell_{q_1}\left(2^{j\delta} \ell_{q_1}^{M_j}\right). \tag{6.73}$$

By (1.290) one can now argue as above and disprove the counterpart of (6.69), reducing it to the previous case. □

Remark 6.21. The above theorem extends $p_2 \ge p_1$ in [Triδ], Theorem 8.2, p. 39, to the more natural restriction $p_* < p_2 \le \infty$. We relied on [HaT05]. But this assertion follows also from the more general Theorems 3 and 4 in [Leo00a].

6.3.2 Modifications

We modify the sequence spaces from Section 6.3.1 so that they are better adapted to our later needs.

Definition 6.22. *Let* $\delta \geq 0$, $\alpha \geq 0$, $0 < p \leq \infty$, *and* $0 < q \leq \infty$. *Let*

$$\lambda = \{\lambda_{jr} \in \mathbb{C} : \ j \in \mathbb{N}_0; \ r \in \mathbb{N}\} \tag{6.74}$$

and

$$\left\| \lambda \,|\ell_q \left(2^{j\delta} \ell_p(\alpha)\right)\right\| = \left(\sum_{j=0}^{\infty} 2^{j\delta q} \left(\sum_{r=1}^{\infty} r^{\alpha p} \, |\lambda_{jr}|^p \right)^{q/p} \right)^{1/q} \tag{6.75}$$

(*with the usual modification if* $p = \infty$ *and/or* $q = \infty$). *Then*

$$\ell_q \left(2^{j\delta} \ell_p(\alpha)\right) = \left\{ \lambda : \ \left\| \lambda \,|\ell_q \left(2^{j\delta} \ell_p(\alpha)\right)\right\| < \infty \right\}. \tag{6.76}$$

Remark 6.23. In case of $\delta = \alpha = 0$ we write $\ell_q(\ell_p)$. It is quite obvious that $\ell_q(2^{j\delta}\ell_p(\alpha))$ is a quasi-Banach space (Banach space if $p \geq 1$, $q \geq 1$).

Theorem 6.24. *Let* $\delta > 0$ *and* $\alpha > 0$. *Let* $0 < p_1 \leq \infty$,

$$\frac{1}{p_*} = \frac{1}{p_1} + \alpha, \tag{6.77}$$

$p_* < p_2 \leq \infty$, $0 < q_1 \leq \infty$ *and* $0 < q_2 \leq \infty$. *Then*

$$\mathrm{id} : \quad \ell_{q_1} \left(2^{j\delta} \ell_{p_1}(\alpha)\right) \hookrightarrow \ell_{q_2}(\ell_{p_2}) \tag{6.78}$$

is compact and

$$e_k(\mathrm{id}) \sim k^{-\alpha + \frac{1}{p_2} - \frac{1}{p_1}}, \qquad k \in \mathbb{N}. \tag{6.79}$$

Proof. Step 1. Let $\alpha > 0$, $\delta > 0$, and $0 < p = q < \infty$ in (6.75). Then

$$\begin{aligned} \sum_{j=0}^{\infty} 2^{j\delta p} \sum_{r=1}^{\infty} r^{\alpha p} |\lambda_{jr}|^p &\sim \sum_{j,l=0}^{\infty} 2^{(j+l)\delta p} \sum_{r \in K_l} |\lambda_{jr}|^p \\ &\sim \sum_{j=0}^{\infty} 2^{j\delta p} \sum_{l=0}^{j} \sum_{r \in K_l} |\lambda_{j-l,r}|^p, \end{aligned} \tag{6.80}$$

where $r \in K_l$ means $r \sim 2^{l\delta/\alpha}$. Hence,

$$\ell_p \left(2^{j\delta} \ell_p(\alpha)\right) \cong \ell_p \left(2^{j\delta} \ell_p^{M_j}\right) \quad \text{with} \quad M_j \sim 2^{jd}, \quad d = \delta/\alpha, \tag{6.81}$$

in the interpretation of (6.80) and Definition 6.18. Here $\cong$ means isomorphic quasi-Banach spaces. Modifying (6.80) appropriately one gets (6.81) also for $p = q = \infty$. Obviously, (6.77) coincides with (6.60). Now (6.79) with $q_1 = p_1$ and $q_2 = p_2$ follows from (6.62).

Step 2. Since (6.79) does not depend on $\delta > 0$ one can replace $q_1 = p_1$ by any couple (p_1, q_1) with $0 < q_1 \le \infty$ (estimates from above and from below at the expense of δ). The same argument applies to the target side, since (6.78) with the outcome (6.79) can be generalised by

$$\mathrm{id}: \quad \ell_{q_1}\left(2^{j\delta_1}\,\ell_{p_1}(\alpha)\right) \hookrightarrow \ell_{q_2}\left(2^{j\delta_2}\ell_{p_2}\right) \tag{6.82}$$

with $\delta_1 > \delta_2$. □

The next modification is motivated by the spaces $b^s_{pq}(w)$ in Definition 6.11 for the special weight $w(x) = (1+|x|^2)^{\alpha/2}$ with $\alpha \ge 0$.

Definition 6.25. *Let $n \in \mathbb{N}$, $\delta \ge 0$, $\alpha \ge 0$, $0 < p \le \infty$ and $0 < q \le \infty$. Let*

$$\lambda = \{\lambda_{jm} \in \mathbb{C}: \; j \in \mathbb{N}_0; \; m \in \mathbb{Z}^n\} \tag{6.83}$$

and

$$\|\lambda\,|\ell_q\left(2^{j\delta}\ell_p(\alpha)\right)_n\| = \left(\sum_{j=0}^\infty 2^{j\delta q}\left(\sum_{m\in\mathbb{Z}^n}\left(1+2^{-j}|m|\right)^{\alpha p}|\lambda_{jm}|^p\right)^{q/p}\right)^{1/q} \tag{6.84}$$

(*with the usual modification if $p = \infty$ and/or $q = \infty$*). *Then*

$$\ell_q\left(2^{j\delta}\ell_p(\alpha)\right)_n = \left\{\lambda: \; \left\|\lambda\,|\ell_q\left(2^{j\delta}\ell_p(\alpha)\right)_n\right\| < \infty\right\}. \tag{6.85}$$

Remark 6.26. If $\delta = \alpha = 0$ then we write $\ell_q(\ell_p)_n$. It is quite obvious that $\ell_q(2^{j\delta}\ell_p(\alpha))_n$ are quasi-Banach spaces (Banach spaces if $p \ge 1$, $q \ge 1$).

Theorem 6.27. (i) *Let $\delta > 0$, $\alpha > 0$, $\delta \ne \alpha$, $0 < p_1 \le \infty$,*

$$\frac{1}{p_*} = \frac{1}{p_1} + \frac{\min(\alpha,\delta)}{n}, \tag{6.86}$$

$p_* < p_2 \le \infty$, $0 < q_1 \le \infty$ and $0 < q_2 \le \infty$. Then

$$\mathrm{id}: \quad \ell_{q_1}\left(2^{j\delta}\ell_{p_1}(\alpha)\right)_n \hookrightarrow \ell_{q_2}(\ell_{p_2})_n \tag{6.87}$$

is compact and

$$e_k(\mathrm{id}) \sim k^{-\frac{\min(\alpha,\delta)}{n}+\frac{1}{p_2}-\frac{1}{p_1}}, \qquad k \in \mathbb{N}. \tag{6.88}$$

Proof. Step 1. We modify the proof of Theorem 6.24. Let, as there, $\alpha > 0$, $\delta > 0$, and $0 < p = q < \infty$ in (6.84). Then

$$\begin{aligned}\sum_{j=0}^\infty 2^{j\delta p}\sum_{m\in\mathbb{Z}^n}(1+2^{-j}|m|)^{\alpha p}\,|\lambda_{jm}|^p &\sim \sum_{j=0}^\infty 2^{j\delta p}\sum_{l=0}^\infty 2^{l\delta p}\sum_{m\in K^j_l}|\lambda_{jm}|^p \\ &\sim \sum_{j=0}^\infty 2^{j\delta p}\sum_{l=0}^j\sum_{m\in K^{j-l}_l}|\lambda_{j-l,m}|^p,\end{aligned} \tag{6.89}$$

where K_l^j collects all terms with $1 + 2^{-j}|m| \sim 2^{l\delta/\alpha}$. The cardinal number of K_l^j is $\sim 2^{(j+l\delta/\alpha)n}$, and, hence the cardinal number of the two last sums on the right-hand side of (6.89) is

$$M_j \sim 2^{jn} \sum_{l=0}^{j} 2^{ln(\frac{\delta}{\alpha}-1)}. \tag{6.90}$$

If $p = q = \infty$ then one has to modify in the usual way.

Step 2. Let $\delta > \alpha$. Then

$$M_j \sim 2^{jd} \qquad \text{with} \quad d = \delta n/\alpha \quad \text{where} \quad j \in \mathbb{N}_0, \tag{6.91}$$

and we have an obvious counterpart of (6.81), where (6.86) coincides with (6.60). Now one gets (6.88) for all cases by the same arguments as in the two steps of the proof of Theorem 6.24.

Step 3. Let $\delta < \alpha$. Then it follows from (6.90) that $M_j \sim 2^{jn}$ and (6.86) coincides with (6.60) where now $d = n$. Then one obtains

$$e_k(\mathrm{id}) \sim k^{-\frac{\delta}{n}+\frac{1}{p_2}-\frac{1}{p_1}}, \qquad k \in \mathbb{N}, \tag{6.92}$$

for the special cases

$$\mathrm{id}: \quad \ell_{p_1}\left(2^{j\delta}\ell_{p_1}(\alpha)\right)_n \hookrightarrow \ell_{p_2}(\ell_{p_2})_n. \tag{6.93}$$

To extend (6.93), (6.92) to all embeddings (6.87) we use the interpolation

$$\left(\ell_u\left(2^{j\delta_1}\ell_{p_1}(\alpha)\right)_n, \ell_v\left(2^{j\delta_2}\ell_{p_1}(\alpha)\right)_n\right)_{\theta,q_1} = \ell_{q_1}\left(2^{j\delta}\ell_{p_1}(\alpha)\right)_n \tag{6.94}$$

with $\delta = (1-\theta)\delta_1 + \theta\delta_2$ where $\delta_1 < \delta_2 < \alpha$, $0 < \theta < 1$ and $u, v, q_1 \in (0, \infty]$. This follows from [BeL76], Theorem 5.6.1, p. 122, first extended to $A = \ell_p$ and then by isomorphic maps to $\ell_p(\alpha)$. Similarly for the target side. Now it follows from the interpolation properties for entropy numbers according to Proposition 1.91 that

$$e_k(\mathrm{id}) \le c\, k^{-\frac{\delta}{n}+\frac{1}{p_2}-\frac{1}{p_1}}, \qquad k \in \mathbb{N}, \tag{6.95}$$

for all id in (6.87) and $\delta < \alpha$. Having (6.95) for all id and (6.92) for the special embeddings in (6.93) one can now argue as in Step 3 of the proof of Theorem 6.20. For this purpose one assumes that one has an assertion of type (6.69) for some id with (6.87). Using (6.94) and again Proposition 1.91 one would get a corresponding assertion for some id with (6.93) in contradiction to (6.92). □

Remark 6.28. In Section 6.3 we followed essentially [HaT05]. All assertions in Theorems 6.20, 6.24, 6.27 rely on [Triδ], Theorem 8.2, pp. 39–41, which coincides with Theorem 6.20 restricted to $p_2 \ge p_1$. Afterwards we used only general means such as Hölder inequalities, interpolation properties for entropy numbers and elementary rearrangements. We avoided, so to say, to go inside the (mixed, weighted) ℓ_p-spaces as we did in [Triδ]. Nearest to the subject considered here are the series of papers [Leo00a], [Leo00b], [Leo00c], and, partly based on them, [KLSS03a], [KLSS03b], [KLSS04], [KLSS05a], [KLSS05b]. There is some overlap with the above theorems. This applies in particular to the spaces considered in Theorems 6.24 and 6.27. A proof of Theorem 6.24, restricted to $1 \leq p_1 < p_2 \leq \infty$, $1 \leq q_1 \leq \infty$, $1 \leq q_2 \leq \infty$, was given in [KLSS03a], Theorem 3, p. 262/263. A proof of Theorem 6.27, again restricted to $p_1 \geq 1$, $p_2 \geq 1$, $q_1 \geq 1$, $q_2 \geq 1$, may be found in [KLSS03b]. The restriction to the case of Banach spaces has been removed in the indicated subsequent papers by the same authors. We refer in particular to [KLSS05a], Corollary 2. But the main aim of the two series of papers mentioned above is not only to recover and to complement existing results, but to deal with more general sequence spaces, replacing $2^{j\delta}$ in (6.58), (6.75) by some sequences of positive numbers β_j and to modify r^α in (6.75) appropriately, everything now based on a new method. Furthermore, special attention is paid to the limiting case $\delta = \alpha$ in Theorem 6.27 which we excluded (but we return to this point later on). A description of these results is beyond the scope of this book. Some of these papers deal with generalised sequence spaces for their own sake. But mostly they prepare via wavelet isomorphisms corresponding assertions in weighted Besov spaces with generalised weights (as reflected by the titles). The close connection between (weighted and unweighted) Besov spaces and sequence spaces of the types as introduced in Definitions 6.18, 6.22, 6.25 has been known for some time (before and independently of the wavelet isomorphisms as described in Theorems 3.5 and 6.15). This might explain the great interest in dealing with mappings between these sequence spaces and especially with entropy numbers of compact embeddings. In addition to the above papers we refer to [Kuhn84], [Mar88], and more recently, [EdH99], [EdH00], [CoK01], [KuS01], [Bel02], [Kuhn03], [Kuhn05].

6.4 Entropy numbers

6.4.1 The main case

We are interested in entropy numbers $e_k(\mathrm{id})$ for compact embeddings between weighted spaces as introduced in Definition 6.3, say,

$$\mathrm{id}: \quad B^{s_1}_{p_1 q_1}(\mathbb{R}^n, w^1) \hookrightarrow B^{s_2}_{p_2 q_2}(\mathbb{R}^n, w^2). \tag{6.96}$$

As for entropy numbers and their properties we again refer to Section 1.10. In case of arbitrary bounded domains we have the satisfactory assertion in Theorem 1.97. First we collect what we know so far, simplify and specify what will be considered.

By elementary embeddings it follows from part (ii) of Theorem 6.7 that for (6.14) also the embedding (6.96) is compact if

$$s_1 - n/p_1 > s_2 - n/p_2 \qquad \text{and} \qquad \frac{w^2(x)}{w^1(x)} \to 0 \quad \text{as} \quad |x| \to \infty. \tag{6.97}$$

One may ask to what extent the conditions $s_2 < s_1$ and $p_1 \le p_2$ in (6.14) are necessary for the compactness assertion and whether q_1 and q_2 play a role now. Furthermore, without restriction of generality one may assume that $w^2 = 1$ in (6.96) which reduces the target spaces in (6.96) to the unweighted spaces $B^{s_2}_{p_2 q_2}(\mathbb{R}^n)$ according to Definition 2.1. Then one has

$$w(x) = w^1(x) \to \infty \quad \text{if} \quad |x| \to \infty \quad \text{in (6.97).}$$

This reduction follows from the isomorphisms in Theorem 6.5(ii). By the same property and (1.299) one has for $s \in \mathbb{R}$, $0 < p < \infty$ and $q, u, v \in (0, \infty]$ the continuous embedding

$$B^s_{pu}(\mathbb{R}^n, w) \hookrightarrow F^s_{pq}(\mathbb{R}^n, w) \hookrightarrow B^s_{pv}(\mathbb{R}^n, w) \tag{6.98}$$

if, and only if, $0 < u \le \min(p, q)$ and $\max(p, q) \le v \le \infty$. In other words, in those cases where the entropy numbers $e_k(\mathrm{id})$ for compact embeddings (6.96) do not depend on q_1 and q_2 one can immediately replace the B-spaces by the F-spaces. (This will always be the case with the exception of limiting situations,) We specify the above weight w by w_α,

$$w_\alpha(x) = (1 + |x|^2)^{\alpha/2} \qquad \text{with} \quad \alpha \in \mathbb{R}, \tag{6.99}$$

hence we are mainly interested in compact embeddings of type

$$\mathrm{id}: \quad B^{s_1}_{p_1 q_1}(\mathbb{R}^n, w_\alpha) \hookrightarrow B^{s_2}_{p_2 q_2}(\mathbb{R}^n). \tag{6.100}$$

Obviously, $w_\alpha \in W^n$ according to Definition 6.1. If $k \in \mathbb{N}$ is large enough then one can apply Theorems 3.5(i) and 6.15(i) to both spaces in (6.100) with the same wavelet isomorphisms. Then the entropy numbers of the compact embedding (6.100) are equivalent to the entropy numbers of the compact embedding

$$\mathrm{id}: \quad b^{s_1}_{p_1 q_1}(w_\alpha) \hookrightarrow b^{s_2}_{p_2 q_2} \tag{6.101}$$

(including that id in (6.100) is compact if, and only if, id in (6.101) is compact). Here b^s_{pq} and $b^s_{pq}(w_\alpha)$ are the sequence spaces according to Definitions 3.1 and 6.11 with $w = w_\alpha$. For the question of whether id in (6.101) is compact and how the entropy numbers are distributed (up to equivalences) the summation over $G \in G^j$ in (3.7) and (6.27) is immaterial. Then we arrive at the spaces in Definition 6.25. In particular, the question of whether id in (6.100) is compact and how the entropy numbers are distributed is equivalent to the corresponding questions for

$$\mathrm{id}: \quad \ell_{q_1}\left(2^{j\delta}\ell_{p_1}(\alpha)\right)_n \hookrightarrow \ell_{q_2}(\ell_{p_2})_n \tag{6.102}$$

with

$$\delta = s_1 - \frac{n}{p_1} - \left(s_2 - \frac{n}{p_2}\right). \tag{6.103}$$

Although the conditions under which id in (6.100) is compact are largely known and covered by [HaT94a] and [ET96], Section 4.3.2, we formulate them again and give a short proof reducing everything to (6.102).

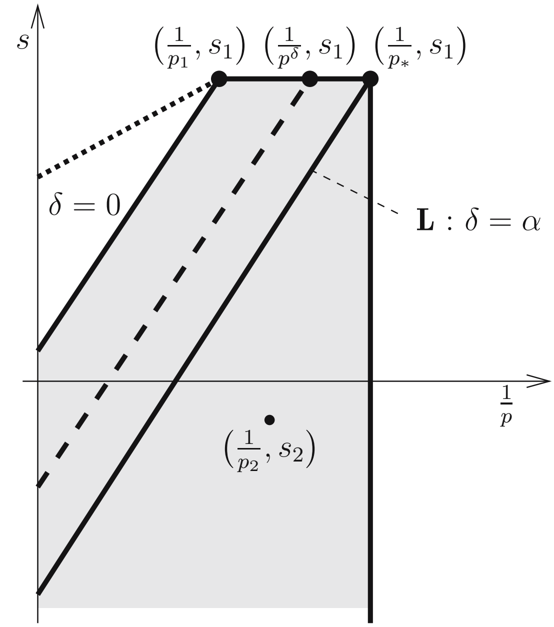

Figure 6.4.1

Proposition 6.29. *Let* $s_1, s_2, \alpha \in \mathbb{R}$ *and* $p_1, p_2, q_1, q_2 \in (0, \infty]$. *Let* δ *be given by* (6.103) *and*

$$\frac{1}{p_*} = \frac{1}{p_1} + \frac{\alpha}{n} . \tag{6.104}$$

Then id *in* (6.100) *is compact if, and only if,*

$$s_1 > s_2, \quad \delta > 0, \quad \alpha > 0, \quad p_2 > p_* , \tag{6.105}$$

(*the shaded area in Figure* 6.4.1).

Proof. The if-part is the subject of the next theorem (and well known by the above references). We prove the only-if-part. Continuous embedding in (6.102) means that there is a number $c > 0$ such that

$$\begin{aligned} &\left(\sum_{j=0}^{\infty} \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{jm}|^{p_2} \right)^{q_2/p_2} \right)^{1/q_2} \\ &\le c \left(\sum_{j=0}^{\infty} 2^{j\delta q_1} \left(\sum_{m \in \mathbb{Z}^n} (1 + |2^{-j} m|)^{\alpha p_1} |\lambda_{jm}|^{p_1} \right)^{q_1/p_1} \right)^{1/q_1} \end{aligned} \tag{6.106}$$

for all sequences $\{\lambda_{jm}\}$. The first two assertions in (6.105) are local. Let

$$\lambda_{jm} = a_j \, 2^{-jn/p_2} \quad \text{if} \quad |m| \le 2^j \quad \text{and zero otherwise.} \tag{6.107}$$

Then (6.106) means

$$\Big(\sum_{j=0}^{\infty} |a_j|^{q_2} \Big)^{1/q_2} \le c \, \Big(\sum_{j=0}^{\infty} 2^{j(s_1-s_2)q_1} |a_j|^{q_1} \Big)^{1/q_1} \tag{6.108}$$

where we used (6.103). But there is no compact embedding if $s_2 \ge s_1$. Similarly one obtains the second assertion in (6.105) inserting $\lambda_{j0} = a_j$ and $\lambda_{jm} = 0$ otherwise, in (6.106). The two remaining assertions in (6.105) are global. Inserting $\lambda_{0m} = \lambda_m$ and $\lambda_{jm} = 0$ otherwise in (6.106) it follows that there is no compact embedding if $\alpha \le 0$. As for the last assertion in (6.105) we insert

$$\lambda_{0m} = a_l \, 2^{-ln/p_2} \quad \text{if} \quad |m| \sim 2^l \quad \text{where} \quad l \in \mathbb{N}, \tag{6.109}$$

and $\lambda_{jm} = 0$ otherwise in (6.106). Then one gets

$$\Big(\sum_{l=0}^{\infty} |a_l|^{p_2} \Big)^{1/p_2} \le c \, \Big(\sum_{l=0}^{\infty} 2^{lp_1(\alpha - \frac{n}{p_2} + \frac{n}{p_1})} \, |a_l|^{p_1} \Big)^{1/p_1} . \tag{6.110}$$

There is no compact embedding if $p_2 \le p_*$. □

Remark 6.30. The simplicity of the above arguments depends on the specific nature of id in (6.102), (6.106). One may generalise $(1 + |2^{-j}m|)^\alpha$ by some positive weights $w_{j,m}$ and $2^{j\delta}$ by some positive numbers β_j. Then the question under which conditions the counterpart of (6.102) is compact is more complicated. A complete solution of this problem has been given in [Leo00a], [KLSS03b] and [KLSS05a, Theorem 1]. Specified to the situation considered here there is also an explicit formulation of the above proposition in [KLSS05a].

Theorem 6.31. *Let*

$$s_1 \in \mathbb{R}, \qquad 0 < p_1 \le \infty, \qquad 0 < q_1 \le \infty, \tag{6.111}$$

and let w_α be given by (6.99) *with $\alpha > 0$. Let*

$$\frac{1}{p_*} = \frac{1}{p_1} + \frac{\alpha}{n}, \tag{6.112}$$

$$-\infty < s_2 < s_1 < \infty, \quad p_* < p_2 \le \infty, \quad 0 < q_2 \le \infty \quad \textit{and} \quad \delta > 0, \tag{6.113}$$

where δ is given by (6.103) (*the shaded area in Figure* 6.4.1). *Then*

$$\text{id}: \quad B^{s_1}_{p_1 q_1}(\mathbb{R}^n, w_\alpha) \hookrightarrow B^{s_2}_{p_2 q_2}(\mathbb{R}^n) \tag{6.114}$$

is compact. Let $e_k(\mathrm{id})$ *be the corresponding entropy numbers. If* $\delta < \alpha$ *then*

$$e_k(\mathrm{id}) \sim k^{-\frac{s_1-s_2}{n}}, \qquad k \in \mathbb{N}. \tag{6.115}$$

If $\delta > \alpha$ *then*

$$e_k(\mathrm{id}) \sim k^{-\frac{\alpha}{n}+\frac{1}{p_2}-\frac{1}{p_1}}, \qquad k \in \mathbb{N}. \tag{6.116}$$

Proof. As mentioned above, id in (6.114) is compact if, and only if, id in (6.102) is compact, and the entropy numbers of these two operators are equivalent to each other. If $\delta > \alpha$ then (6.86) coincides with (6.112) and (6.88) coincides with (6.116). Let $0 < \delta < \alpha$ and let, temporarily, p^δ be the corresponding number on the left-hand side of (6.86), hence

$$\frac{1}{p^\delta} = \frac{1}{p_1} + \frac{\delta}{n} = \frac{s_1 - s_2}{n} + \frac{1}{p_2}. \tag{6.117}$$

We are on the broken line in Figure 6.4.1 with $(\frac{1}{p^\delta}, s_1)$ as an endpoint. In particular, $p_2 > p^\delta$ covers the whole line and one gets by (6.88) and (6.103) that

$$e_k(\mathrm{id}) \sim k^{-\frac{\delta}{n}+\frac{1}{p_2}-\frac{1}{p_1}} = k^{-\frac{s_1-s_2}{n}}, \qquad k \in \mathbb{N}. \tag{6.118}$$

□

Remark 6.32. Since both (6.115) and (6.116) are independent of q_1, q_2 it follows that one can replace B on the left-hand side of (6.114) by F if $p_1 < \infty$ and independently B on the right-hand side of (6.114) by F if $p_2 < \infty$. The theorem has a little history. It goes back to [HaT94a], which may also be found in [ET96], Section 4.3.2, where we proved (6.115) as stated but (6.116) only under the additional restriction $p_* < p_2 \le p_1$. If $p_2 > p_1$ then we obtained only the estimate

$$c\, k^{-\frac{\alpha}{n}+\frac{1}{p_2}-\frac{1}{p_1}} \le e_k(\mathrm{id}) \le c_\varepsilon\, k^{-\frac{\alpha}{n}+\frac{1}{p_2}-\frac{1}{p_1}} \, (\log k)^{\varepsilon+\frac{1}{p_1}-\frac{1}{p_2}} \tag{6.119}$$

for all $\varepsilon > 0$ and some $c > 0$, $c_\varepsilon > 0$. But it had been conjectured in [Har97b], Section 2.5, that one has (6.116) also if $p_2 > p_1$. A first affirmative answer was given in [KLSS03b], Theorem 7, under the restriction that p_1, p_2, q_1, q_2 are larger than or equal to 1. This restriction has been removed in [KLSS05a]. We followed here again [HaT05]. With exception of some rearrangements as outlined in Section 6.5.1 we restrict ourselves exclusively to the weights w_α according to (6.99). It is the main aim of [KLSS05a] to deal with more general weights, preferably of type

$$w(x) = (1 + |x|^2)^{\alpha/2}\, \psi(|x|), \qquad \alpha > 0, \tag{6.120}$$

where ψ is a perturbation. Then one gets again assertions of type (6.115), (6.116), but now the function ψ comes in. We refer for details to [KLSS05a]. If $w(x)$ tends to infinity if $|x| \to \infty$, but rather slowly, say as $(\log |x|)^\beta$ for some $\beta > 0$, then the situation is different. The breaking point $\alpha = \delta$ in Theorem 6.31 disappears and the entropy numbers of related compact embeddings depend only on $w(x)$. We refer for details to [KLSS05b].

6.4.2 The limiting case

So far we excluded in Theorem 6.31 the limiting case $\alpha = \delta$ which corresponds to the line L in Figure 6.4.1 Then the two exponents in (6.115), (6.116) coincide,

$$\frac{\alpha}{n} + \frac{1}{p_1} - \frac{1}{p_2} = \frac{\delta}{n} + \frac{1}{p_1} - \frac{1}{p_2} = \frac{s_1 - s_2}{n}. \tag{6.121}$$

But in general the entropy numbers $e_k(\mathrm{id})$ for id in (6.114) with $\alpha = \delta$ behave differently. For example, if

$$-\infty < s_2 < s_1 < \infty \qquad \text{and} \qquad \alpha = s_1 - s_2 \tag{6.122}$$

then

$$e_k\left(\mathrm{id} : \ \mathcal{C}^{s_1}(\mathbb{R}^n, w_\alpha) \hookrightarrow \mathcal{C}^{s_2}(\mathbb{R}^n)\right) \sim \left(\frac{k}{\log k}\right)^{-\frac{s_1 - s_2}{n}}, \quad 1 < k \in \mathbb{N}, \tag{6.123}$$

where $\mathcal{C}^s(\mathbb{R}^n)$ are the Hölder-Zygmund spaces according to (1.10) and

$$\mathcal{C}^s(\mathbb{R}^n, w) = B^s_{\infty\infty}(\mathbb{R}^n, w), \qquad s \in \mathbb{R}, \quad w \in W^n, \tag{6.124}$$

are their weighted counterparts. We refer to [HaT94a] and [ET96], p. 179. There one finds also further assertions both for B-spaces and F-spaces with $\alpha = \delta$. But they are mostly estimates. This tricky problem attracted some attention over the years. We formulate here a result as it came out quite recently.

Theorem 6.33. *Let s_1, p_1, q_1, w_α, with $\alpha > 0$, and p_* be as in* (6.111), (6.112). *Let s_2, p_2, q_2 be as in* (6.113), *now with*

$$\alpha = \delta = s_1 - \frac{n}{p_1} - \left(s_2 - \frac{n}{p_2}\right), \tag{6.125}$$

the line L in Figure 6.4.1. *Let*

$$\varrho = \frac{s_1 - s_2}{n} + \frac{1}{q_2} - \frac{1}{q_1}. \tag{6.126}$$

Then

$$\mathrm{id} : \quad B^{s_1}_{p_1 q_1}(\mathbb{R}^n, w_\alpha) \hookrightarrow B^{s_2}_{p_2 q_2}(\mathbb{R}^n) \tag{6.127}$$

is compact. If $\varrho > 0$ then

$$e_k(\mathrm{id}) \sim k^{-\frac{s_1 - s_2}{n}} (\log k)^{\varrho}, \qquad 1 < k \in \mathbb{N}. \tag{6.128}$$

If $\varrho < 0$ then

$$e_k(\mathrm{id}) \sim k^{-\frac{s_1 - s_2}{n}}, \qquad k \in \mathbb{N}. \tag{6.129}$$

Remark 6.34. The most surprising assertion is the genuine q-dependence of $e_k(\mathrm{id})$ in (6.128). This effect had been discovered in [Har95a], [Har97a], [Har97b] for some couples (p_1, p_2) and (q_1, q_2). The next step was done in [KLSS03b], Theorem 10, proving (6.128) with $\varrho > 0$ in case of Banach spaces and under some additional more handsome restrictions for the parameters involved. Taking these results as starting point we gave in [HaT05], Theorem 4.3, a proof of (6.128) with $\varrho > 0$ under the above conditions, relying essentially on some interpolation. Furthermore, in [HaT05], Conjecture 4.11, we formulated (6.129) with $\varrho < 0$. But we could not prove it. A complete new proof of the above theorem is due to [KLSS04], Theorem 2. There one finds also for the interesting limit-limit case $\alpha = \delta$, $\varrho = 0$,

$$k^{-\frac{s_1-s_2}{n}} \preceq e_k(\mathrm{id}) \preceq k^{-\frac{s_1-s_2}{n}} (\log \log k)^{1/q_1}, \qquad 4 \le k \in \mathbb{N}, \tag{6.130}$$

(log taken to base 2).

Remark 6.35. There are some assertions in [HaT94a], [ET96, Theorem 4.3.2] and [HaT05] for F-spaces in the limiting situation $\alpha = \delta > 0$. But they are less complete than for B-spaces. However according to [HaT94a], [ET96] one has for all parameters covered by Theorem 6.33 (now with $p_1 < \infty$, $p_2 < \infty$)

$$e_k \left(\mathrm{id} : \; F^{s_1}_{p_1 q_1}(\mathbb{R}^n, w_\alpha) \hookrightarrow F^{s_2}_{p_2 q_2}(\mathbb{R}^n) \right) \succeq k^{-\frac{s_1-s_2}{n}} (\log k)^{\alpha/n}, \tag{6.131}$$

$1 < k \in \mathbb{N}$. *One may conjecture that this estimate is an equivalence in all cases.* This is supported by (6.127), (6.128) with $p_1 = q_1$, $p_2 = q_2$, and, hence, $\varrho = \alpha/n$. Furthermore we quote the following special case from [HaT05], Corollary 4.7. Let $H^s_p(\mathbb{R}^n)$ with $1 < p < \infty$, $s \in \mathbb{R}$, be the Sobolev spaces as introduced in (1.7), (1.8) and let

$$H^s_p(\mathbb{R}^n, w) = F^s_{p,2}(\mathbb{R}^n, w), \quad w \in W^n, \quad 1 < p < \infty, \quad s \in \mathbb{R}, \tag{6.132}$$

be the weighted counterpart with the classical Sobolev spaces $W^m_p(\mathbb{R}^n, w)$ according to (6.21) as a special case. Let

$$-\infty < s_2 < s_1 < \infty, \quad 1 < p_1 < \infty, \quad \alpha > 0, \tag{6.133}$$

p_* as in (6.112), $\max(1, p_*) < p_2 \le p_1$ and (6.125) (limiting case). Then

$$e_k \left(\mathrm{id} : \; H^{s_1}_{p_1}(\mathbb{R}^n, w_\alpha) \hookrightarrow H^{s_2}_{p_2}(\mathbb{R}^n) \right) \sim k^{-\frac{s_1-s_2}{n}} (\log k)^{\alpha/n}, \tag{6.134}$$

$1 < k \in \mathbb{N}$.

Remark 6.36. Instead of entropy numbers one can measure the degree of compactness in terms of approximation numbers. What is meant by approximation numbers may be found in Definitions 1.87 and 4.43. We described in Theorem 1.107 what is known about approximation numbers of compact embeddings in (unweighted) function spaces in bounded Lipschitz domains. What can be said about approximation numbers for compact embeddings between weighted function spaces, again with id in (6.100) as the case of preference? The first decisive step was taken in [Har95b]. A description of these results may also be found in [ET96], Section 4.3.3. The outcome is now more complicated than in the case of entropy numbers. These results have been complemented in [Cae98] and quite recently in [Skr05].

6.5 Complements

6.5.1 The transference method

In connection with anisotropic spaces we described in Section 5.3 a method which allows us to transfer assertions for isotropic Besov spaces to corresponding anisotropic Besov spaces. This did not work equally well for the F-spaces. We reduced this problem to corresponding sequence spaces quasi-normed according to (5.114), (5.115) and applied Proposition 5.26. Now we are in an even better position since the transference argument works here both for B-spaces and F-spaces.

Proposition 6.37. *Let $w \in W^n$ and $\tilde{w} \in W^n$ according to Definition* 6.1 *such that for some permutation P of the points of $\mathbb{Z}^n$,*

$$\tilde{w}(Pm) \sim w(m), \qquad m \in \mathbb{Z}^n. \tag{6.135}$$

Let either $A = B$ or $A = F$ and let

$$\mathrm{id}_w : \quad A^{s_1}_{p_1 q_1}(\mathbb{R}^n, w) \hookrightarrow A^{s_2}_{p_2 q_2}(\mathbb{R}^n) \tag{6.136}$$

be compact. Then

$$\mathrm{id}_{\tilde{w}} : \quad A^{s_1}_{p_1 q_1}(\mathbb{R}^n, \tilde{w}) \hookrightarrow A^{s_2}_{p_2 q_2}(\mathbb{R}^n) \tag{6.137}$$

is also compact and

$$e_k(\mathrm{id}_w) \sim e_k(\mathrm{id}_{\tilde{w}}), \qquad k \in \mathbb{N}. \tag{6.138}$$

Proof. By Theorem 6.15 one can shift this problem to the corresponding sequence spaces. In case of the F-spaces, say $F^{s_1}_{p_1 q_1}(\mathbb{R}^n, w)$, we denote the integrand in (6.28) by $\lambda(x)$. Let Q_l be cubes with side-length 1 centred at $l \in \mathbb{Z}^n$ (and with sides parallel to the axes of coordinates) and let χ_l be the characteristic function of Q_l. Each Q_l is naturally divided into 2^{jn} subcubes of side-length 2^{-j}. We may assume that χ_{jm} in (6.28) are the characteristic functions of these subcubes. This minor technical modification is quite obvious, but is also a consequence of Proposition 1.33. Then

$$\|\lambda \,|f^{s_1}_{p_1 q_1}(\mathbb{R}^n, w)\|^p = \sum_{l \in \mathbb{Z}^n} \int_{\mathbb{R}^n} |\lambda(x)\chi_l(x)|^p \,\mathrm{d}x = \sum_{l \in \mathbb{Z}^n} \int_{Q_l} |\lambda(x)|^p \,\mathrm{d}x. \tag{6.139}$$

If $2^{-j}m \in Q_l$ then one can replace $w(2^{-j}m)$ by $w(l)$ and hence by $\tilde{w}(Pl)$. One transfers $\lambda(x)$ from Q_l to Q_{Pl}, and changes the integration over Q_l by an integration over Q_{Pl}. As for the sequence λ one gets a permutated sequence $P\lambda$ and a linear and isomorphic map

$$\|P\lambda \,|f^{s_1}_{p_1 q_1}(\tilde{w})\| \sim \|\lambda \,|f^{s_1}_{p_1 q_1}(w)\|. \tag{6.140}$$

In case of the B-spaces one has (6.27). One can argue in the same way as above for each level j. Now (6.136) can be transferred to the sequence side. In case of the unweighted spaces on the right-hand side of (6.136) one can apply any permutation such as P^{-1} with (6.135). This reduces $\mathrm{id}_{\tilde{w}}$ to id_w and vice versa. One obtains in particular (6.138). □

Remark 6.38. The replacement of quasi-norms in the above function spaces by equivalent ones does not influence assertions about embeddings (continuous or compact) and equivalences for entropy numbers. In particular by Remark 6.2 one may assume that the weights in Proposition 6.37 are step functions which are constant on the above cubes Q_l satisfying the compatibility condition (6.2) which means that these steps cannot oscillate too wildly. But otherwise one may assume that $w(x)$ on Q_l equals 2^{aj} for some (large) $a > 0$ and $J \le j \in \mathbb{N}$. In particular, one can apply the above proposition to two such step functions w and $\tilde{w}$, both satisfying (6.2), and for which the cardinal numbers of the sets

$$\big\{l \in \mathbb{Z}^n : \; w(x) = 2^{aj} \text{ on } Q_l\big\} \quad \text{and} \quad \big\{l \in \mathbb{Z}^n : \; \tilde{w}(x) = 2^{aj} \text{ on } Q_l\big\} \tag{6.141}$$

coincide, $a > 0$, $j \ge J$. In other words, for weight functions satisfying the compatibility condition (6.2) the distribution of the entropy numbers of embeddings of type (6.136) is governed by the decay properties of the decreasing rearrangement $(w^{-1})^*(t)$, $t \ge 0$, of $w^{-1}(x)$. As for rearrangement we refer to (1.210), (1.211) and Remark 1.76.

6.5.2 Radial spaces

Let $H^s_p(\mathbb{R}^n)$ with $s > 0$, $1 < p < \infty$, be the Sobolev space according to (1.7), (1.8), and let $RH^s_p(\mathbb{R}^n)$ be its subspace consisting of all radial symmetric functions, hence $f(x) = F(|x|)$. Let $n \ge 2$. Then functions $f \in RH^s_p(\mathbb{R}^n)$ show a specific behavior near the origin and at infinity. Furthermore, if $p < q \le \infty$ and $\frac{1}{q} > \frac{1}{p} - \frac{s}{n}$ then

$$\mathrm{id}: \quad RH^s_p(\mathbb{R}^n) \hookrightarrow L_q(\mathbb{R}^n), \tag{6.142}$$

is compact in sharp contrast to the corresponding behavior of the full space $H^s_p(\mathbb{R}^n)$. Observations of this type came up at the end of the 1970s and the beginning of the 1980s in connection with the study of radially symmetric solutions of some semi-linear elliptic equations. We refer to [Stra77], [Lio82], the references in this paper, and [KuP97], Chapter II, §8. Other papers reflecting the situation around 1980 may be found in the literature mentioned below in the Remarks 6.46, 6.47. In recent times questions of this type attracted some attention, especially in connection with radial symmetric subspaces of spaces of type $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$. We do not deal here systematically with these spaces. We give the necessary definitions, formulate some results, outline the proofs and give references. We wish to make clear how closely related these spaces are to the weighted spaces considered in this Chapter 6 and to the methods developed so far. We try to provide a better understanding of some remarkable effects.

As usual, $SO(n)$ stands for the group of all rotations in $\mathbb{R}^n$ around the origin (hence the group of all real orthogonal $n \times n$ matrices having determinant 1). In particular, $SO(n)$ acts transitively on the unit sphere $\{x : \ |x| = 1\}$. If $\varphi \in S(\mathbb{R}^n)$ and $g \in SO(n)$ then $\varphi \circ g$ means

$$(\varphi \circ g)(x) = \varphi(gx) \qquad \text{for all } x \in \mathbb{R}^n.$$

We say that $f \in S'(\mathbb{R}^n)$ is *radial* if

$$f\,(\varphi \circ g) = f(\varphi) \quad \text{for all } \varphi \in S(\mathbb{R}^n) \text{ and all } g \in SO(n). \tag{6.143}$$

Of course, if f is a continuous function, radial means $f(x) = F(|x|)$. Let $B(r)$ be the open ball centred at the origin and of radius $r > 0$.

Definition 6.39. *Let $n \geq 2$, let A be either B or F and let $s \in \mathbb{R}$, $0 < p \leq \infty$ ($p < \infty$ in the F-case), $0 < q \leq \infty$. Let $\alpha \in \mathbb{R}$ and $w_\alpha(x) = (1 + |x|^2)^{\alpha/2}$ as in* (6.99). *Then*

$$RA^s_{pq}(\mathbb{R}^n, w_\alpha) = \left\{ f \in A^s_{pq}(\mathbb{R}^n, w_\alpha),\ f \ \textit{radial} \right\}, \tag{6.144}$$

$$R^0 A^s_{pq}(\mathbb{R}^n) = \left\{ f \in RA^s_{pq}(\mathbb{R}^n, w_\alpha),\ \operatorname{supp} f \subset \overline{B(2)} \right\}, \tag{6.145}$$

$$R^\infty A^s_{pq}(\mathbb{R}^n, w_\alpha) = \left\{ f \in RA^s_{pq}(\mathbb{R}^n, w_\alpha),\ \operatorname{supp} f \subset \mathbb{R}^n \backslash B(1) \right\}. \tag{6.146}$$

Remark 6.40. These are subspaces of the spaces introduced in Definition 6.3 with respect to the rotationally invariant weights w_α. Of course, $R^0 A^s_{pq}(\mathbb{R}^n)$ is independent of α. They are quasi-Banach spaces with respect to the quasi-norm of $A^s_{pq}(\mathbb{R}^n, w_\alpha)$. The full radial space in (6.144) can be reduced to the two subspaces in (6.145), (6.146). It comes out that the behavior of these spaces near the origin and at infinity is somewhat different. We formulate some results and sketch the proofs. But we are mainly interested to demonstrate the interplay of diverse ingredients and techniques and to discuss the phenomena which may occur. Again we use the abbreviation (6.103) indicating now the dimension, hence

$$\delta_n = s_1 - \frac{n}{p_1} - \left(s_2 - \frac{n}{p_2} \right), \qquad n \in \mathbb{N}. \tag{6.147}$$

Theorem 6.41. *Let $n \geq 2$. Let $\alpha_1 \in \mathbb{R}$, $\alpha_2 \in \mathbb{R}$, and $p_1, p_2, q_1, q_2 \in (0, \infty]$ (with $p_1 < \infty$, $p_2 < \infty$ for the F-spaces). Let $s_1 \in \mathbb{R}$, $s_2 \in \mathbb{R}$ with*

$$s_1 > s_2 \quad \text{and} \quad \delta_1 = s_1 - \frac{1}{p_1} - \left(s_2 - \frac{1}{p_2} \right) > 0. \tag{6.148}$$

Let

$$\frac{1}{p_1} > \begin{cases} \frac{1}{p_2} + \frac{|\alpha|}{n-1} & \text{if } \alpha < 0, \\ \frac{1}{p_2} - \frac{\alpha}{n} & \text{if } \alpha \geq 0, \end{cases} \tag{6.149}$$

where $\alpha = \alpha_1 - \alpha_2$. Then

$$\mathrm{id}: \quad R^\infty A^{s_1}_{p_1 q_1}(\mathbb{R}^n, w_{\alpha_1}) \hookrightarrow A^{s_2}_{p_2 q_2}(\mathbb{R}^n, w_{\alpha_2}) \tag{6.150}$$

is compact and

$$e_k(\mathrm{id}) \sim \begin{cases} k^{-\alpha - n(\frac{1}{p_1} - \frac{1}{p_2})} & \text{if } \alpha < \delta_n, \\ k^{-(s_1 - s_2)} & \text{if } \alpha > \delta_n, \end{cases} \tag{6.151}$$

where $k \in \mathbb{N}$.

Proof. Step 1. One can insert R^∞ on the right-hand side of (6.150). First we deal with the F-spaces. We wish to show that under the given circumstances

$$\mathrm{id}^n: \quad R^\infty F^{s_1}_{p_1 q_1}(\mathbb{R}^n, w_{\alpha_1}) \hookrightarrow R^\infty F^{s_2}_{p_2 q_2}(\mathbb{R}^n, w_{\alpha_2}) \tag{6.152}$$

is compact if, and only if,

$$\mathrm{id}^1: \quad F^{s_1}_{p_1 q_1}(\mathbb{R}, w_\gamma) \hookrightarrow F^{s_2}_{p_2 q_2}(\mathbb{R}) \tag{6.153}$$

is compact for the corresponding spaces on the real line $\mathbb{R}$ as treated in Theorem 6.31, Remark 6.32 where

$$\gamma = \alpha + (n-1) \left(\frac{1}{p_1} - \frac{1}{p_2} \right), \tag{6.154}$$

and that

$$e_k(\mathrm{id}^n) \sim e_k(\mathrm{id}^1), \qquad k \in \mathbb{N}. \tag{6.155}$$

First we assume that s_1 and s_2 are large,

$$s_1 > \sigma_{p_1 q_1}, \qquad s_2 > \sigma_{p_2 q_2}, \tag{6.156}$$

with σ_{pq} as in (2.6). In particular, $\tilde{f}(t) = f(x)$ with $t = |x|$ makes sense. We apply the localisation principle according to Proposition 6.13 to this radial function restricted to an annulus around $|x| = j \in \mathbb{N}$ large, say of width 2. Then one can replace ϱ_m in (6.33) by product functions $\psi_j(r)\Theta_{j,m}(\theta)$ indicating the radial and the angular directions (polar coordinates) such that for any fixed j the C^∞ functions $Q_{j,m}$ can be transformed into each other by suitable rotations. Furthermore, $\psi_j \geq 0$ are C^∞ functions with

$$\psi_j(t) = 1 \quad \text{if } |t-j| \leq 1, \qquad \psi_j(t) = 0 \quad \text{if } |t-j| \geq 2. \tag{6.157}$$

After this modification for fixed j all $\sim j^{n-1}$ terms in (6.33) for the radial function f are equal and one can collect them, say, in the x_n-direction around (x', x_n) with $x' = 0$ and $x_n \sim j$. By a harmless diffeomorphic map one can flatten the outcome, getting products $\Psi(x')\psi_j(x_n)$, where ψ_j as above and Ψ is a C^∞ function in $\mathbb{R}^{n-1}$ with a compact support in, say, the unit ball in $\mathbb{R}^{n-1}$. By (6.156) we can apply the *Fubini property* according to [Triϵ], Theorem 4.4, p. 36, to $F^{s_1}_{p_1q_1}(\mathbb{R}^n)$. It follows that for fixed j,

$$\|f\psi_j \sum_m \Theta_{j,m} \,|F^{s_1}_{p_1q_1}(\mathbb{R}^n)\|^p \sim j^{n-1} \,\|\tilde{f}\psi_j \,|F^{s_1}_{p_1q_1}(\mathbb{R})\|^p. \tag{6.158}$$

The arguments work also in the reverse direction, rotating a function on $\mathbb{R}$, supported around j, to the above annulus around $|x| \sim j$ multiplied with $\sim j^{-(n-1)}$. Then one gets an isomorphic map of

$$R^\infty F^{s_1}_{p_1q_1}(\mathbb{R}^n, w_{\alpha_1}) \quad \text{onto} \quad R^\infty F^{s_1}_{p_1q_1}(\mathbb{R}, w_{\gamma_1}) \quad \text{with} \quad \gamma_1 = \alpha_1 + \frac{n-1}{p_1}. \tag{6.159}$$

Using the mapping according to Theorem 6.5(ii) which respects the support properties, then one can transfer id^n in (6.152) to the R^∞-version of id^1 in (6.153). But in the one-dimensional case the behavior at the origin is harmless. Hence one gets the equivalence of id^n and id^1 for the question of compactness and also (6.155) with (6.154). By Theorem 6.31 and (6.148) we have compactness of id^1 if $\gamma > 0$ and $\frac{1}{p_2} < \frac{1}{p_1} + \gamma$, hence

$$\alpha + (n-1)\left(\frac{1}{p_1} - \frac{1}{p_2}\right) > 0 \quad \text{and} \quad \alpha + n\left(\frac{1}{p_1} - \frac{1}{p_2}\right) > 0. \tag{6.160}$$

But this is the same as in (6.149). We have (6.115) with $n = 1$, which coincides with the second line in (6.151) if $\delta_1 < \gamma$, hence $\alpha > \delta_n$. If $\delta_1 > \gamma$, hence $\delta_n > \alpha$, then

$$\gamma + \frac{1}{p_1} - \frac{1}{p_2} = \alpha + n\left(\frac{1}{p_1} - \frac{1}{p_2}\right) \tag{6.161}$$

is the corresponding exponent in (6.151).

Step 2. First we remove the restriction (6.156) in case of the F-spaces. All assertions in the theorem depend only on the difference $s_1 - s_2$. We wish to apply the radial lift I_σ in (6.12). But it does not preserve the support property in connection with (6.152). However one can replace the support assumption in (6.146) by a regularity assumption for f near the origin, say, $f \in \mathcal{C}^\varkappa(B(2))$. If $\varkappa > 0$ is large then such a replacement does not influence the question of compactness and the distribution of entropy numbers. Since I_σ is elliptic this local smoothness is the preserved mapping $\mathcal{C}^\varkappa$ in $\mathcal{C}^{\varkappa-\sigma}$. Then one gets the above theorem for all F-spaces. As for the B-spaces we first remark that the above arguments apply also to $p=\infty$ and $F^s_{\infty\infty} = B^s_{\infty\infty}$. Otherwise we use the interpolation formula (6.53). By a look at Peetre's K-functional it follows that related optimal decompositions of radial functions result also in radial functions. Relaxing the support assumptions in (6.146) for the K-functional one obtains for

$$0<\theta<1, \quad 0<p\le\infty, \quad s_1 < s = (1-\theta)s_1 + \theta s_2 < s_2$$

that

$$R^\infty B^s_{pq}(\mathbb{R}^n, w_\alpha) \subset \Big(\tilde{R}^\infty F^{s_1}_{pp}(\mathbb{R}^n, w_\alpha),\, \tilde{R}^\infty F^{s_2}_{pp}(\mathbb{R}^n, w_\alpha)\Big)_{\theta,q} \tag{6.162}$$

where $\tilde{R}^\infty$ refers to the corresponding spaces in (6.146) with $\mathbb{R}^n \backslash B(1/2)$ in place of $\mathbb{R}^n\backslash B(1)$. We use the interpolation property for entropy numbers according to Proposition 1.91. Then one gets for the B-spaces an estimate of $e_k(\mathrm{id})$ from above by the right-hand side of (6.151). The sharpness follows from the sharpness for the F-spaces by the same arguments as in Step 3 of the proof of Theorem 6.20. □

Remark 6.42. We discuss the outcome. Let $\alpha > 0$. Then for given p_1 the restrictions for p_2 in (6.149) are the same as in Theorem 6.31. Furthermore we have in (6.151) the same breaking point as in Theorem 6.31, the line L in Figure 6.4.1. But the behavior of the entropy numbers is different. Also the area covered by (6.148) is larger than the shaded area in Figure 6.4.1. One has to add for given $p_1 < \infty$ the region

$$\delta_1 - (n-1)\left(\frac{1}{p_1} - \frac{1}{p_2}\right) = \delta_n \le 0 < \delta_1, \qquad p_1 < p_2 \le \infty, \tag{6.163}$$

indicated by the dotted line of slope 1 in Figure 6.4.1. Hence the restriction to radial functions does not only improve the behavior of the entropy numbers but id in (6.150) is compact in some cases where according to Proposition 6.29 the embedding id in (6.114) is not compact or even does not exist. The most interesting case might be $\alpha = 0$, unweighted radial function spaces, where the theory started from. Then one has for $p_1 < \infty$ that $p_1 < p_2 \le \infty$, which means $p_* = p_1$ in (6.112) and in Figure 6.4.1, and

$$e_k(\mathrm{id}) \sim \begin{cases} k^{-n\left(\frac{1}{p_1}-\frac{1}{p_2}\right)} & \text{if } \delta_n > 0, \\ k^{-(s_1-s_2)} & \text{if } \delta_n < 0, \end{cases} \tag{6.164}$$

where $k \in \mathbb{N}$ and with (6.163) in the latter case. Finally for any $\alpha < 0$ there are admitted couples (p_1, p_2) with (6.149), (6.151). It might be a little bit surprising that the above theorem covers also cases with $\delta_n < 0$ in sharp contrast to Proposition 6.29 and Theorem 6.31. But this is no longer the case if one deals with the spaces $R^0 A^s_{pq}(\mathbb{R}^n, w_\alpha)$ or $RA^s_{pq}(\mathbb{R}^n, w_\alpha)$ according to Definition 6.39.

Again σ_{pq} has the same meaning as in (2.6).

Theorem 6.43. *Let $n \geq 2$. Let $p_1, p_2, q_1, q_2 \in (0,\infty]$ (with $p_1 < \infty$, $p_2 < \infty$ for F-spaces). Let $s_1 \in \mathbb{R}$, $s_2 \in \mathbb{R}$ with*

$$s_1 > s_2 \qquad \text{and} \qquad \delta_n = s_1 - \frac{n}{p_1} - \left(s_2 - \frac{n}{p_2}\right) > 0. \tag{6.165}$$

Then

$$\mathrm{id}: \quad R^0 A^{s_1}_{p_1 q_1}(\mathbb{R}^n) \hookrightarrow A^{s_2}_{p_2 q_2}(\mathbb{R}^n) \tag{6.166}$$

is compact and

$$e_k(\mathrm{id}) \sim k^{-(s_1-s_2)}, \qquad k \in \mathbb{N}. \tag{6.167}$$

Proof. Step 1. The estimate from below is one-dimensional and covered by the proof of Theorem 6.41. We prove the estimate from above. Let $\psi \in C^\infty$ be radial, $\psi(x) = 1$ if $|x| \leq 2$ and $\psi(y) = 0$ if $|y| \geq 4$. Let

$$\tilde{\psi}(x) = \psi(x) - \psi(2x), \qquad \text{hence} \quad \operatorname{supp} \tilde{\psi} \subset \{y:\ 1 \leq |y| \leq 4\}. \tag{6.168}$$

Let $J \in \mathbb{N}$. Then

$$\psi(x) = \psi(2^J x) + \sum_{j=0}^{J-1} \tilde{\psi}(2^j x) = \psi^J(x) + \sum_{j=0}^{J-1} \psi_j(x) \tag{6.169}$$

is a resolution of unity near the origin. For given $J \in \mathbb{N}$ we decompose id in (6.166) by

$$\mathrm{id} = \psi^J \,\mathrm{id} + \sum_{j=0}^{J-1} \psi_j \,\mathrm{id} = \mathrm{id}^J + \sum_{j=0}^{J-1} \mathrm{id}_j. \tag{6.170}$$

Let, in addition to (6.165),

$$A_1 = F^{s_1}_{p_1 q_1} \text{ with } s_1 > \sigma_{p_1 q_1} \quad \text{and} \quad A_2 = F^{s_2}_{p_2 q_2} \text{ with } s_2 > \sigma_{p_2 q_2}, \tag{6.171}$$

complemented by $A_1 = \mathcal{C}^{s_1}$ with $s_1 > 0$ and $A_2 = \mathcal{C}^{s_2}$ with $s_2 > 0$. Then one can apply the homogeneity assertion in [Triϵ], Corollary 5.16, p. 66. One gets by (6.165),

$$\begin{aligned} \|\psi(2^J \cdot) f \,|A_2(\mathbb{R}^n)\| &\sim 2^{J(s_2 - \frac{n}{p_2})} \,\|\psi f(2^{-J}\cdot)\,|A_2(\mathbb{R}^n)\| \\ &\preceq 2^{J(s_2 - \frac{n}{p_2})} \,\|\psi f(2^{-J}\cdot)\,|A_1(\mathbb{R}^n)\| \\ &\preceq 2^{-J\delta_n} \,\|\psi(2^J\cdot) f\,|A_1(\mathbb{R}^n)\|. \end{aligned} \tag{6.172}$$

Hence,

$$\|\mathrm{id}^J\| \preceq 2^{-J\delta_n} \qquad \text{and} \qquad \|\mathrm{id}_j\| \preceq 2^{-j\delta_n}. \tag{6.173}$$

By elementary properties of entropy numbers according to Proposition 1.89, the above homogeneity arguments and Theorem 6.41 it follows that

$$e_k\left(\mathrm{id}_j:\ R^0 A^{s_1}_{p_1 q_1}(\mathbb{R}^n) \hookrightarrow A^{s_2}_{p_2 q_2}(\mathbb{R}^n)\right) \preceq 2^{-j\delta_n}\, k^{-(s_1-s_2)}, \tag{6.174}$$

$k \in \mathbb{N}$. Using that $A^{s_2}_{p_2q_2}(\mathbb{R}^n)$ is a p-Banach space with $p = \min(1, p_2, q_2)$ it follows by (1.286) that

$$e_k(\mathrm{id})^p \preceq 2^{-J\delta_n p} + \sum_{j=0}^{J-1} e_{k_j}(\mathrm{id}_j)^p, \qquad k = \sum_{j=0}^{J-1} k_j. \tag{6.175}$$

Let

$$k_j \sim 2^{\frac{\delta_n}{s_1-s_2}(J-j)}\, 2^{j\varepsilon} \qquad \text{with} \quad 0 < \varepsilon < \frac{\delta_n}{s_1 - s_2}. \tag{6.176}$$

Then

$$\sum_{j=0}^{J-1} k_j \sim 2^{\frac{\delta_n}{s_1-s_2}J} \sim k \tag{6.177}$$

and by (6.174), (6.175),

$$e_k(\mathrm{id})^p \preceq k^{-(s_1-s_2)p} \left[1 + \sum_{j=0}^{J-1} 2^{-j\varepsilon p(s_1-s_2)} \right] \preceq k^{-(s_1-s_2)p}. \tag{6.178}$$

This gives the desired estimate from above under the assumption (6.171).

Step 2. The rest is now the same as in Step 2 of the proof of Theorem 6.41. This applies to the lifting argument and also to an appropriate counterpart of the interpolation formula (6.162). □

Remark 6.44. In contrast to Theorem 6.41 we needed now that $\delta_n > 0$. But this is not only a technical matter. By the references given below in Remark 6.46, $\delta_n > 0$ is also necessary. Otherwise it is quite clear that one can clip together Theorems 6.41 and 6.43 to get corresponding assertions for the full radial spaces (6.144). We formulate the outcome.

Corollary 6.45. *Let* $n \geq 2$. *Let* $\alpha_1 \in \mathbb{R}$, $\alpha_2 \in \mathbb{R}$ *and* $p_1, p_2, q_1, q_2 \in (0, \infty]$ (*with* $p_1 < \infty$, $p_2 < \infty$ *for* F-*spaces*). *Let* $s_1 \in \mathbb{R}$, $s_2 \in \mathbb{R}$ *with*

$$s_1 > s_2 \quad \text{and} \quad \delta_n = s_1 - \frac{n}{p_1} - \left(s_2 - \frac{n}{p_2} \right) > 0. \tag{6.179}$$

Let p_1, p_2, $\alpha = \alpha_1 - \alpha_2$ *be restricted as in* (6.149). *Then*

$$\mathrm{id}: \quad RA^{s_1}_{p_1q_1}(\mathbb{R}^n, w_{\alpha_1}) \hookrightarrow A^{s_2}_{p_2q_2}(\mathbb{R}^n, w_{\alpha_2}) \tag{6.180}$$

is compact and one has (6.151) *for the corresponding entropy numbers.*

Proof. If $\alpha < \delta_n$ then

$$s_1 - s_2 > \alpha + n \left(\frac{1}{p_1} - \frac{1}{p_2} \right). \tag{6.181}$$

Then the corollary follows from Theorems 6.41, 6.43. □

Remark 6.46. As mentioned at the beginning of this Section 6.5.2, radial subspaces of Sobolev spaces have been considered in connection with radially-symmetric solutions of some semi-linear elliptic equations in $\mathbb{R}^n$, where $n \geq 2$. Of interest are the decay properties of elements of these spaces near the origin and at infinity on the one hand and compact embeddings of type (6.142) on the other hand. The literature mentioned there can be complemented by further papers at this time which can be found in the references given below. The extension of these studies to general B-spaces and F-spaces started some 20 years later with [SiSk00]. In addition to decay properties it had been proved there that

$$\mathrm{id}: \quad RA^{s_1}_{p_1 q_1}(\mathbb{R}^n) \hookrightarrow A^{s_2}_{p_2 q_2}(\mathbb{R}^n), \qquad n \geq 2, \tag{6.182}$$

is compact if, and only if, $0 < p_1 < p_2 \leq \infty$ and $\delta_n > 0$. These considerations have been extended in [Skr02] to spaces invariant under other subgroups of the group of all isometries in $\mathbb{R}^n$. The next step was carried out in [KLSS03a] with the outcome

$$e_k(\mathrm{id}) \sim k^{-n(\frac{1}{p_1} - \frac{1}{p_2})}, \qquad k \in \mathbb{N}, \quad n \geq 2, \tag{6.183}$$

for id in (6.182), restricted to the case of Banach spaces. This assertion is now extended by Corollary 6.45 with $\alpha_1 = \alpha_2 = 0$ to the full range of the parameters, which corresponds also to the upper line in (6.164). Approximation numbers of id in (6.182) (restricted to Banach spaces) have been considered recently in [SkT04]. In Sections 1.9.1, 1.9.2 we described limiting embeddings of spaces of type $A^s_{pq}(\mathbb{R}^n)$ with $s = n/p$ into some Orlicz spaces. One can ask what happens with these embeddings if one replaces $A^{n/p}_{pq}(\mathbb{R}^n)$ by $RA^{n/p}_{pq}(\mathbb{R}^n)$, $n \geq 2$. According to [SkT00] some of these embeddings are compact (in dependence on the parameters). Assertions about related entropy numbers can be found in [Skr04], including applications to the spectral theory of some elliptic operators. The arguments in [KLSS03a], and also in the other papers of these authors mentioned in this remark, rely on atomic decompositions adapted to radial functions, somewhat in contrast to our more qualitative considerations. On the other hand we took over one of the basic ideas in these papers to reduce (unweighted) radial spaces in $\mathbb{R}^n$ with $n \geq 2$ to weighted spaces in $\mathbb{R}$.

Remark 6.47. As indicated in Remark 6.4 one can replace $w \in W^n$ in the spaces $A^s_{pq}(\mathbb{R}^n, w)$ by more general weights with (6.8), extended to $\beta = 1$, in place of (6.2). One can ask for compact embeddings both for the spaces themselves and their radial subspaces. All that one needs for the foundation of these spaces is covered by the literature mentioned in Remark 6.4, especially by [Sco98a], [Sco98b]. Let, for example, $w^\beta(x)$ be a corresponding smooth positive weight function with

$$w^\beta(x) = e^{|x|^\beta}, \qquad |x| \geq 1, \qquad 0 < \beta \leq 1. \tag{6.184}$$

Let $s_1 \in \mathbb{R}$, $s_2 \in \mathbb{R}$, and $p_1, p_2, q_1, q_2 \in (0, \infty]$ (with $p_1 < \infty$, $p_2 < \infty$ for F-spaces) with (6.179). Let $0 < \beta \leq 1$. Then both

$$\mathrm{id}: \quad A^{s_1}_{p_1 q_1}(\mathbb{R}^n, w^\beta) \hookrightarrow A^{s_2}_{p_2 q_2}(\mathbb{R}^n) \tag{6.185}$$

and

$$\mathrm{id}^R: \quad RA^{s_1}_{p_1q_1}(\mathbb{R}^n, w^\beta) \hookrightarrow A^{s_2}_{p_2q_2}(\mathbb{R}^n) \tag{6.186}$$

are compact. Furthermore,

$$e_k(\mathrm{id}) \sim k^{-\frac{s_1-s_2}{n}}, \qquad k \in \mathbb{N}, \tag{6.187}$$

and

$$e_k(\mathrm{id}^R) \sim k^{-(s_1-s_2)}, \qquad k \in \mathbb{N}. \tag{6.188}$$

These assertions follow from

$$A^{s_1}_{p_1q_1}(\mathbb{R}^n, w^\beta) \hookrightarrow A^{s_1}_{p_1q_1}(\mathbb{R}^n, w_\alpha) \hookrightarrow A^{s_2}_{p_2q_2}(\mathbb{R}^n), \tag{6.189}$$

and its radial counterpart for any $\alpha > 0$, Theorem 6.31 with (6.115) and Corollary 6.45 with the lower line in (6.151). This might be of some interest for its own sake. But more interesting is the application of the exponential case, $w^1(x) \sim e^{c|x|}$ with $c > 0$, to (unweighted) function spaces of the above type on hyperbolic manifolds and non-compact symmetric spaces of rank one (for example the Poincaré circle) subject to some symmetries (being radial, for example). Then exponential weights are coming in naturally. First remarks in this direction may be found at the end of [SiSk00]. This theory has been elaborated in [Skr03] and especially in the substantial recent paper [SkT05]. In particular, one finds there assertions of type (6.188), again with applications to the spectral theory of Schrödinger operators. Approximation numbers of these types of compact embeddings have been considered quite recently in [SkT06].

Chapter 7

Fractal Analysis: Measures, Characteristics, Operators

7.1 Measures

7.1.1 Definitions, basic properties

Sections 1.12–1.16, 1.18, and parts of Section 1.17 dealt with fractal measures in $\mathbb{R}^n$, their relations to typical characteristics in fractal geometry and analysis, and corresponding connections to elliptic operators. Now we return to some aspects of this subject in greater detail. Recall that the self-contained survey Chapter 1 of a few developments of the recent theory of function spaces on the one hand and the other chapters of this book on the other hand should be readable independently. This causes a mild overlapping of some basic definitions. But we restrict ourselves to the bare minimum and use otherwise Chapter 1 as a source of references.

As for basic notation we refer to Section 2.1. We mostly assume that μ is a positive Radon measure in $\mathbb{R}^n$ with

$$\Gamma = \operatorname{supp} \mu \subset \{x : |x| < 1\} \quad \text{and} \quad 0 < \mu(\mathbb{R}^n) < \infty. \tag{7.1}$$

Since μ is Radon it can be interpreted uniquely in the standard way as an element of $S'(\mathbb{R}^n)$ and we write in slight abuse of notation $\mu \in S'(\mathbb{R}^n)$. Let $0 < p \leq \infty$. Then $L_p(\Gamma, \mu)$ is the usual complex-valued quasi-Banach space. If $1 \leq p \leq \infty$ then $f \in L_p(\Gamma, \mu)$, considered as a finite complex Radon measure, can again be uniquely interpreted as $f\mu \in S'(\mathbb{R}^n)$. For some details we refer to Section 1.12.2. The assumption that the compact support Γ of μ is located in a ball of radius 1 is unimportant. It simplifies some assertions which otherwise may depend on the diameter of the compact support of μ.

Again let Q_{jm} with $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$ be the closed cubes in $\mathbb{R}^n$ with sides parallel to the axes of coordinates centred at $2^{-j}m$ and with side-length 2^{-j+1}.

Definition 7.1. *Let μ be a Radon measure in $\mathbb{R}^n$ with (7.1). Let $0 < p \le \infty$, $0 < q \le \infty$, and $\lambda \in \mathbb{R}$. Then*

$$\mu^\lambda_{pq} = \left(\sum_{j=0}^{\infty} 2^{j\lambda q} \left(\sum_{m \in \mathbb{Z}^n} \mu(Q_{jm})^p \right)^{q/p} \right)^{1/q} \tag{7.2}$$

with the obvious modifications if p and/or q are infinite.

Remark 7.2. We repeated Definition 1.125 for sake of independence as far as basic notation is concerned. Otherwise we refer to Remark 1.126 where we gave references and discussed in some detail connections with fractal geometry. One may also consult [Triϵ, Section 9] for further information. We collect a few properties.

(i) For any μ with (7.1) one has

$$\mu \in B^0_{1,\infty}(\mathbb{R}^n) \quad \text{and} \quad \|\mu \,|B^0_{1,\infty}(\mathbb{R}^n)\| \sim \mu(\mathbb{R}^n), \tag{7.3}$$

where the equivalence constants are independent of μ.

(ii) Let

$$\sigma^-_p = \min(0, n - n/p) \quad \text{where} \quad 0 < p \le \infty. \tag{7.4}$$

Then

$$\mu^\lambda_{pq} \sim \mu(\mathbb{R}^n) \quad \text{if} \quad \begin{cases} \text{either} & 0 < q \le \infty,\ \lambda < \sigma^-_p, \\ \text{or} & q = \infty,\ \lambda = \sigma^-_p, \end{cases} \tag{7.5}$$

where again the equivalence constants are independent of μ.

This coincides essentially with Proposition 1.127. There one finds also a proof of these assertions.

Definition 7.3. *Let $0 < q \le \infty$, $s \in \mathbb{R}$, and*

$$s - n/p = \lambda - n. \tag{7.6}$$

(i) *Let $0 < p \le \infty$. Then $B^s_{pq}(\mathbb{R}^n)$ has the μ-property means that*

$$\mu \in B^s_{pq}(\mathbb{R}^n) \quad \textit{if, and only if,} \quad \mu^\lambda_{pq} < \infty \tag{7.7}$$

for all positive Radon measures μ with (7.1).

(ii) *Let $0 < p < \infty$. Then $F^s_{pq}(\mathbb{R}^n)$ has the μ-property means that*

$$\mu \in F^s_{pq}(\mathbb{R}^n) \quad \textit{if, and only if,} \quad \mu^\lambda_{pp} < \infty \tag{7.8}$$

for all positive Radon measures μ with (7.1).

Remark 7.4. This coincides with Definition 1.129. The remarkable independence of the right-hand side of (7.8) of q will be commented on later. Of course, if $q = p$ then (7.8) coincides with (7.7). By (7.3) the distinguished space $B^0_{1,\infty}(\mathbb{R}^n)$ has the μ-property. Next we prove an extended version of Theorem 1.131. Again we write $A^s_{pq}(\mathbb{R}^n)$ if the corresponding assertion applies both to $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ under the indicated restrictions for the parameters s, p, q.

Theorem 7.5.

(i) *Let* $s \in \mathbb{R}$, $0 < q \le \infty$, $0 < p \le \infty$ *with* $p < \infty$ *in the* F*-case and* $(s,q) \neq (0,\infty)$ *in the* B*-case (this means* $q < \infty$ *if* $s = 0$*). Then* $A^s_{pq}(\mathbb{R}^n)$ *has the* μ*-property according to Definition* 7.3 *if, and only if,* $s < 0$.

(ii) *The spaces* $B^0_{p\infty}(\mathbb{R}^n)$ *with* $0 < p \le 1$ *have the* μ*-property.*

(iii) *If* $B^s_{pq}(\mathbb{R}^n)$ *has the* μ*-property according to parts* (i) *or* (ii) *then*

$$\|\mu \,|B^s_{pq}(\mathbb{R}^n)\| \sim \mu^\lambda_{pq} \quad \text{with} \quad s - n/p = \lambda - n, \tag{7.9}$$

where the equivalence constants are independent of μ *with* (7.1).

(iv) *If* $s < 0$, $0 < p < \infty$, $0 < q \le \infty$, *then*

$$\|\mu \,|F^s_{pq}(\mathbb{R}^n)\| \sim \mu^\lambda_{pp} \quad \text{with} \quad s - n/p = \lambda - n, \tag{7.10}$$

where the equivalence constants are independent of μ *with* (7.1).

Proof. Step 1. Let $\mu = \varphi\mu_L$, where μ_L is Lebesgue measure, $\varphi \in S(\mathbb{R}^n)$ with $\varphi(x) > 0$ if $|x| < 1$ and $\varphi(y) = 0$ if $|y| \ge 1$. Then μ is an element of all spaces $A^s_{pq}(\mathbb{R}^n)$, but

$$\mu^\lambda_{pq} \sim \left(\sum_{j=0}^\infty 2^{jq(\lambda + n(\frac{1}{p} - 1))} \right)^{1/q} = \infty \tag{7.11}$$

if $s = \lambda - n + n/p > 0$ or $s = 0, q < \infty$. This proves the only-if-assertion of (i).

Step 2. We prove the if-assertion of part (i) for the B-spaces with $s < 0$ and the equivalence (7.9). We apply the characterisation of $B^s_{pq}(\mathbb{R}^n)$ according to Corollary 1.12, Remark 1.13 and (1.51) based on the non-negative compactly supported C^∞ kernels k and the related means

$$k\left(2^{-j}, \mu\right)(x) = 2^{jn} \int_{\mathbb{R}^n} k\left(2^j(y - x)\right) \mu(\mathrm{d}y), \quad j \in \mathbb{N}_0. \tag{7.12}$$

Let Q'_{jm} be the cubes in $\mathbb{R}^n$ centred at $2^{-j}m$ and with side-length 2^{-j}. Hence in obvious notation $Q_{jm} = 2Q'_{jm}$. Choosing appropriate kernels k_1 and k_2 of the above type one gets for the related means

$$k_1\left(2^{-j}, \mu\right)(x) \ge 2^{jn}\mu(Q_{jm}) \ge k_2\left(2^{-j}, \mu\right)(x), \quad x \in Q'_{jm}. \tag{7.13}$$

Hence by (7.13) and (1.51),

$$\|\mu \,|B^s_{pq}(\mathbb{R}^n)\| \sim \left(\sum_{j=0}^{\infty} 2^{jsq+jnq-jnq/p} \left(\sum_{m\in\mathbb{Z}^n} \mu(Q_{jm})^p \right)^{q/p} \right)^{1/q} \tag{7.14}$$

as a characterisation. This proves (7.9) (equivalent quasi-norms) and the if-part of (i) for $A = B$.

Step 3. We prove part (ii) and (7.9) with $s = 0, 0 < p \le 1, q = \infty$, hence $\lambda = n-n/p$ and $\mu^{\lambda}_{p\infty} \sim \mu(\mathbb{R}^n)$ according to (7.4), (7.5). The case $p = 1$ is covered by (7.3). If $p < 1$ then one gets, for example by (1.44) and Hölder's inequality,

$$\|\mu \,|B^0_{p\infty}(\mathbb{R}^n)\| \preceq \|\mu \,|B^0_{1,\infty}(\mathbb{R}^n)\| \sim \mu(\mathbb{R}^n). \tag{7.15}$$

The converse follows from (1.422) in the same way as in the case of $p = 1$.

Step 4. It remains to prove the if-assertion of part (i) for the F-spaces and the equivalence (7.10). But this follows from part (i) for the spaces $B^s_{pp}(\mathbb{R}^n) = F^s_{pp}(\mathbb{R}^n)$ with $0 < p < \infty, s < 0$, and Proposition 1.133. □

Remark 7.6. The proof is surprisingly short. But it relies on a substantial characterisation of the spaces considered in Theorem 1.10 and Corollary 1.12 as well as on independence of the positive cones in the spaces $F^s_{pq}(\mathbb{R}^n)$ with $s < 0$ of q according to Proposition 1.133. All these assertions have their own history which we described in Remarks 1.14, 1.134 and the references given there. Otherwise we followed [Tri03b]. The above theorem covers in particular Theorem 1.131.

Remark 7.7. According to (7.5) and the above theorem we have for $0 < p \le \infty$,

$$\|\mu \,|B^s_{p\infty}(\mathbb{R}^n)\| \sim \mu(\mathbb{R}^n) \quad \text{if} \quad s = \min\left(0, n(\frac{1}{p} - 1)\right). \tag{7.16}$$

Furthermore for $0 < p \le \infty$ ($p < \infty$ in the case of F-spaces) and $0 < q \le \infty$ it follows both for $A^s_{pq}(\mathbb{R}^n) = B^s_{pq}(\mathbb{R}^n)$ and $A^s_{pq}(\mathbb{R}^n) = F^s_{pq}(\mathbb{R}^n)$ that

$$\|\mu \,|A^s_{pq}(\mathbb{R}^n)\| \sim \mu(\mathbb{R}^n) \quad \text{if} \quad s < \min\left(0, n(\frac{1}{p} - 1)\right). \tag{7.17}$$

This justifies (1.429) and (1.430). In other words, in the above theorem those spaces $A^s_{pq}(\mathbb{R}^n)$ are of special interest where $(t = 1/p, s)$ is located in the triangle in Figure 1.17.1 with the corner points $(0,-n)$, $(0,0)$, $(1,0)$.

7.1.2 Potentials and Fourier transforms

Bessel potentials, Riesz potentials, truncated Riesz potentials and Fourier transforms of Radon measures of type (7.1) attracted a lot of attention, especially in case of self-similar fractals. Some references can be found in the literature mentioned in Remark 1.126 and at the beginning of Section 1.17.5. One may also consult [Triδ, Sections 17.10, 17.11], [Str94], [Mat95, Section 12] and [Zah04] for further information, references and typical assertions. In [Tri03b] we applied Theorem 7.5 to mapping properties of the indicated potentials and Fourier transforms of measures of type (7.1). This will not be repeated here in detail. We restrict ourselves to a few almost immediate applications of Theorem 7.5 to questions of this type.

According to (1.5), (1.6) the classical Bessel potentials

$$J_\sigma = (\mathrm{id} - \Delta)^{-\sigma/2} : \quad A^s_{pq}(\mathbb{R}^n) \hookrightarrow A^{\sigma+s}_{pq}(\mathbb{R}^n) \tag{7.18}$$

map, for all $\sigma \in \mathbb{R}$ and all admitted parameters s, p, q and $A = B$ or $A = F$, the indicated spaces isomorphically onto each other. Of course, $\Delta = \sum_{j=1}^n \partial^2/\partial^2 x_j$ is the Laplacian. If $0 < \sigma < n$, then

$$(J_\sigma f)(x) = \int_{\mathbb{R}^n} G_\sigma(x-y)\, f(y)\, \mathrm{d}y, \qquad x \in \mathbb{R}^n, \tag{7.19}$$

where the kernels $G_\sigma(x)$ decay exponentially if $|x| \to \infty$ and have the well-known behavior

$$G_\sigma(x) \sim \frac{c}{|x|^{n-\sigma}} \quad \text{if} \quad |x| < 1, \tag{7.20}$$

for some $c > 0$ near the origin, [AdH96, pp. 10–13]. Extending the right-hand side of (7.20) to $x \in \mathbb{R}^n$ one gets the Riesz potentials and its local version, the truncated Riesz potentials,

$$(I_{\varrho,\sigma} f)(x) = \int_{\mathbb{R}^n} \frac{\chi_\varrho(x-y)}{|x-y|^{n-\sigma}}\, f(y)\, \mathrm{d}y, \quad 0 < \sigma < n, \quad x \in \mathbb{R}^n, \tag{7.21}$$

where χ_ϱ is the characteristic function of a ball of radius $\varrho > 0$ centred at the origin. Of interest is the local and global behavior of the Bessel potentials $J_\sigma \mu$ and the (truncated) Riesz potentials $I_{\sigma,\varrho}\mu$ for (self-similar) measures of type (7.1). Detailed studies may be found in the above references and the literature in these references. We wish to apply Theorem 7.5. This suggests by Remark 7.7 a restriction of p and s by

$$1 < p \le \infty, \quad 0 > s \ge -\frac{n}{p'}, \quad \text{where} \quad \frac{1}{p} + \frac{1}{p'} = 1\,, \tag{7.22}$$

and hence by (7.6) with $\lambda = s + n/p'$ by

$$1 < p \le \infty, \qquad 0 \le \lambda < n/p'\,. \tag{7.23}$$

Corollary 7.8. *Let $0 < q \le \infty$ and let p and λ as in* (7.23) (*with $p < \infty$ for the F-spaces*). *Let $\sigma \in \mathbb{R}$. Then*

$$\Big\| J_\sigma \mu \,|B^{\lambda+\sigma-n/p'}_{pq}(\mathbb{R}^n) \Big\| \sim \mu^\lambda_{pq} \tag{7.24}$$

and

$$\left\| J_\sigma \mu \,|F^{\lambda+\sigma-n/p'}_{pq}(\mathbb{R}^n)\right\| \sim \mu^\lambda_{pp}\,, \tag{7.25}$$

where the equivalence constants are independent of the Radon measures μ with (7.1)*.*

Proof. This follows immediately from Theorem 7.5, (7.18) and the above explanations. □

Remark 7.9. Corresponding equivalences for truncated Riesz potentials now with $0 < \sigma < n$ can be reduced to the above assertions, having in mind that the corresponding kernels $G_\sigma(x)$ in (7.19) decay exponentially at infinity. We refer for details to [Tri03b]. Here are a few special cases.

(i) Let $H^\sigma_p(\mathbb{R}^n)$ with $1 < p < \infty$, $\sigma \in \mathbb{R}$, be the Sobolev spaces according to (1.7), (1.8), and let $\mathcal{C}^\sigma(\mathbb{R}^n)$ with $\sigma \in \mathbb{R}$ be the Hölder-Zygmund spaces according to (1.10). Then

$$\left\| J_\sigma \mu \,|H^{\sigma-n/p'}_p(\mathbb{R}^n)\right\|^p \sim \sum_{j=0}^\infty \sum_{m\in\mathbb{Z}^n} \mu(Q_{jm})^p \tag{7.26}$$

where $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\sigma \in \mathbb{R}$, and

$$\left\| J_\sigma \mu \,|\mathcal{C}^{\lambda-n+\sigma}(\mathbb{R}^n)\right\| \sim \sup_{j,m} 2^{j\lambda}\, \mu(Q_{jm}) \tag{7.27}$$

where $0 \le \lambda < n$, $\sigma \in \mathbb{R}$. This follows from (7.25) with $\lambda = 0$ and (7.24) with $p = q = \infty$. In particular,

$$\left\| J_{n/p'}\, \mu \,|L_p(\mathbb{R}^n)\right\|^p \sim \sum_{j=0}^\infty \sum_{m\in\mathbb{Z}^n} \mu(Q_{jm})^p, \quad 1 < p < \infty. \tag{7.28}$$

(ii) There are counterparts of (7.26)–(7.28) for the truncated Riesz potentials $I_{\sigma,\varrho}$ with $\varrho_0 \le \varrho < \infty$, where ϱ_0 is sufficiently large. One has (7.26) with $I_{\sigma,\varrho}$ in place of J_σ under the additional restriction $0 < \sigma \le n/p'$, in particular,

$$\left\| I_{n/p',\varrho}\, \mu \,|L_p(\mathbb{R}^n)\right\|^p \sim \sum_{j=0}^\infty \sum_{m\in\mathbb{Z}^n} \mu(Q_{jm})^p, \quad 1 < p < \infty, \tag{7.29}$$

and also (7.27) with $I_{\sigma,\varrho}$ in place of J_σ under the additional restriction $0 < \sigma < n - \lambda$.

Similarly one can employ Theorem 7.5 to say something about the decay of the Fourier transform $\widehat{\mu}(\xi)$ if $|\xi| \to \infty$ for measures with (7.1). Recall that

$$\|f\,|H^s_2(\mathbb{R}^n)\|^2 \sim \int_{\mathbb{R}^n} (1+|\xi|^2)^s\, |\widehat{f}(\xi)|^2\, \mathrm{d}\xi, \quad s \in \mathbb{R}, \tag{7.30}$$

as a consequence of (1.7) (and that the Fourier transform generates a unitary operator in $L_2(\mathbb{R}^n)$).

Corollary 7.10. *Let $0 \le 2\lambda < n$. Then for all measures μ with* (7.1),

$$\int_{\mathbb{R}^n} (1+|\xi|)^{2\lambda-n}\, |\widehat{\mu}(\xi)|^2 \,\mathrm{d}\xi \sim \sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} 2^{2j\lambda}\, \mu(Q_{jm})^2 \tag{7.31}$$

where the equivalence constants are independent of μ.

Proof. This follows from (7.9) with $s = \lambda - n/2 < 0$ and (7.30) with $H^s_2(\mathbb{R}^n) = B^s_{2,2}(\mathbb{R}^n)$. □

Remark 7.11. The supremum of all λ such that (7.31) is finite is near to the so-called *Fourier dimension* of Γ admitting all μ with (7.1). We refer to the corresponding discussion in [Triδ, Section 17.10] and the generalisation of the above corollary in [Tri03b].

7.1.3 Traces: general measures

Now we assume that the Radon measure μ according to (7.1) is singular,

$$\Gamma = \operatorname{supp}\mu \subset \{x : |x| < 1\}, \quad 0 < \mu(\mathbb{R}^n) < \infty, \quad |\Gamma| = 0, \tag{7.32}$$

where $|\Gamma|$ is the Lebesgue measure of Γ. Again it is convenient but unimportant that the support of μ, assumed to be compact, is located in the unit ball in $\mathbb{R}^n$. Recall that $S(\mathbb{R}^n)$ is dense in $B^s_{pq}(\mathbb{R}^n)$ if both $p < \infty, q < \infty$. Let $\mathring{B}^s_{p\infty}(\mathbb{R}^n)$ be the completion of $S(\mathbb{R}^n)$ in $B^s_{p\infty}(\mathbb{R}^n)$. More details and references about this point may be found in Remark 2.30.

Definition 7.12. *Let μ be a Radon measure in $\mathbb{R}^n$ with* (7.32). *Let $1 < p < \infty$, $1 \le q \le \infty$, $s > 0$, and $1 \le r < \infty$. Let for some $c > 0$,*

$$\Big(\int_\Gamma |\varphi(\gamma)|^r \,\mu(\mathrm{d}\gamma) \Big)^{1/r} \le c\, \|\varphi \,|B^s_{pq}(\mathbb{R}^n)\| \quad \textit{for all} \quad \varphi \in S(\mathbb{R}^n). \tag{7.33}$$

Then the trace operator tr_μ,

$$\mathrm{tr}_\mu : \quad B^s_{pq}(\mathbb{R}^n) \hookrightarrow L_r(\Gamma,\mu) \tag{7.34}$$

(*with $\mathring{B}^s_{p\infty}(\mathbb{R}^n)$ in place of $B^s_{p\infty}(\mathbb{R}^n)$ if $q=\infty$*) *is the completion of the pointwise trace $(\mathrm{tr}_\mu \varphi)(\gamma) = \varphi(\gamma)$ where $\varphi \in S(\mathbb{R}^n)$.*

Remark 7.13. Of course, $L_r(\Gamma,\mu)$ is the usual complex Banach space, normed by

$$\|g\,|L_r(\Gamma,\mu)\| = \Big(\int_\Gamma |g(\gamma)|^r \,\mu(\mathrm{d}\gamma) \Big)^{1/r} = \Big(\int_{\mathbb{R}^n} |g(x)|^r \,\mu(\mathrm{d}x) \Big)^{1/r}. \tag{7.35}$$

According to Sections 1.12.2 and 1.17.2 one can interpret $L_r(\Gamma,\mu)$ as a subset of $S'(\mathbb{R}^n)$ identifying $g \in L_r(\Gamma,\mu)$ with the complex finite measure $g\mu \in S'(\mathbb{R}^n)$,

$$(\mathrm{id}_\mu g)(\varphi) = \int_\Gamma g(\gamma)\,\varphi(\gamma)\,\mu(\mathrm{d}\gamma), \quad \varphi \in S(\mathbb{R}^n). \tag{7.36}$$

We call id_μ the *identification operator*. We studied in [Triϵ, Section 9] in detail trace operators of type (7.34) with $F^s_{pq}(\mathbb{R}^n)$ in place of $B^s_{pq}(\mathbb{R}^n)$. But it is not our aim to do here the same for the B-spaces as in [Triϵ] for the F-spaces. We restrict ourselves here to a few more specific points needed later on. But some arguments can be taken over. In particular, according to [Triϵ, Section 9.2, pp. 122–124], id_μ is the dual of tr_μ,

$$\mathrm{id}_\mu = \mathrm{tr}'_\mu : \quad L_{r'}(\Gamma,\mu) \hookrightarrow B^{-s}_{p',q'}(\mathbb{R}^n), \tag{7.37}$$

where p, q, s, r have the same meaning as in the above definition and

$$\frac{1}{r} + \frac{1}{r'} = \frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1. \tag{7.38}$$

This follows from the dual pairings

$$\big(L_r(\Gamma,\mu)\big)' = L_{r'}(\Gamma,\mu) \quad \text{and} \quad \big(B^s_{pq}(\mathbb{R}^n)\big)' = B^{-s}_{p',q'}(\mathbb{R}^n) \tag{7.39}$$

with $q<\infty$ and $\Big(\mathring{B}^s_{p\infty}(\mathbb{R}^n)\Big)' = B^{-s}_{p',1}(\mathbb{R}^n)$ according to [Triβ, pp. 178, 180].

Remark 7.14. It might be desirable to define $\mathrm{tr}_\mu f \in L_r(\Gamma,\mu)$ with $f \in B^s_{pq}(\mathbb{R}^n)$ more explicitly, if it exists. There are several possibilities. One can use a refined version of the theory of Lebesgue points. This has been described in [Triϵ, pp. 260/261] with a reference to [AdH96]. But it seems to be more effective to expand $f \in B^s_{pq}(\mathbb{R}^n)$ in wavelet frames or wavelet bases, for example as in (1.116), and to ask whether its restriction to Γ converges absolutely in $L_r(\Gamma,\mu)$. This applies also to $B^s_{p\infty}(\mathbb{R}^n)$. We follow this way later on.

Remark 7.15. If $r=1$ in (7.34), hence

$$\mathrm{tr}_\mu : \quad B^s_{pq}(\mathbb{R}^n) \hookrightarrow L_1(\Gamma,\mu), \tag{7.40}$$

and $1<p<\infty$, $1 \le q < \infty$, $s>0$, then we have the elegant Theorem 1.174. In particular one gets for tr_μ in (7.40) by this assertion and its proof,

$$\|\mathrm{tr}_\mu\| \sim \Bigg(\sum_{j=0}^\infty 2^{-j(s-n/p)q'} \Big(\sum_{m\in\mathbb{Z}^n} \mu(Q_{jm})^{p'} \Big)^{q'/p'} \Bigg)^{1/q'} \tag{7.41}$$

(with the usual modification if $q'=\infty$) where the equivalence constants are independent of μ with (7.32). Only the cases with $0<s\le n/p$ are of interest. Otherwise, hence $s>n/p$, the right-hand side of (7.41) is equivalent to $\mu(\mathbb{R}^n)$. We are interested here in (7.34) with $1<p=r<\infty$. Then there is no such decisive answer as for tr_μ in (7.40). As for (7.34) with $F^s_{pq}(\mathbb{R}^n)$ in place of $B^s_{pq}(\mathbb{R}^n)$ we refer to [Triϵ, pp. 125–127] for some implicit characterisations. On the other hand, if tr_μ in (7.34) exists for some $r>1$, then it follows from $L_r(\Gamma,\mu) \hookrightarrow L_1(\Gamma,\mu)$

that also tr_μ in (7.40) exists and, hence, that the right-hand side of (7.41) is finite (necessary condition). To find effective sufficient conditions we put

$$\mu_j = \sup_{m \in \mathbb{Z}^n} \mu(Q_{jm}), \qquad j \in \mathbb{N}_0, \tag{7.42}$$

for measures μ with (7.32). We have

$$\Big(\sum_{m \in \mathbb{Z}^n} \mu(Q_{jm})^{p'} \Big)^{1/p'} \le \mu_j^{\frac{p'-1}{p'}} \Big(\sum_{m \in \mathbb{Z}^n} \mu(Q_{jm}) \Big)^{1/p'} \sim \mu_j^{1/p} \, \mu(\mathbb{R}^n)^{1/p'}. \tag{7.43}$$

If μ is isotropic then (7.43) is an equivalence. But in any case if one substitutes in (7.41) the left-hand side of (7.43) by the right-hand side then one gets the following sufficient condition for the existence of tr_μ in (7.34) with $p = r$. As before $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$. We can incorporate $B^s_{p\infty}(\mathbb{R}^n)$ in (7.34) by the direct approach we are using now.

Theorem 7.16. *Let* $1 < p < \infty$, $1 \le q \le \infty$, $s > 0$. *Let* μ *be a Radon measure according to* (7.32).

(i) *Then the trace operator* tr_μ,

$$\mathrm{tr}_\mu : \quad B^s_{pq}(\mathbb{R}^n) \hookrightarrow L_p(\Gamma, \mu) \tag{7.44}$$

exists if

$$\Big(\sum_{j=0}^{\infty} 2^{-jq'(s-n/p)} \, \mu_j^{q'/p} \Big)^{1/q'} < \infty. \tag{7.45}$$

Furthermore there is a positive constant c *such that for all admitted* μ,

$$\|\mathrm{tr}_\mu\| \le c \Big(\sum_{j=0}^{\infty} 2^{-jq'(s-n/p)} \mu_j^{q'/p} \Big)^{1/q'} \tag{7.46}$$

(*usual modification if* $q' = \infty$). *If, in addition,* $q > 1$, *then* tr_μ *is compact.*

(ii) *Then the identification operator* id_μ,

$$\mathrm{id}_\mu : \quad L_p(\Gamma, \mu) \hookrightarrow B^{-s}_{pq}(\mathbb{R}^n), \tag{7.47}$$

according to (7.36) *exists if*

$$\Big(\sum_{j=0}^{\infty} 2^{-jq(s-n/p')} \mu_j^{q/p'} \Big)^{1/q} < \infty. \tag{7.48}$$

Furthermore there is a positive number c such that for all admitted μ,

$$\|\mathrm{id}_\mu\| \le c \left(\sum_{j=0}^{\infty} 2^{-jq(s-n/p')} \mu_j^{q/p'} \right)^{1/q} \tag{7.49}$$

(*usual modification if $q=\infty$*). *If, in addition, $q<\infty$, then* id_μ *is compact.*

Proof. Step 1. We expand $f \in B^s_{pq}(\mathbb{R}^n)$ according to the wavelet frame (1.116), based on (1.117) and Corollary 1.42, hence

$$f = \sum_{\beta \in \mathbb{N}_0^n} f_\beta, \qquad f_\beta(x) = \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda^\beta_{jm}(f) \cdot (\beta\text{-qu})_{jm}(x) \tag{7.50}$$

with $\|\lambda(f)\,|b_{pq}\|_\varrho < \infty$ for some $\varrho \ge 0$. Let χ_{jm} be the characteristic function of the ball centred at $2^{-j-J}m$ and of radius 2^{-j}. Then it follows by (1.107) that

$$\begin{aligned} &\|f_\beta\,|L_p(\Gamma,\mu)\| \\ &\le c \sum_{j=0}^{\infty} \left(\int_\Gamma \sum_m |\lambda^\beta_{jm}(f)|^p \, 2^{-j(s-n/p)p} \chi_{jm}(\gamma)\, \mu(\mathrm{d}\gamma) \right)^{1/p} \\ &\le c \sum_{j=0}^{\infty} 2^{-j(s-n/p)} \mu_j^{1/p} \left(\sum_m |\lambda^\beta_{jm}(f)|^p \right)^{1/p} \\ &\le c \left(\sum_{j=0}^{\infty} 2^{-j(s-n/p)q'} \mu_j^{q'/p} \right)^{1/q'} \left(\sum_{j=0}^{\infty} \left(\sum_m |\lambda^\beta_{jm}(f)|^p \right)^{q/p} \right)^{1/q}. \end{aligned} \tag{7.51}$$

By (1.64), (1.108) and (1.117) one gets for some $\varrho > 0$ and $c > 0$,

$$\|f_\beta\,|L_p(\Gamma,\mu)\| \le c \left(\sum_{j=0}^{\infty} 2^{-j(s-n/p)q'} \mu_j^{q'/p} \right)^{1/q'} 2^{-\varrho|\beta|} \|f\,|B^s_{pq}(\mathbb{R}^n)\|. \tag{7.52}$$

Summation over $\beta \in \mathbb{N}_0^n$ results in (7.46) including $B^s_{p\infty}(\mathbb{R}^n)$ and $\mathring{B}^s_{p\infty}(\mathbb{R}^n)$.

Step 2. We prove the compactness of tr_μ if $q>1$. Let $B \in \mathbb{N}$, $N \in \mathbb{N}$ and

$$\mathrm{tr}^{B,N}_\mu f = \sum_{|\beta| \le B} \sum_{j \le N} \sum_{m \in \mathbb{Z}^n}{}^{\Gamma} \lambda^\beta_{jm}(f) \cdot (\beta\text{-qu})_{jm} \tag{7.53}$$

where for given j the last summation is restricted to those $m \in \mathbb{Z}^n$ such that the support of $(\beta\text{-qu})_{jm}$ has a non-empty intersection with Γ. In particular, $\mathrm{tr}^{B,N}_\mu$ is an operator of finite rank. Then one gets by the above arguments for $f \in B^s_{pq}(\mathbb{R}^n)$ having norm of at most 1 that

$$\|(\mathrm{tr}_\mu - \mathrm{tr}_\mu^{B,N}) f \,|L_p(\Gamma,\mu)\| \\ \le c \sum_{|\beta|>B} 2^{-\varrho|\beta|} + c \left(\sum_{|\beta|\le B} 2^{-\varrho|\beta|} \right) \cdot \left(\sum_{j\ge N} 2^{-j(s-n/p)q'} \mu_j^{q'/p} \right)^{1/q'} . \tag{7.54}$$

Since $q' < \infty$ the right-hand side tends to zero if $B \to \infty$ and $N \to \infty$. Hence tr_μ is compact. This proves part (i).

Step 3. As for part (ii) we rely on the duality (7.37), hence

$$\mathrm{id}_\mu : \quad L_{p'}(\Gamma,\mu) \hookrightarrow B^{-s}_{p'q'}(\mathbb{R}^n) \tag{7.55}$$

where we used in case of $q = \infty$, hence $q' = 1$, the indicated modification at the end of Remark 7.13. Replacing p', q' by p, q one gets part (ii) including the compactness assertion. □

Remark 7.17. The above theorem is a combination of corresponding assertions in [Tri04c, Tri04e].

7.1.4 Traces: isotropic measures

We continue our considerations about isotropic measures from Section 1.15. A non-negative function h on the unit interval $[0,1]$ is called strictly increasing if $h(t_1) < h(t_2)$ for $0 \le t_1 < t_2 \le 1$. Let $h_j = h(2^{-j})$ for $j \in \mathbb{N}_0$. Our use of $\sim$ has been explained in Remark 1.98. Again a ball in $\mathbb{R}^n$ centred at $x \in \mathbb{R}^n$ and of radius $r > 0$ is denoted by $B(x,r)$.

Definition 7.18. *Let μ be a Radon measure in $\mathbb{R}^n$ according to* (7.32).

(i) *Then μ is called isotropic if there is a continuous strictly increasing function h on the interval* $[0,1]$ *with $h(0) = 0$, $h(1) = 1$, and*

$$\mu\,(B(\gamma,r)) \sim h(r) \quad \text{with} \quad \gamma \in \Gamma \quad \text{and} \quad 0 < r < 1 \tag{7.56}$$

(*where the equivalence constants are independent of γ and r*).

(ii) *The isotropic measure μ according to part* (i) *is called strongly isotropic if there is a number $k \in \mathbb{N}$ such that*

$$2h_{j+k} \le h_j \qquad \text{for all} \quad j \in \mathbb{N}_0. \tag{7.57}$$

Remark 7.19. This coincides essentially with the parts (i) and (ii) of Definition 1.151 (recall that Chapter 1 on the one hand and the rest of this book on the other hand are considered to be self-contained as far as basic definitions are concerned). Otherwise we refer to Section 1.15.1 where one finds further information, assertions, examples and references. In particular Theorem 1.155 clarifies which functions h generate isotropic measures in $\mathbb{R}^n$. Furthermore according to Proposition 1.153 any isotropic measure is doubling. The constants in the equivalence (7.41) and in the corresponding estimates in Theorem 7.16 are independent of μ with (7.32). But now we give up this type of independence. In other words, constants may now depend on the chosen isotropic measure μ.

Proposition 7.20. *Let μ be an isotropic Radon measure according to Definition* 7.18 *with the generating function h.*

(i) *Then μ is strongly isotropic if, and only if,*

$$\sum_{j\geq J} h_j \sim h_J \qquad \textit{for all} \quad J \in \mathbb{N}_0. \tag{7.58}$$

(ii) *Then μ is strongly isotropic if, and only if,*

$$\sum_{j\leq J} h_j^{-1} \sim h_J^{-1} \qquad \textit{for all} \quad J \in \mathbb{N}_0. \tag{7.59}$$

Proof. Step 1. Let μ be strongly isotropic. Then it follows from (7.57) for $l \in \mathbb{N}_0$,

$$h_{J+lk} \leq 2^{-l} h_J, \qquad J \in \mathbb{N}_0, \tag{7.60}$$

and with $J - lk \in \mathbb{N}_0$,

$$h^{-1}_{J-lk} \leq 2^{-l} h_J^{-1}, \qquad J \in \mathbb{N}_0. \tag{7.61}$$

Now one gets (7.58) from (7.60) and the monotonicity of h_j. Similarly for (7.59) where (7.61) covers all terms (using again the monotonicity of h_j) with the possible exception of the first k terms. But they can be incorporated by $h \sim h^*$ and (1.497).

Step 2. Assume that we have (7.58) and that for some $J \in \mathbb{N}_0$ and some $L \in \mathbb{N}$,

$$2h_{J+l} \geq h_J \qquad \text{for} \quad l = 0, \ldots, L-1. \tag{7.62}$$

Then

$$L\, h_J \leq 2 \sum_{m=0}^{\infty} h_{J+m} \leq c\, h_J. \tag{7.63}$$

Hence $L \leq c$. Then (7.57) with $k = L > c$ follows from the monotonicity of h_j.

Step 3. Assume that we have (7.59) and that for some $L \in \mathbb{N}$ and some $J \geq L$,

$$2h^{-1}_{J-l} \geq h_J^{-1} \qquad \text{for} \quad l = 0, \ldots, L-1. \tag{7.64}$$

Then

$$L\, h_J^{-1} \leq 2 \sum_{m=0}^{J} h_m^{-1} \leq c h_J^{-1}. \tag{7.65}$$

Now one obtains (7.57) as above. □

We will use this proposition in the theorem below. But first we specialize the assertions of Section 7.1.3 to arbitrary isotropic measures. Recall again $\frac{1}{p} + \frac{1}{p'} = 1$ if $1 \le p \le \infty$.

Corollary 7.21. *Let μ be an isotropic Radon measure according to Definition 7.18 with the generating function h. Let*

$$1 < p < \infty, \quad 1 \le q < \infty, \quad 1 \le r \le p \quad \text{and} \quad s > 0. \tag{7.66}$$

Then

$$\mathrm{tr}_\mu : \quad B^s_{pq}(\mathbb{R}^n) \hookrightarrow L_r(\Gamma, \mu) \tag{7.67}$$

exists if, and only if,

$$\left(\sum_{j=0}^{\infty} 2^{-jq'(s-\frac{n}{p})} \, h_j^{q'/p} \right)^{1/q'} < \infty \tag{7.68}$$

(*with the usual modification if $q' = \infty$*). *If, in addition, $q > 1$, then* tr_μ *is compact.*

Proof. If $r = 1$ then one has the necessary and sufficient condition (7.41). In case of isotropic measures this coincides with (7.45) and (7.68). This follows from (7.43). Then the above assertion is a consequence of

$$L_p(\Gamma, \mu) \hookrightarrow L_r(\Gamma, \mu) \hookrightarrow L_1(\Gamma, \mu)$$

and Theorem 7.16. □

It is the main aim of this Section 7.1.4 to have a closer look at the compactness of tr_μ where $1 < p = q = r < \infty$ in dependence on (a qualified version) of (7.68). Only $0 < s \le n/p$ is of interest. Otherwise, hence $s > n/p$, (7.68) is always satisfied. As before we abbreviate

$$B^s_p(\mathbb{R}^n) = B^s_{pp}(\mathbb{R}^n), \qquad 0 < p \le \infty, \quad s \in \mathbb{R}. \tag{7.69}$$

Furthermore, let H be the inverse function of h, generating an isotropic measure according to Definition 7.18,

$$h(t) = \tau \quad \text{if, and only if,} \quad H(\tau) = t, \quad \text{where} \quad 0 \le t \le 1, \quad 0 \le \tau \le 1. \tag{7.70}$$

We express the degree of compactness of tr_μ in terms of the approximation numbers $a_k(\mathrm{tr}_\mu)$ according to Definition 4.43(ii) with $A = B^s_p(\mathbb{R}^n)$ and $B = L_p(\Gamma, \mu)$. Further information about approximation numbers may be found in Section 1.10. Again $\frac{1}{p} + \frac{1}{p'} = 1$ if $1 \le p \le \infty$.

Theorem 7.22. *Let $1 < p < \infty$ and $0 < s \le n/p$. Let μ be a strongly isotropic Radon measure in $\mathbb{R}^n$ according to Definition 7.18(ii) with the generating function h and the inverse function H, satisfying*

$$\sum_{j \ge J} 2^{-jp'(s-\frac{n}{p})} h_j^{p'/p} \sim 2^{-Jp'(s-\frac{n}{p})} h_J^{p'/p}, \qquad J \in \mathbb{N}, \tag{7.71}$$

(*where the equivalence constants are independent of* J). *Then the operator* tr_μ*,*

$$\mathrm{tr}_\mu : \quad B^s_p(\mathbb{R}^n) \hookrightarrow L_p(\Gamma,\mu), \tag{7.72}$$

is compact. Let $a_k = a_k(\mathrm{tr}_\mu)$ *be the corresponding approximation numbers. Then*

$$a_k \sim k^{-1/p}\, H\left(k^{-1}\right)^{s-\frac{n}{p}}, \qquad k \in \mathbb{N}. \tag{7.73}$$

Proof. Step 1. By Corollary 7.21 the operator tr_μ is compact. We prove that there is a number $c>0$ such that

$$a_k(\mathrm{tr}_\mu) \le c k^{-1/p}\, H\left(k^{-1}\right)^{s-\frac{n}{p}}, \qquad k \in \mathbb{N}. \tag{7.74}$$

Again we use the wavelet expansion (7.50). Let for fixed $\beta \in \mathbb{N}^n_0$,

$$\mathrm{tr}^\beta_\mu f = \sum_{j=0}^\infty \sum_{m\in\mathbb{Z}^n} \lambda^\beta_{jm}(f)\cdot(\beta\text{-qu})_{jm} \tag{7.75}$$

and for $N \in \mathbb{N}$,

$$\mathrm{tr}^{\beta,N}_\mu f = \sum_{j\le N} \sum_{m\in\mathbb{Z}^n}{}^{\Gamma} \lambda^\beta_{jm}(f)\cdot(\beta\text{-qu})_{jm}, \tag{7.76}$$

where the second sum in (7.76) has the same meaning as in (7.53). Then it follows as in (7.54) for $f \in B^s_p(\mathbb{R}^n)$ having norm of at most 1 that

$$\begin{aligned} \|(\mathrm{tr}^\beta_\mu - \mathrm{tr}^{\beta,N}_\mu) f\,|L_p(\Gamma,\mu)\| &\le c\,2^{-\varrho|\beta|}\left(\sum_{j>N} 2^{-j(s-\frac{n}{p})p'}\, h_j^{p'/p}\right)^{1/p'} \\ &\le c'\,2^{-\varrho|\beta|}\,2^{-N(s-\frac{n}{p})}\,h_N^{1/p}, \end{aligned} \tag{7.77}$$

where we used (7.71). Here $\varrho>0$ is at our disposal and c, c' are independent of $\beta \in \mathbb{N}^n_0$ and $N \in \mathbb{N}_0$ (but may depend on ϱ). Since μ is strongly isotropic (and by Proposition 1.153 also doubling) it follows from (7.59) that

$$\text{rank }(\mathrm{tr}^{\beta,N}_\mu) \sim \sum_{j\le N} h_j^{-1} \sim h_N^{-1}. \tag{7.78}$$

Then one obtains from (7.77) and (7.78) that there are two positive numbers c and c' such that for all $\beta \in \mathbb{N}^n_0$ and $N \in \mathbb{N}_0$,

$$a_{ch_N^{-1}}(\mathrm{tr}^\beta_\mu) \le c'\,2^{-\varrho|\beta|}\,2^{-N(s-\frac{n}{p})}\,h_N^{1/p}. \tag{7.79}$$

By (1.497) and $h \sim h^*$ it follows that $h(2^{-j-1}) \sim h(2^{-j})$ where $j \in \mathbb{N}_0$. Hence for $k \in \mathbb{N}$ there are numbers $N_k \in \mathbb{N}$ such that

$$h(2^{-N_k})^{-1} \sim k \quad \text{with} \quad N_1 \le N_2 \le \cdots \le N_k \to \infty \tag{7.80}$$

if $k \to \infty$ (with equivalence constants which are independent of k). Inserted in (7.79) one obtains that

$$a_k(\mathrm{tr}^\beta_\mu) \le c\, 2^{-\varrho|\beta|}\, 2^{-N_k(s-\frac{n}{p})}\, k^{-1/p}, \qquad k \in \mathbb{N}. \tag{7.81}$$

Let $\varepsilon > 0$. For given $k \in \mathbb{N}$ we apply (7.81) to $k_\beta \in \mathbb{N}$ with $k_\beta \sim 2^{-\varepsilon|\beta|}k$ (this means 1 if the latter number is between 0 and 1). Then it follows from (7.81) and the additivity property for approximation numbers according to Proposition 1.89 that

$$\begin{aligned} a_{ck}(\mathrm{tr}_\mu) &\le \sum_{\beta\in\mathbb{N}^n_0} a_{k_\beta}(\mathrm{tr}^\beta_\mu) \\ &\le c' \sum_{\beta\in\mathbb{N}^n_0} 2^{-\varrho|\beta|}\, 2^{\varepsilon|\beta|/p}\, 2^{-N_{k_\beta}(s-\frac{n}{p})}\, k^{-1/p} \\ &\le c''\, 2^{-N_k(s-\frac{n}{p})}\, k^{-1/p}, \end{aligned} \tag{7.82}$$

where we used $s \le n/p$ and $N_k \ge N_{k_\beta}$ according to (7.80). Now (7.74) follows from (7.82) and (7.80). As for the replacement of ck by k one may also consult Corollary 7.24 and its proof below.

Step 2. We prove that for two suitable positive numbers c and c',

$$a_{ch_j^{-1}}(\mathrm{tr}_\mu) \ge c'\, 2^{-j(s-\frac{n}{p})}\, h_j^{1/p}, \qquad j \in \mathbb{N}_0. \tag{7.83}$$

Recall $h_j = h(2^{-j})$. Using $h_j \sim h_{j+1}$, its counterpart for H as explicitly stated in (7.96) below, and (7.80) then (7.83) proves the converse of (7.74). Hence it remains to justify (7.83). First we remark that

$$\mathbb{R}^n = \bigcup_{r=1}^{2^n} R_{j,r}\,, \qquad j \in \mathbb{N}_0, \tag{7.84}$$

where each $R_{j,r}$ is the union of closed cubes of side-length 2^{-j} having pairwise distance of at most 2^{-j}. Then one has

$$\mu\,(\Gamma \cap R_{j,r}) \ge 2^{-n}\,\mu(\Gamma) \quad \text{for at least one } r,$$

denoted temporarily by Γ_j. Now one finds points

$$\gamma^{j,l} \in \Gamma_j \quad \text{and disjoint balls} \quad B\left(\gamma^{j,l}, c2^{-j}\right) \tag{7.85}$$

where $l = 1, \ldots, M_j$ with $M_j \sim h_j^{-1}$. Here $c > 0$ is sufficiently small and in each cube of Γ_j is at most one point $\gamma^{j,l}$. Let $\varkappa$ be a non-trivial non-negative C^∞ function in $\mathbb{R}^n$ supported by the unit ball and let

$$f_j(x) = \sum_{l=1}^{M_j} c_{jl}\, 2^{-j(s-\frac{n}{p})}\, \varkappa\left(2^j(x-\gamma^{j,l})\right), \quad c_{jl} \in \mathbb{C}, \quad x \in \mathbb{R}^n. \tag{7.86}$$

We apply the localisation property for $B^s_p(\mathbb{R}^n)$ according to [ET96, Section 2.3.2, pp. 35–36] (there we assumed that $\gamma^{j,l}$ are lattice points belonging to $2^{-j}\mathbb{Z}^n$, but this is immaterial as follows from the proof given there). Then

$$\|f_j \,|B^s_p(\mathbb{R}^n)\| \sim \left(\sum_{l=1}^{M_j} |c_{jl}|^p \right)^{1/p} \tag{7.87}$$

and

$$\|f_j \,|L_p(\Gamma,\mu)\| \sim 2^{-j(s-\frac{n}{p})}\, h_j^{1/p} \left(\sum_{l=1}^{M_j} |c_{jl}|^p \right)^{1/p}. \tag{7.88}$$

All equivalence constants are independent of $j \in \mathbb{N}_0$. Hence

$$\|f_j \,|L_p(\Gamma,\mu)\| \sim 2^{-j(s-\frac{n}{p})}\, h_j^{1/p} \quad \text{if} \quad \|f_j \,|B^s_p(\mathbb{R}^n)\| \sim 1. \tag{7.89}$$

Let T be a linear operator,

$$T: \quad B^s_p(\mathbb{R}^n) \hookrightarrow L_p(\Gamma,\mu) \quad \text{with} \quad \operatorname{rank} T \le M_j - 1. \tag{7.90}$$

Then one finds a function f_j according to (7.86) with norm 1 in $B^s_p(\mathbb{R}^n)$ and $Tf_j = 0$. Now we have by (7.89) and (4.227),

$$a_{M_j}(\mathrm{tr}_\mu) = \inf \|\mathrm{tr}_\mu - T\| \ge c\, 2^{-j(s-\frac{n}{p})}\, h_j^{1/p}, \qquad j \in \mathbb{N}_0, \tag{7.91}$$

where the infimum is taken over all T with (7.90). Here c is a positive constant which is independent of j. Now (7.83) follows from (7.91) and $M_j \sim h_j^{-1}$. □

Remark 7.23. We followed essentially [Tri04c]. In the preceding chapters of this book but also in [ET96, Triδ, Triϵ] we expressed the degree of compactness of (embedding) operators preferably in terms of entropy numbers. One may ask what can be said about the entropy numbers of tr_μ in (7.72) expecting that they behave in the same way as the approximation numbers in (7.73). At least a corresponding estimate can be obtained quite easily. As for definitions and properties of entropy numbers we refer to Definition 4.43 and Section 1.10.

Corollary 7.24. *Let the hypotheses of Theorem* 7.22 *be satisfied. Let* a_k *be the approximation numbers of* tr_μ *and let* e_k *be the corresponding entropy numbers. Then*

$$a_{2^j} \sim a_{2^{j-1}}, \qquad j \in \mathbb{N} \tag{7.92}$$

(*with equivalence constants which are independent of* j) *and for some* $c > 0$,

$$e_k \le c\, a_k \sim k^{-1/p} H(k^{-1})^{s-\frac{n}{p}}, \qquad k \in \mathbb{N}. \tag{7.93}$$

Proof. First we remark that for any b with $0 < b < 1$ there is a $j \in \mathbb{N}$ such that

$$2h_{j+k} \le h_j < b \le h_{j-1}, \tag{7.94}$$

where the left-hand side comes from (7.57). Then

$$H(b/2) \ge 2^{-j-k} \ge 2^{-k-1} H(b). \tag{7.95}$$

In particular,

$$H(2^{-j}) \sim H(2^{-j+1}), \qquad j \in \mathbb{N}. \tag{7.96}$$

Now (7.92) follows from (7.73), (7.96). Based on (7.92) one can apply [ET96, Section 1.3.3, p. 15] resulting in the inequality in (7.93). □

Example 7.25. In the references in Remarks 1.154, 1.156 related to the criterion in Theorem 1.155 one finds lists and examples of functions h generating isotropic measures according to Definition 7.18. One may ask which of these functions generate strongly isotropic measures and under which conditions Theorem 7.22 can be applied. This will not be done. We restrict ourselves to the most distinguished h-sets, these are the d-sets with $0 < d < n$. Then Γ is a compact set in $\mathbb{R}^n$,

$$\mu\left(B(\gamma, r)\right) \sim r^d, \quad \gamma \in \Gamma, \quad 0 < r < 1, \tag{7.97}$$

and one may identify μ with $\mathcal{H}^d|\Gamma$, the restriction of the Hausdorff measure $\mathcal{H}^d$ in $\mathbb{R}^n$ to Γ. Then one has $h(t) = t^d$ and $H(t) = t^{1/d}$ where $0 \le t \le 1$. Let

$$1 < p < \infty, \qquad \frac{n-d}{p} < s \le \frac{n}{p}\,. \tag{7.98}$$

Then (7.71) is satisfied and one gets both for the approximation numbers a_k and the entropy numbers e_k of tr_μ in (7.72) that

$$e_k \sim a_k \sim k^{-1/p}\, k^{-\frac{1}{d}(s-\frac{n}{p})} = k^{-\frac{1}{d}(s-\frac{n-d}{p})}, \qquad k \in \mathbb{N}. \tag{7.99}$$

This follows for a_k from (7.73), whereas the corresponding assertion for the entropy numbers e_k is covered by [Triδ, Sections 20.2, 20.6, pp. 159/166]. At least in this case one has equivalence in (7.93). An anisotropic counterpart of this theory may be found in [Tam06].

7.2 Characteristics

7.2.1 Characteristics of measures

We return now in greater detail to fractal characteristics of measures as considered so far in Sections 1.14 and 1.17.1. In particular we prove Theorem 1.167. Again let μ be a Radon measure in $\mathbb{R}^n$ with

$$\Gamma = \mathrm{supp}\,\mu \quad \text{compact}, \quad 0 < \mu(\mathbb{R}^n) < \infty, \quad |\Gamma| = 0, \tag{7.100}$$

and let μ^λ_{pq} with $0 < p \le \infty$, $0 < q \le \infty$, $\lambda \in \mathbb{R}$, be given by (7.2) (obviously, the assumption that Γ in (7.1) is a subset of the unit ball is immaterial and of no use in what follows). Let $\mu^\lambda_p = \mu^\lambda_{p\infty}$, hence

$$\mu^\lambda_p = \begin{cases} \sup_{j\in\mathbb{N}_0} 2^{j\lambda} \Big(\sum_{m\in\mathbb{Z}^n} \mu(Q_{jm})^p \Big)^{1/p} & \text{if } 0 < p < \infty, \\ \sup_{j\in\mathbb{N}_0, m\in\mathbb{Z}^n} 2^{j\lambda} \mu(Q_{jm}) & \text{if } p = \infty, \end{cases} \tag{7.101}$$

where Q_{jm} are the same cubes as in Definition 7.1. Then

$$\mu_p^{\sigma_p^-} < \infty \qquad \text{where} \quad \sigma_p^- = \min(0, n - \frac{n}{p}). \tag{7.102}$$

This follows from (7.5) with a reference to Proposition 1.127.

Definition 7.26. *Let μ be a Radon measure in $\mathbb{R}^n$ with* (7.100). *Let $0 \le t = 1/p < \infty$. Then*

$$\lambda_\mu(t) = \sup \big\{ \lambda : \ \mu^\lambda_p < \infty \big\} \tag{7.103}$$

are multifractal characteristics of μ and

$$s_\mu(t) = \sup \big\{ s : \ \mu \in B^s_{p\infty}(\mathbb{R}^n) \big\} \tag{7.104}$$

are Besov characteristics of μ.

Remark 7.27. This coincides with Definition 1.165 (recall that Chapter 1 on the one hand and the rest of this book on the other hand should be readable independently). By elementary embeddings one can replace $B^s_{p\infty}(\mathbb{R}^n)$ in (7.104) by $B^s_{pq}(\mathbb{R}^n)$ for some q or all q with $0 < q \le \infty$. By (7.102) and $2^{-jn} \preceq \sup_m \mu(Q_{jm})$ it follows that

$$\min(0, n(1-t)) \le \lambda_\mu(t) \le n, \ \text{where } 0 \le t < \infty \quad \text{and} \quad \lambda_\mu(1) = 0. \tag{7.105}$$

Since μ is singular we have $s_\mu(1) = 0$ as a consequence of (1.418). We refer also to Figure 1.17.1.

Recall that increasing means non-decreasing.

Theorem 7.28. *Let μ be a Radon measure in $\mathbb{R}^n$ with* (7.100). *Let $\lambda_\mu(t)$ and $s_\mu(t)$ be the characteristics according to Definition* 7.26. *Then $s_\mu(t)$ with $0 \le t < \infty$ is a continuous increasing concave function in the (t,s)-diagram in Figure* 1.17.1 *with a slope of at most n. Furthermore, $s_\mu(1) = \lambda_\mu(1) = 0$,*

$$n(t-1) \le \lambda_\mu(t) + n(t-1) = s_\mu(t) \le 0 \quad \textit{if} \quad 0 \le t \le 1, \tag{7.106}$$

and

$$0 \le \lambda_\mu(t) + n(t-1) \le s_\mu(t) \le n(t-1) \quad \textit{if} \quad t \ge 1. \tag{7.107}$$

Proof. So far we know $\lambda_\mu(1) = s_\mu(1) = 0$ and the left-hand sides of (7.106), (7.107). Since Γ in (7.100) is compact it follows from Theorem 1.10 and Hölder's inequality that

$$\mu \in B^s_{p_1,\infty}(\mathbb{R}^n) \quad \text{if} \quad \mu \in B^s_{p_0,\infty}(\mathbb{R}^n) \text{ and } p_1 \le p_0.$$

Hence $s_\mu(t)$ is increasing. If

$$\mu \in B^{s_0}_{p_0,\infty}(\mathbb{R}^n) \quad \text{and} \quad \mu \in B^{s_1}_{p_1,\infty}(\mathbb{R}^n) \tag{7.108}$$

then it follows again by Hölder's inequality and (2.12) that for $0 \le \theta \le 1$,

$$\mu \in B^s_{p\infty}(\mathbb{R}^n) \quad \text{with} \quad s = (1-\theta)s_0 + \theta s_1, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}. \tag{7.109}$$

Hence $s_\mu(t)$ is concave in the (t,s)-diagram in Figure 1.17.1. If $0 \le t \le 1$ then $s_\mu(t) \le 0$. Now the equality in (7.106) is a consequence of (7.9). The proof is based on (7.13). If $s \ge 0$ then one cannot apply Corollary 1.12 any longer, but Theorem 1.10. However this is sufficient for the absolute value of the right-hand side of (7.13) which results in the middle inequality in (7.107). It follows that $s_\mu(t)$ is continuous. By well-known limiting embeddings, [Triβ, Section 2.7.1], or (7.159) below, $s_\mu(t)$ has a slope of at most n. This assertion extends to $t > 1$, including the right-hand side of (7.107). One may again consult Figure 1.17.1. □

Remark 7.29. We followed essentially [Tri03b], complemented by [Tri03c, Tri04f]. So far it is unclear whether the equality in (7.106) for $0 \le t \le 1$ can always be extended to $t > 1$. Presumably not, but we have no counter-example. As usual in mathematics one turns this unsatisfactory situation into a notation.

Definition 7.30. *A Radon measure μ with* (7.100) *and the characteristics $\lambda_\mu(t)$ and $s_\mu(t)$ according to Definition* 7.26 *is called tame if*

$$s_\mu(t) = \lambda_\mu(t) + n(t-1) \qquad \text{for all} \quad 0 \le t < \infty. \tag{7.110}$$

Remark 7.31. First candidates of measures μ with (7.100) to be checked whether they are tame or not tame are the isotropic measures according to Definition 7.18. This has been done in [Tri04b, p. 190]. We restrict ourselves here to d-sets in $\mathbb{R}^n$, hence

$$\operatorname{supp} \mu = \Gamma \quad \text{compact}, \quad \mu\left(B(\gamma, r)\right) \sim r^d, \quad 0 \le d < n, \tag{7.111}$$

where again $B(\gamma, r)$ is a ball centred at $\gamma \in \Gamma$ and of radius r. Then one has $|\Gamma| = 0$ according to Proposition 1.153(iii). Recall that one may choose $\mu = \mathcal{H}^d|\Gamma$, the restriction of the Hausdorff measure $\mathcal{H}^d$ in $\mathbb{R}^n$ to Γ.

Proposition 7.32. *Let Γ be a d-set according to* (7.111) *with $\mu = \mathcal{H}^d|\Gamma$. Then μ is tame and*

$$s_\mu(t) = \lambda_\mu(t) + n(t-1) = (n-d)(t-1), \qquad 0 \le t < \infty. \tag{7.112}$$

Proof. We have

$$\left(\sum_{m \in \mathbb{Z}^n} \mu(Q_{jm})^p \right)^{1/p} \sim (2^{-jdp} 2^{jd})^{1/p} = 2^{-jd(1-t)} \tag{7.113}$$

and hence by (7.101), (7.103) one obtains that $\lambda_\mu(t) = d(1-t)$. It follows that

$$s_\mu(t) \geq \lambda_\mu(t) + n(t-1) = (n-d)(t-1) \tag{7.114}$$

with equality if $0 \leq t \leq 1$. Since $s_\mu(t)$ is concave one gets (7.112). □

Remark 7.33. As mentioned above for general isotropic measures according to Definition 7.18 the situation is more sophisticated. On the other hand one may ask for subclasses of measures μ with (7.100) which are tame. The geometry of Γ seems to play a role. We give a description of some assertions.

Definition 7.34. *Let μ be a Radon measure in $\mathbb{R}^n$ with* (7.100)*. Then μ is called lacunary if there are two positive numbers c_1 and c_2 and closed cubes Q^l_j where $j \in \mathbb{N}_0$ and $l = 1, \ldots, N_j$, with sides parallel to the axes of coordinates and of side-length $c_1 2^{-j}$ with the following properties:*

(i) *If $j \in \mathbb{N}_0$ and $l \neq m$ then*

$$\text{dist } (Q^l_j, Q^m_j) \geq c_2 2^{-j} \quad \text{and} \quad \Gamma = \bigcap_{j=0}^{\infty} \bigcup_{l=1}^{N_j} Q^l_j. \tag{7.115}$$

(ii) *For any cube Q^l_j with $j \in \mathbb{N}$ and $1 \leq l \leq N_j$ there is a cube Q^m_{j-1} with*

$$Q^l_j \subset Q^m_{j-1} \quad \text{and} \quad \mu(Q^l_j) \sim \mu(Q^m_{j-1}). \tag{7.116}$$

Remark 7.35. Quite obviously, $\bigcup_{l=1}^{N_j} Q^l_j$ is a decreasing sequence of sets converging to Γ. By (7.115) and the porosity property according to (2.119) the set Γ is disconnected and (in a qualified way) porous. One can apply the well-known mass distribution procedure to the above construction generating the Radon measure μ, where (7.116) ensures that the mass distribution from level $j-1$ to level j is not too irregular. Details about the mass distribution procedure may be found in [Fal90, pp. 13–15]. One may also consult [Triδ, Section 4] for a description, further references, and applications to isotropic and anisotropic (self-similar) sets. Furthermore we return to this technique in Example 7.40 below.

Theorem 7.36. *Any lacunary measure as introduced in Definition* 7.34 *is a tame measure according to Definition* 7.30*.*

Remark 7.37. Both the definition of lacunary measures and a proof of Theorem 7.36 may be found in [Tri04f]. The proof is based on a characterisation of $B^s_{pq}(\mathbb{R}^n)$ by some specific local means which will be considered in Section 7.2.2 below. We refer in particular to Remark 7.43.

Example 7.38. One may extend Proposition 7.32 and its proof to $\Gamma = \Gamma_1 \cup \Gamma_2$ where Γ_1 is a d_1-set in $\mathbb{R}^n$ and Γ_2 is a d_2-set in $\mathbb{R}^n$ with $0 \le d_1 < d_2 < n$. One may think about $\Gamma_1 \cap \Gamma_2 = \emptyset$, but this is not necessary since $\mathcal{H}^{d_2}(\Gamma_1) = 0$. Let $\mu = \mathcal{H}^{d_1}|\Gamma_1 + \mathcal{H}^{d_2}|\Gamma_2$. Then μ is tame according to Definition 7.30 and

$$s_\mu(t) = \begin{cases} (n-d_1)(t-1) & \text{if } 0 \le t \le 1, \\ (n-d_2)(t-1) & \text{if } 1 \le t < \infty. \end{cases} \tag{7.117}$$

Remark 7.39. Let $s(t)$ be a function with the properties as described in Theorem 7.28. There is the question whether one finds a measure μ with $s_\mu(t) = s(t)$. First assertions in this direction were given in Theorem 1.199. We return to this point in greater detail in Section 7.2.3. In particular for given $s(t)$ with the properties as in Theorem 7.28 one always finds a compactly supported Radon measure μ with $s_\mu(t) = s(t)$ and $|\text{sing supp } \mu| = 0$. We refer to Remark 7.53. But it is not clear whether $|\text{supp } \mu| = 0$ can be obtained or whether such a measure is tame. Under these circumstances it might be of interest to complement Proposition 7.32 and Example 7.38 by the following more sophisticated tame measure μ and its Besov characteristics $s_\mu(t)$.

Example 7.40. Let Q be the closed cube in $\mathbb{R}^n$, centred at the origin with sides parallel to the axes of coordinates and of side-length 1. Let $\varrho > 0$ and let T_k be contractions in $\mathbb{R}^n$, where $k = 1, \ldots, N$, with $2 \le N \in \mathbb{N}$,

$$T_k : \ \mathbb{R}^n \ni x \mapsto \varrho x + x^k \quad \text{where} \quad x^k \in \mathbb{R}^n, \tag{7.118}$$

such that

$$Q^k = T_k Q \subset Q, \qquad Q^k \cap Q^l = \emptyset \quad \text{if} \quad k \ne l. \tag{7.119}$$

Then $0 < \varrho < 1/2$ and the cubes Q^k have a positive distance from each other. Next we recall how to create a self-similar fractal set Γ. Let

$$TQ = (TQ)^1 = \bigcup_{k=1}^N Q^k, \qquad (TQ)^0 = Q, \tag{7.120}$$

$$(TQ)^l = T\left((TQ)^{l-1}\right) = \bigcup_{1 \le j_r \le N} T_{j_1} \cdots T_{j_l} Q, \qquad l \in \mathbb{N}, \tag{7.121}$$

and

$$\Gamma = (TQ)^\infty = \bigcap_{l \in \mathbb{N}} (TQ)^l = \lim_{l \to \infty} (TQ)^l. \tag{7.122}$$

Quite obviously, the above cubes and Γ fit in the scheme of Definition 7.34, maybe with ϱ^j in place of 2^{-j}. But this is immaterial. Let

$$w = (w_1, \ldots, w_N), \quad w_1 \ge \cdots \ge w_N > 0, \quad \sum_{l=1}^N w_l = 1, \tag{7.123}$$

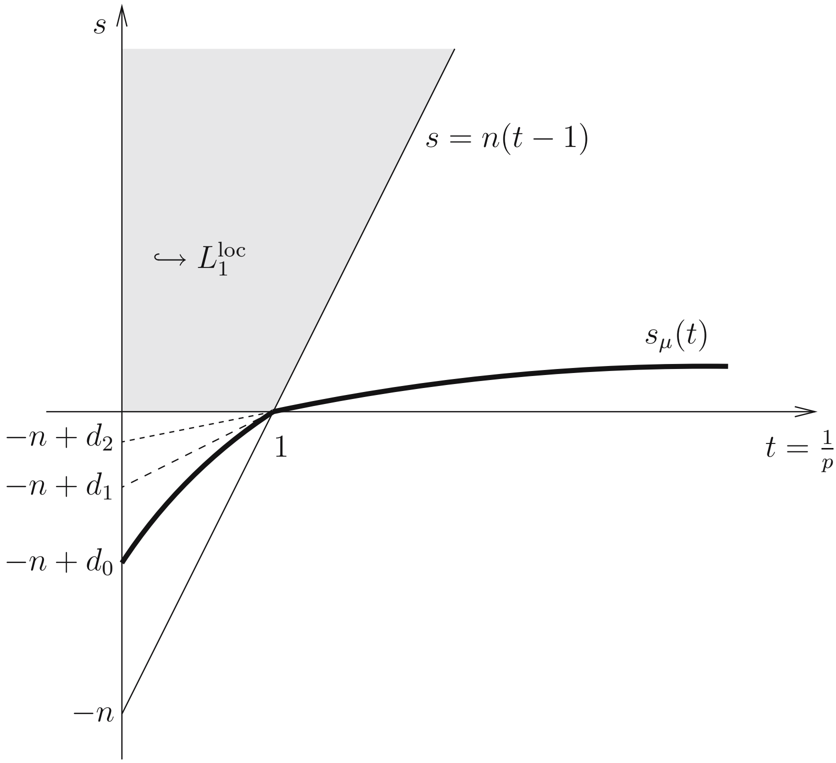

Figure 7.2.1

be a probability. We distribute the unit mass in Q by the mass distribution procedure with respect to the above probability and the above transforms. References were given in Remark 7.35. One gets a Radon measure μ with

$$\text{supp } \mu = \Gamma, \qquad \mu(T_{j_1} \cdots T_{j_l} Q) = w_{j_1} \cdots w_{j_l}. \tag{7.124}$$

Then μ satisfies (7.100). One checks quite easily that μ is lacunary according to Definition 7.34 and hence by Theorem 7.36 tame. In particular we have (7.110) where $\lambda_\mu(t)$ and $s_\mu(t)$ have the same meaning as in Definition 7.26. It comes out that

$$\lambda_\mu(t) = (\log \varrho)^{-1} \log \left(\sum_{l=1}^N w_l^p \right)^{1/p}, \qquad 0 \le t = 1/p < \infty, \tag{7.125}$$

(usual modification if $p = \infty$) and

$$s_\mu(t) = n(t-1) + \lambda_\mu(t), \qquad 0 \le t < \infty. \tag{7.126}$$

We refer to [Tri04f, Theorem 19], where one finds also a proof. One can strengthen (7.126) by

$$\mu \in B^{s_\mu(t)}_{pq}(\mathbb{R}^n) \quad \text{if, and only if,} \quad q = \infty. \tag{7.127}$$

In [Tri04f] we discussed the curve $s_\mu(t)$. In the regular case $w_1 = N^{-1}$ (and hence $w_j = N^{-1}$ for $j = 1, \ldots, N$) it comes out that Γ is a d-set and one gets by (7.112)

$$s_\mu(t) = (n-d)(t-1) \qquad \text{with} \quad d = |\log \varrho|^{-1} \log N, \quad 0 \le t < \infty. \tag{7.128}$$

In the non-regular case, $N^{-1} < w_1 < 1$ one gets a genuinely bent curve $s_\mu(t)$ as shown in Figure 7.2.1 with

$$0 < n - d_2 < s'_\mu(1) = n - d_1 < n - d_0 = -s_\mu(0) < n, \tag{7.129}$$

where

$$d_0 = |\log \varrho|^{-1} |\log w_1|, \qquad d_1 = |\log \varrho|^{-1} \sum_{j=1}^N w_j |\log w_j| \tag{7.130}$$

and

$$s'_\mu(t) \to n - d_2 \quad \text{if } t \to \infty \quad \text{with} \quad d_2 = |\log \varrho|^{-1} \log N. \tag{7.131}$$

Details may be found in [Tri04f].

7.2.2 A digression: Adapted local means

So far we have the multifractal characteristics $\lambda_\mu(t)$ and the Besov characteristics $s_\mu(t)$ for measures μ with (7.100) as introduced in Definition 7.26. According to Definition 7.30 a measure is called *tame* if the equality in (7.106) is valid for all $0 \le t < \infty$. Presumably most of these measures (whatever this means) are not tame (but we have no examples to substantiate this vague opinion) and one may ask for a substitute of (7.110) valid for all measures. Furthermore one would like to extend these considerations to all (compactly supported) distributions. Both will be done in the next Section 7.2.3. To prepare what follows we have a closer look at some characterisations of $B^s_{pq}(\mathbb{R}^n)$ in terms of local means as described in Section 1.4.

Let K be a real non-negative C^∞ function in $\mathbb{R}^n$ with

$$K(x) > 0 \text{ if } |x| < c, \quad K(y) = 0 \text{ if } |y| \ge c, \quad \widehat{K}(0) = 1, \tag{7.132}$$

for some $c > 0$. Let $e_r = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the rth position be the unit vector pointing in the rth direction of the axes of coordinates, where $r = 1, \ldots, n$. Let

$$(\mathbb{D}^1 g)(x) = \frac{1}{4} \sum_{r=1}^n \big(g(x+e_r) - 2g(x) + g(x-e_r) \big), \quad x \in \mathbb{R}^n, \tag{7.133}$$

be the sum of the second symmetric differences and, by iteration,

$$\mathbb{D}^{M+1} g = \mathbb{D}^1 \mathbb{D}^M g, \qquad M \in \mathbb{N}. \tag{7.134}$$

Proposition 7.41. *Let K be the above function, let $M \in \mathbb{N}$ and let $\mathbb{Z}^n_M$ be the rhombohedron*

$$\mathbb{Z}^n_M = \left\{ m \in \mathbb{Z}^n : \sum_{r=1}^n |m_r| \le M \right\}. \tag{7.135}$$

Then

$$(\mathbb{D}^M K)^\wedge(\xi) = (-1)^M \widehat{K}(\xi) \left(\sum_{r=1}^n \sin^2(\xi_r/2) \right)^M, \quad \xi \in \mathbb{R}^n, \tag{7.136}$$

and

$$(\mathbb{D}^M K)(x) = 2^{-2M} \sum_{m \in \mathbb{Z}^n_M} d^M_m \, K(x+m), \quad x \in \mathbb{R}^n, \quad M \in \mathbb{N}_0, \tag{7.137}$$

where $d^M_m \in \mathbb{Z}$.

Proof. Let $M = 1$. Then

$$\begin{aligned} (\mathbb{D}^1 K)^\wedge(\xi) &= \frac{1}{4} \sum_{r=1}^n \left(e^{i\xi_r} - 2 + e^{-i\xi_r} \right) \widehat{K}(\xi) \\ &= \frac{1}{4} \sum_{r=1}^n \left(e^{i\xi_r/2} - e^{-i\xi_r/2} \right)^2 \widehat{K}(\xi) \\ &= - \left(\sum_{r=1}^n \sin^2(\xi_r/2) \right) \widehat{K}(\xi). \end{aligned} \tag{7.138}$$

Iteration gives (7.136). By (7.133), (7.134) one gets (7.137). □

We use the local means (1.41) with K and $\mathbb{D}^M K$ in place of k, hence

$$K(t,f)(x) = \int_{\mathbb{R}^n} K(y)\, f(x+ty)\, \mathrm{d}y = t^{-n} \int_{\mathbb{R}^n} K\left(\frac{y-x}{t} \right) f(y)\, \mathrm{d}y, \tag{7.139}$$

and for $M \in \mathbb{N}$,

$$\begin{aligned} K^M(t,f)(x) &= \int_{\mathbb{R}^n} (\mathbb{D}^M K)(y)\, f(x+ty)\, \mathrm{d}y \\ &= 2^{-2M} \sum_{m \in \mathbb{Z}^n_M} d^M_m \int_{\mathbb{R}^n} K(y+m)\, f(x+ty)\, \mathrm{d}y \\ &= 2^{-2M} \sum_{m \in \mathbb{Z}^n_M} d^M_m \, K(t,f)(x-tm), \end{aligned} \tag{7.140}$$

where $x \in \mathbb{R}^n$, $t > 0$, and $f \in S'(\mathbb{R}^n)$ (usual interpretation). We put $K^0(t,f) = K(t,f)$. Let Q_{jm} be the same cubes as above, Section 2.1.2.

Theorem 7.42. *Let* $0 < p \le \infty$, $0 < q \le \infty$, *and* $s \in \mathbb{R}$. *Let* $K(t, f)$ *and* $K^M(t, f)$ *be the above means based on* (7.132). *Let* $M \in \mathbb{N}_0$ *with* $2M > s$ *and*

$$V^M_{jm} f = \begin{cases} \sup\limits_{x \in Q_{jm}} |K^M(2^{-j}, f)(x)| & \text{if } j \in \mathbb{N},\ m \in \mathbb{Z}^n, \\ \sup\limits_{x \in Q_{jm}} |K(1, f)(x)| & \text{if } j = 0,\ m \in \mathbb{Z}^n. \end{cases} \tag{7.141}$$

Then

$$B^s_{pq}(\mathbb{R}^n) = \big\{ f \in S'(\mathbb{R}^n) :\ \| f \,| B^s_{pq}(\mathbb{R}^n)\|^M < \infty \big\} \tag{7.142}$$

with

$$\| f \,| B^s_{pq}(\mathbb{R}^n)\|^M = \left(\sum_{j=0}^{\infty} 2^{j(s-\frac{n}{p})q} \left(\sum_{m \in \mathbb{Z}^n} (V^M_{jm} f)^p \right)^{q/p} \right)^{1/q} \tag{7.143}$$

(*equivalent quasi-norms, usual modification if* p *and/or* q *is infinite*).

Proof. We apply Theorem 1.10 with $k_0 = K$ and $k = \mathbb{D}^M K$. By (7.136) we have (1.42). Furthermore, if $j \in \mathbb{N}$ then

$$k(2^{-j}, f)(x) \le V^M_{jm} f \le c\, k^*(2^{-j}, f)_a(x), \qquad x \in Q_{jm}, \tag{7.144}$$

where $k^*(2^{-j}, f)_a(x)$ is the maximal function according to (1.43). In particular,

$$\begin{aligned} \| k(2^{-j}, f) \,| L_p(\mathbb{R}^n)\| &\le c \left(\sum_{m \in \mathbb{Z}^n} 2^{-jn} \left(V^M_{jm} f \right)^p \right)^{1/p} \\ &\le c' \, \| k^*(2^{-j}, f)_a \,| L_p(\mathbb{R}^n)\|, \end{aligned} \tag{7.145}$$

(modification if $p = \infty$). Then the above theorem follows from Theorem 1.10 with $a > n/p$. □

Remark 7.43. If $s < 0$ then one may choose $M = 0$ and hence (7.139) with $t = 2^{-j}$ for all $j \in \mathbb{N}_0$ with the non-negative kernels (7.132). If, in addition, μ is a (non-negative) measure then it follows by (7.12), (7.13),

$$\| \mu \,| B^s_{pq}(\mathbb{R}^n)\| \sim \left(\sum_{j=0}^{\infty} 2^{j(s+n-\frac{n}{p})q} \left(\sum_{m \in \mathbb{Z}^n} \mu(Q_{jm})^p \right)^{q/p} \right)^{1/q}, \tag{7.146}$$

which coincides with (7.9). According to Theorem 7.28 and also by the Figures 1.17.1 and 7.2.1, $s < 0$ (and $M = 0$) is a good choice for measures of type (7.100) if $1 \le p \le \infty$. But if $p < 1$ then one needs also spaces $B^s_{pq}(\mathbb{R}^n)$ with $s \ge 0$ (and hence $M \in \mathbb{N}$) and (7.146) is no longer justified. If $M \in \mathbb{N}$ then, based on (7.13), there might be a temptation to introduce differences of measures μ,

$$(\mathbb{D}^M \mu)(Q_{jm}) = 2^{-2M} \sum_{l \in \mathbb{Z}^n_M} d^M_l 2^{jn} \mu(Q_{j,m+l}), \quad j \in \mathbb{N}, \quad m \in \mathbb{Z}^n, \tag{7.147}$$

in analogy to (7.137), (7.140), and to compare

$$K^M(2^{-j}, f)(x) \qquad \text{with} \qquad (\mathbb{D}^M \mu)(Q_{jm}) \quad \text{if} \quad x \in Q_{jm}. \tag{7.148}$$

But the actual situation might be more tricky. It is not clear whether $V^M_{jm}\mu$ for $f = \mu$ with (7.100) can be replaced in the above theorem by $(\mathbb{D}^M \mu)(Q_{jm})$ in generalisation of (7.13). However if the measure μ is lacunary according to Definition 7.34 then it comes out that $K^M(2^{-j}, f)(x)$ in (7.140) has on the relevant part of $\mathbb{R}^n$ only one non-vanishing term. Then the situation is similar to the case of $M = 0$ and one gets Theorem 7.36. We refer for details to [Tri04f, Section 3.1].

7.2.3 Characteristics of distributions

For Radon measures μ with (7.100) we introduced in Definition 7.26 both the multifractal characteristics $\lambda_\mu(t)$ and the Besov characteristics $s_\mu(t)$ with the outcome (7.106), (7.107). We discussed in Section 7.2.1 the somewhat unsatisfactory situation if $t > 1$. We modify now $\lambda_\mu(t)$ by $\lambda^\mu(t)$ so that we always have equality in the middle terms of (7.106), (7.107) (hence $\lambda_\mu(t) = \lambda^\mu(t)$ if $0 \le t \le 1$). In addition we extend these considerations from measures μ with (7.100) to distributions with

$$f \in S'(\mathbb{R}^n), \quad \text{supp}\, f \text{ compact}, \quad \text{sing supp}\, f \neq \emptyset. \tag{7.149}$$

Recall that the *singular support* of $f \in S'(\mathbb{R}^n)$ is given by

$$\text{sing supp}\, f = \{x \in \mathbb{R}^n : \ f|B(x,r) \neq C^\infty \text{ for any } r > 0\}, \tag{7.150}$$

where again $B(x,r)$ is the open ball centred at $x \in \mathbb{R}^n$ and of radius r, and $f|\Omega \in D'(\Omega)$ is the restriction of f to Ω. The counterpart of (7.101) is now given by

$$V f^{\lambda,M}_p = \begin{cases} \sup\limits_{j \in \mathbb{N}_0} 2^{j(\lambda-n)} \Big(\sum\limits_{m \in \mathbb{Z}^n} (V^M_{jm} f)^p \Big)^{1/p} & \text{if } 0 < p < \infty, \\ \sup\limits_{j \in \mathbb{N}_0, m \in \mathbb{Z}^n} 2^{j(\lambda-n)} V^M_{jm} f & \text{if } p = \infty, \end{cases} \tag{7.151}$$

where $V^M_{jm} f$ has the same meaning as in (7.141). If $M = 0$ and $f = \mu$ according to (7.100) then we have (7.13) and hence by (7.101),

$$V \mu^{\lambda,0}_p = \mu^\lambda_p, \qquad 0 < p \le \infty, \quad \lambda \in \mathbb{R}. \tag{7.152}$$

Definition 7.44. *Let $f \in S'(\mathbb{R}^n)$ with* (7.149). *Let $0 \le t = 1/p < \infty$. Then*

$$\lambda^f(t) = \sup\bigl\{\lambda : \ V f_p^{\lambda,M} < \infty \textit{ for some } M \in \mathbb{N}_0\bigr\} \tag{7.153}$$

are multifractal characteristics of f and

$$s_f(t) = \sup\bigl\{s : \ f \in B^s_{p\infty}(\mathbb{R}^n)\bigr\} \tag{7.154}$$

are Besov characteristics of f.

Remark 7.45. By (7.134) and the definition of $K^M(t,f)$ in (7.140) it follows that for given p and $M \in \mathbb{N}_0$,

$$V f_p^{\lambda,N} < \infty \quad \text{if} \quad V f_p^{\lambda,M} < \infty, \qquad M \le N \in \mathbb{N}. \tag{7.155}$$

Again one can replace $B^s_{p\infty}(\mathbb{R}^n)$ in (7.154) by $B^s_{pq}(\mathbb{R}^n)$ for some q or all q with $0 < q \le \infty$.

Recall that increasing means non-decreasing.

Theorem 7.46.

(i) *Let $f \in S'(\mathbb{R}^n)$ as in* (7.149). *Let $\lambda^f(t)$ and $s_f(t)$ be the characteristics according to Definition* 7.44. *Then $s_f(t)$ with $0 \le t < \infty$ is a continuous increasing concave function in the (t,s)-diagram in Figures* 1.17.1 *or* 7.2.1 *with a slope of at most n. Furthermore,*

$$s_f(t) = \lambda^f(t) + n(t-1), \qquad 0 \le t < \infty. \tag{7.156}$$

(ii) *Let μ be a Radon measure with* (7.100) *and let $\lambda_\mu(t)$ be the multifractal characteristics according to Definition* 7.26. *Then*

$$\lambda_\mu(t) \le \lambda^\mu(t) \qquad \textit{if} \quad 0 \le t < \infty \tag{7.157}$$

with

$$\lambda_\mu(t) = \lambda^\mu(t) \qquad \textit{if} \quad 0 \le t \le 1. \tag{7.158}$$

Proof. It follows by the same arguments as in the proof of Theorem 7.28 that $s_f(t)$ is increasing, concave, and hence, continuous. By the limiting embedding

$$B^{s_0}_{p_0 q}(\mathbb{R}^n) \hookrightarrow B^{s_1}_{p_1 q}(\mathbb{R}^n) \quad \text{if} \quad s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}, \quad s_0 \ge s_1, \tag{7.159}$$

$0 < q \le \infty$, [Triβ, Section 2.7.1, p. 129], it follows that the slope of $s_f(t)$ is at most n. Now (7.156) follows from (7.151), (7.155), and Theorem 7.42. Finally, if $f = \mu$ is a Radon measure with (7.100) then both (7.157) and (7.158) follow from (7.156) and Theorem 7.28. □

Remark 7.47. We followed essentially [Tri03c] where one finds also a few additional explanations. Some further assertions and references may also be found in Section 1.18. In particular by part (i) of the above theorem it is now clear what is meant by part (i) of Theorem 1.199. We are now going to prove part (ii) of this theorem. Recall that again $|\Gamma|$, $\mathcal{H}^d(\Gamma)$ and $\dim_H \Gamma$ are the corresponding Lebesgue measure, the Hausdorff measure and the Hausdorff dimension of the set Γ in $\mathbb{R}^n$. Some relevant references may be found in Section 1.18. There we discussed also properties of continuous increasing concave functions $s(t)$ in a (t,s)-diagram. Recall that increasing means non-decreasing.

Theorem 7.48. *Let $s(t)$ for $0 \le t < \infty$ be a real continuous increasing concave function of slope smaller than or equal to n. Then there is an $f \in S'(\mathbb{R}^n)$ with* (7.149) *such that*

$$s_f(t) = s(t), \qquad 0 \le t < \infty, \tag{7.160}$$

where $s_f(t)$ is the Besov characteristics according to Definition 7.44. *Furthermore with $\Gamma = \operatorname{sing\,supp} f$ and $p = 1/t$,*

$$|\Gamma| = 0 \qquad \text{and} \qquad f \in B^{s(t)}_{p\infty}(\mathbb{R}^n). \tag{7.161}$$

Let, in addition, $s'(\infty) < n$. Then

$$\mathcal{H}^d(\Gamma) = 0 \quad \text{if, and only if,} \quad d \ge n - s'(\infty), \tag{7.162}$$

and, in particular, $\dim_H \Gamma = n - s'(\infty)$.

Proof. Step 1. Let Γ be a d-set in $\mathbb{R}^n$ with $0 < d < n$ and $\mu = \mathcal{H}^d|\Gamma$. Then we have

$$s_\mu(t) = (n-d)(t-1), \quad \mu \in B^{s_\mu(t)}_{p\infty}(\mathbb{R}^n), \quad 0 \le t < \infty, \tag{7.163}$$

where again $p = 1/t$. The first assertion is covered by Proposition 7.32. Furthermore, it follows from (7.113) that $\lambda_\mu(t) = d(1-t)$ is a maximum (not only a supremum). Then one gets the second assertion in (7.163) as at the end of the proof of Theorem 7.28 again with a reference to Theorem 1.10. Let I_σ be the lift according to (1.5), hence

$$I_\sigma g = \left(\langle\xi\rangle^\sigma \widehat{g}\right)^\vee, \qquad \sigma \in \mathbb{R}. \tag{7.164}$$

With $f = I_\sigma \mu$ we have by (1.6) that

$$s_f(t) = (n-d)(t-1) - \sigma, \qquad f \in B^{s_f(t)}_{p\infty}(\mathbb{R}^n), \qquad 0 \le t < \infty. \tag{7.165}$$

Hence one gets (7.160) for any straight line $s(t)$ of slope strictly between 0 and n. An arbitrary admitted curve $s(t)$ will be approximated from above by a sequence of polygonal lines. This makes it necessary to have control of the dependence of the quasi-norms of $f = I_\sigma \mu$ in $B^{s_f(t)}_{p\infty}(\mathbb{R}^n)$ on d, σ and t. For this purpose we first furnish $B^s_{p\infty}(\mathbb{R}^n)$ according to (2.8), (2.9), (2.12) with the quasi-norm

$$\|f \,|B^s_{p\infty}(\mathbb{R}^n)\| = \sup_{j \in \mathbb{N}_0} 2^{js} \|(\varphi_j \widehat{f})^\vee \,|L_p(\mathbb{R}^n)\|, \tag{7.166}$$

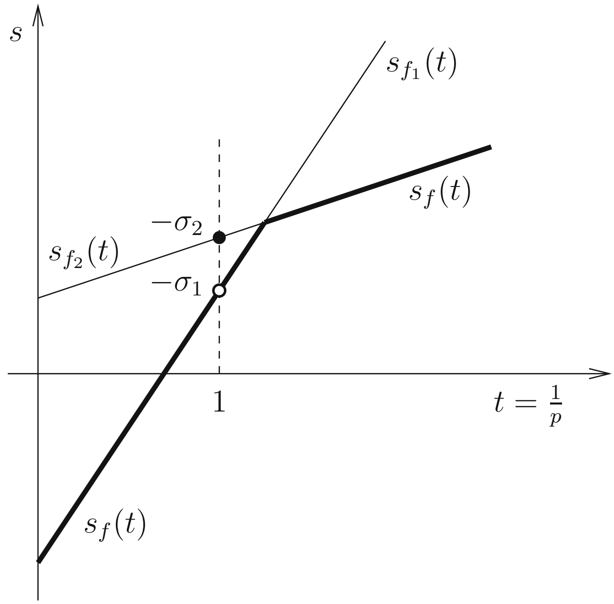

Figure 7.2.3(a)

where we assume that φ_0, and hence also φ_j, is fixed in what follows. Then for any $N > 0$ and any t with $0 \le t < \infty$ there is a constant $c(N,t)$ (depending also on φ_0) with the following property: For any ε with $0 < \varepsilon \le 1$, any d with $0 < d < n$ and any $\sigma \in \mathbb{R}$ with $|\sigma| \le N$ there is a d-set and a related measure μ with

$$\operatorname{supp} \mu \subset \{x : |x| < \varepsilon\}, \quad \| I_\sigma \mu \, | B^{(n-d)(t-1)-\sigma}_{p\infty}(\mathbb{R}^n) \| \le c(N,t), \tag{7.167}$$

where again $p = 1/t$. We shift the proof of this technical assertion to the Remarks 7.49 and 7.50 below.

Step 2. As in Example 7.38 we assume $0 < d_1 < d_2 < n$. Let $\Gamma_1 = \operatorname{supp} \mu_1$ with $\mu_1 = \mathcal{H}^{d_1}|\Gamma_1$ a d_1-set and let Γ_2 be a corresponding d_2-set with the related measure μ_2 and $\Gamma_1 \cap \Gamma_2 = \emptyset$. Let

$$f_1 = I_{\sigma_1}\mu_1, \quad f_2 = I_{\sigma_2}\mu_2 \quad \text{and} \quad f = f_1 + f_2. \tag{7.168}$$

Then we have for $s_{f_1}(t)$ and $s_{f_2}(t)$ the counterparts of (7.165) and

$$s_f(t) = \min\left(s_{f_1}(t), s_{f_2}(t)\right), \qquad 0 \le t < \infty, \tag{7.169}$$

as shown in Figure 7.2.3(a). Furthermore we claim that

$$\operatorname{sing\,supp} f = \Gamma_1 \cup \Gamma_2, \qquad \| f \, | B^{s_f(t)}_{p\infty}(\mathbb{R}^n) \| \le c(N,t) \tag{7.170}$$

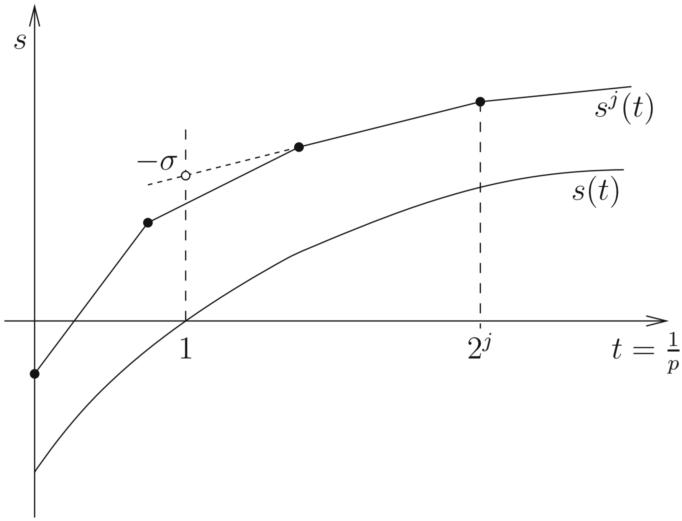

Figure 7.2.3(b)

if $|\sigma_1| \le N$, $|\sigma_2| \le N$. The latter follows from (7.167) and the monotonicity of the fixed quasi-norms in (7.166). Since I_σ is an elliptic pseudodifferential operator one has

$$\text{sing supp } I_\sigma g = \text{sing supp } g, \qquad g \in S'(\mathbb{R}^n), \tag{7.171}$$

[Tay91, §0.4]. But in this simple case one can prove this assertion directly. Then one gets also the first assertion in (7.170). We iterate this procedure and get (7.160) if $s(t)$ is a polygonal line with finitely many breaking points. Then f is a finite sum of terms of type (7.168).

Step 3. After these preparations we now prove the theorem. If $s'(\infty) = n$ then $s(t) = a + nt$ for some $a \in \mathbb{R}$, and one gets $s(t) = s_f(t)$ with $f = I_\sigma \delta$ for some $\sigma \in \mathbb{R}$ and the δ-distribution. Hence we may assume $s'(\infty) < n$ and also without restriction of generality $s(1) = 0$. We approximate $s(t)$ by a sequence of concave polygonal lines $s^j(t)$ according to Step 2 as indicated in Figure 7.2.3(b), $j \in \mathbb{N}$, such that

$$s(t) < s^{j+1}(t) < s^j(t), \qquad 0 \le t < \infty, \tag{7.172}$$

and

$$s(t) + 2^{-j-1} \le s^j(t) \le s(t) + 2^{-j} \quad \text{if} \quad 0 \le t \le 2^j, \tag{7.173}$$

whereas $s^j(t)$ for $t > 2^j$ is a straight line of slope

$$s'(\infty) < (s^j)'_+(2^j) < n. \tag{7.174}$$

We may assume that for fixed j all the (finitely many) counterparts of Γ_1, Γ_2 from Step 2 have disjoint supports and this applies also to the singular supports of the resulting $f^j \in S'(\mathbb{R}^n)$ with $s_{f^j} = s^j$,

$$\text{sing supp } f^j \cap \text{sing supp } f^k = \emptyset \quad \text{if} \quad j \neq k. \tag{7.175}$$

Furthermore for fixed t with $0 \le t < \infty$ one can estimate the numbers σ needed in the liftings by I_σ as in (7.168) and indicated in Figures 7.2.3(a),(b). Since we assume $s(1) = 0$ it follows by elementary reasoning that $|\sigma| \le N(t)$. Then we have by (7.170) that

$$\|f^j \,|B^{s(t)}_{p\infty}(\mathbb{R}^n)\| \le \|f^j \,|B^{s^j(t)}_{p\infty}(\mathbb{R}^n)\| \le c(t), \qquad 0 \le t < \infty. \tag{7.176}$$

We may assume that all sing supp f^j are subsets of the unit ball. Let ψ be a C^∞ cut-off function, identically 1 on the unit ball. Then

$$f = \psi \sum_{j=1}^{\infty} 2^{-j} f^j \in B^{s(t)}_{p\infty}(\mathbb{R}^n) \tag{7.177}$$

is the distribution we are looking for with (7.160), (7.161) and, if $s'(\infty) < n$ also with (7.162) by (7.174) and the above construction. □

Remark 7.49. We have to justify (7.167). This will be done in this remark and in the following one. We prove a little bit more than needed. First we construct specific d-sets. Let $Q = [0,1]^n$ be the unit cube in $\mathbb{R}^n$. Then

for any d with $0 \le d \le n$ there is d-set Γ in $\mathbb{R}^n$ and a corresponding Radon measure μ with

$$\text{supp } \mu = \Gamma \subset Q, \qquad \mu(Q) = \mu(\Gamma) = 1, \tag{7.178}$$

and

$$2^{-2n} r^d \le \mu(Q_r) \le 2^{2n} r^d \tag{7.179}$$

for any closed cube Q_r centred at some point $\gamma \in \Gamma$ with sides parallel to the axes of coordinates and of side-length r where $0 < r \le 1$.

In case of $d = 0$ one can choose the Dirac measure, and in case of $d = n$ the Lebesgue measure, restricted to Q, hence the characteristic function of Q. So we may assume $0 < d < n$. Let $1 < \varkappa < \infty$ such that $n = \varkappa d$. We divide Q naturally into 2^n cubes with side-length 1/2. Let $Q_{1,l}$ where $l = 1, \ldots, 2^n$, the cubes concentric to these 2^n cubes with side-length $2^{-\varkappa}$. By the mass distribution procedure of fractal geometry, [Fal90, pp. 13, 14], one iterates this subdivision, starting now with $Q_{1,l}$, and gets a (totally disconnected) d-set Γ and a Radon measure μ with (7.178) and

$$\mu(Q_{j,l}) = 2^{-jn} = (2^{-\varkappa j})^d, \qquad l = 1, \ldots, 2^{jn}. \tag{7.180}$$

Details about this self-similar construction may be found in [Fal85, Section 8.3]. Let Q^j be a cube of side-length $2^{-j\varkappa}$ centred at some point $\gamma \in \Gamma$. Of course, $\gamma \in Q_{j,l}$ for some l. By the above specific construction it follows that

$$2^{-n}(2^{-j\varkappa})^d = 2^{-n}\mu(Q_{j,l}) \le \mu(Q^j) \le 2^n (2^{-j\varkappa})^d. \tag{7.181}$$

Let Q_r be a cube with side-length r, where $0 < r \le 1$, centred at some point $\gamma \in \Gamma$, and let $j \in \mathbb{N}_0$ such that

$$2^{-(j+1)\varkappa} < r \le 2^{-j\varkappa}. \tag{7.182}$$

Then it follows from $n = \varkappa d$ and (7.181) that

$$2^{-2n} r^d \le 2^{-n}\, 2^{-(j+1)\varkappa d} \le \mu(Q_r) \le 2^n\, 2^{-jd\varkappa} \le 2^{2n} r^d. \tag{7.183}$$

This proves (7.179). Of course one can replace the cubes Q_r by corresponding balls of radius r (with constants depending only on n). Furthermore if Γ_J is the intersection of Γ with one of the above cubes $Q_{J,l}$ where $J \in \mathbb{N}$, then one gets a sub-fractal and the related measure μ_J has the total mass 2^{-Jn}. One has again (7.179) with $0 < r \le 2^{-J\varkappa}$ and supp $\Gamma_J \subset Q_{J,l}$ (for the selected l). We choose this (sub-)fractal in the first part of (7.167) and prove now the second part with $\sigma = 0$, hence

$$\left\| \mu \,|B^{(n-d)(t-1)}_{p\infty}(\mathbb{R}^n) \right\| \le c(t), \qquad 0 \le t < \infty, \tag{7.184}$$

where again $p = 1/t$, for some $c(t) > 0$ which is independent of d (and $\varepsilon > 0$). It is sufficient to justify (7.184) for the d-set Γ we started from, hence (7.178), (7.179). Recall that $B^s_{p\infty}(\mathbb{R}^n)$ is quasi-normed by (7.166). We have

$$\varphi_j(x) = \varphi(2^{-j}x) \quad \text{with} \quad \varphi(x) = \varphi_0(x) - \varphi_0(2x) \quad \text{if } j \in \mathbb{N}.$$

Let

$$\Gamma_j = \{x \in \mathbb{R}^n : \text{ dist } (x, \Gamma) \le 2^{-j}\}, \qquad j \in \mathbb{N}_0, \tag{7.185}$$

and

$$\Gamma^j = \{x \in \mathbb{R}^n : \ 2^{-j-1} \le \text{dist } (x, \Gamma) \le 2^{-j}\}, \qquad j \in \mathbb{N}_0. \tag{7.186}$$

Then

$$\Gamma' = \Gamma_j \cup \bigcup_{k=0}^{j-1} \Gamma^k \subset \{y : \ |y| \le c\} \tag{7.187}$$

for some $c > 0$ depending only on n. Let N_j be the minimal number of cubes of side-length 2^{-j} centred at Γ needed to cover Γ. Since $\mu(\Gamma) = 1$ it follows that $N_j \le c 2^{jd}$ with $j \in \mathbb{N}$, where c depends only on n. Concentric cubes with side-length $3 \cdot 2^{-j}$ of the above cubes cover also Γ_j, hence

$$|\Gamma^j| \le |\Gamma_j| \le 3^n\, 2^{-jn} N_j \le c\, 2^{-j(n-d)}, \tag{7.188}$$

where c depends only on n. Let $j \in \mathbb{N}$ (if $j = 0$ one has to modify appropriately what follows). Then for $x \in \mathbb{R}^n$,

$$\begin{aligned}(\varphi_j\, \widehat{\mu})^\vee(x) &= c \int_{\mathbb{R}^n} \varphi_j^\vee(x-y)\, \mu(\mathrm{d}y) \\ &= c' 2^{jn} \int_\Gamma \varphi^\vee \left(2^j(x-\gamma)\right)\, \mu(\mathrm{d}\gamma).\end{aligned} \tag{7.189}$$

We have for any $a > 0$,

$$|\varphi^\vee\left(2^j(x-\gamma)\right)| \le \frac{c_a}{(1+2^j|x-\gamma|)^a}, \quad j \in \mathbb{N}, \quad \gamma \in \Gamma, \tag{7.190}$$

and hence for $x \in \mathbb{R}^n \backslash \Gamma'$ and $\gamma \in \Gamma$,

$$|\varphi^\vee\left(2^j(x-\gamma)\right)| \le c'_a\, 2^{-ja}(1+|x|)^{-a}. \tag{7.191}$$

With $ap > n$ and again $t = 1/p$ one gets

$$\|(\varphi_j\,\widehat{\mu})^\vee\,|L_p(\mathbb{R}^n\backslash\Gamma')\| \le c(t)\,2^{j(n-a)}. \tag{7.192}$$

Let $k = 0, \ldots, j-1$, and $x \in \Gamma^k$, $\gamma \in \Gamma$ with dist $(x,\gamma) \sim 2^{-l}$ where $l = 0, \ldots, k$. Then for any $b > 0$,

$$|\varphi^\vee\left(2^j(x-\gamma)\right)| \le c_b\,2^{-jb}\,2^{lb}. \tag{7.193}$$

Then one gets for $x \in \Gamma^k$ by (7.189) and, so far, $b > n+1$, that

$$\begin{aligned} |(\varphi_j\,\widehat{\mu})^\vee(x)| &\le c'_b\,2^{j(n-b)} \sum_{l=-\infty}^{k} 2^{lb}\,2^{-ld} \\ &\le c''_b\,2^{j(n-b)}\,2^{k(b-d)} \\ &= \tilde{c}_b\,2^{j(n-d)}\,2^{-(j-k)(b-d)} \end{aligned} \tag{7.194}$$

with constants depending only on b and n. Similarly for $x \in \Gamma_j$, getting (7.194) with $k = j$. Let $p < \infty$. Then one obtains by (7.188), (7.189), (7.192), (7.194) that

$$\begin{aligned} &\|(\varphi_j\,\widehat{\mu})^\vee\,|L_p(\mathbb{R}^n)\|^p \\ &\le c2^{j(n-a)p} + \int_{\Gamma_j} |(\varphi_j\widehat{\mu})^\vee(x)|^p\,\mathrm{d}x + \sum_{k=0}^{j-1}\int_{\Gamma^k} |(\varphi_j\,\widehat{\mu})^\vee(x)|^p\,\mathrm{d}x \\ &\le c'\left[2^{j(n-a)p} + 2^{j(n-d)p}|\Gamma_j| + 2^{j(n-d)p}\sum_{k=0}^{j-1} 2^{-(j-k)(b-d)p}|\Gamma^k|\right] \\ &\le c''\left[2^{j(n-a)p} + 2^{j(n-d)(p-1)} + 2^{j(n-d)(p-1)}\sum_{k=0}^{j-1} 2^{-(j-k)[(b-d)p-n+d]}\right] \\ &\le \tilde{c}\,2^{j(n-d)(p-1)} \end{aligned} \tag{7.195}$$

if a and b are chosen sufficiently large in dependence on n and p. Since $\frac{p-1}{p} = 1-t$ one gets

$$\|(\varphi_j\,\widehat{\mu})^\vee\,|L_p(\mathbb{R}^n)\| \le c\,2^{j(n-d)(1-t)} \tag{7.196}$$

and hence (7.184) where c depends only on t (and n). If $p = \infty$ then one gets (7.196) and hence (7.184) with $t = 0$ from (7.192) and (7.194).

Remark 7.50. Let $B^s_{pq}(\mathbb{R}^n)$ be quasi-normed according to (2.12) with respect to a fixed system $\varphi = \{\varphi_k\}_{k=0}^\infty$ in (2.8)–(2.10). Then

$$\varphi_j(x) = \varphi_j(x) \sum_{l=-1}^{1} \varphi_{j+l}(x), \qquad j \in \mathbb{N}_0, \quad x \in \mathbb{R}^n, \tag{7.197}$$

$\varphi_{-1} = 0$. We prove the following assertion:

Let $N > 0$ and $0 \le t = 1/p < \infty$. Then there is a constant $c(N,t)$ (depending also on φ and n) such that for all $s \in \mathbb{R}$, all $0 < q \le \infty$, all $\sigma \in \mathbb{R}$ with $|\sigma| \le N$ and all $f \in B^s_{pq}(\mathbb{R}^n)$,

$$\| I_\sigma f \, | B^{s-\sigma}_{pq}(\mathbb{R}^n) \| \le c(N,t) \, \| f \, | B^s_{pq}(\mathbb{R}^n) \|, \tag{7.198}$$

where I_σ is the same lift as in (1.5), *hence*

$$I_\sigma f = \Big((1 + |\xi|^2)^{\sigma/2} \, \widehat{f} \Big)^\vee . \tag{7.199}$$

Afterwards one gets (7.167) as a consequence of (7.184) and (7.198) which completes the proof of Theorem 7.48. By (2.12), (7.197), (7.199) the assertion (7.198) follows from

$$\| \Big(\varphi_j(\xi) \, (1 + |\xi|^2)^{\sigma/2} \widehat{f} \Big)^\vee \, | L_p(\mathbb{R}^n) \| \le c(N,t) \, 2^{\sigma j} \sum_{l=-1}^{1} \| \Big(\varphi_{j+l} \widehat{f} \Big)^\vee \, | L_p(\mathbb{R}^n) \| \tag{7.200}$$

and hence from

$$\| \Big(\varphi_j(\xi) \, (1 + |\xi|^2)^{\sigma/2} \widehat{f_{j+l}} \Big)^\vee \, | L_p(\mathbb{R}^n) \| \le c(N,t) \, 2^{\sigma j} \, \| f_{j+l} \, | L_p(\mathbb{R}^n) \| \tag{7.201}$$

with $f_j = (\varphi_j \widehat{f})^\vee$. But this is a matter of Fourier multipliers for analytic L_p-functions $g \in S'(\mathbb{R}^n)$ with compact support of $\widehat{g}$ in balls of radius $c2^j$. It is sufficient to deal with $j \in \mathbb{N}$ and $l = 0$ in (7.201). Let $f^j(x) = f_j(2^{-j}x)$. Then $\widehat{f^j}$ has a compact support in a standard ball and

$$\widehat{f_j}(\xi) = 2^{-jn} \widehat{f^j}(2^{-j}\xi), \qquad j \in \mathbb{N}. \tag{7.202}$$

With $\varphi(\xi) = \varphi_0(\xi) - \varphi_0(2\xi)$ and $\varphi_j(\xi) = \varphi(2^{-j}\xi)$ one gets

$$\begin{aligned} &\Big(\varphi_j(\xi)(1 + |\xi|^2)^{\sigma/2} \widehat{f_j}(\xi) \Big)^\vee (x) \\ &= 2^{\sigma j - jn} \Big(\varphi(2^{-j}\xi) \, \big(2^{-2j} + |2^{-j}\xi|^2\big)^{\sigma/2} \, \widehat{f^j} \, (2^{-j}\xi) \Big)^\vee (x) \\ &= 2^{\sigma j} \Big(\varphi(\xi) \, \big(2^{-2j} + |\xi|^2\big)^{\sigma/2} \, \widehat{f^j} \, (\xi) \Big)^\vee (2^j x) \end{aligned} \tag{7.203}$$

and

$$\begin{aligned}
&\left\| \left(\varphi_j(\xi) \left(1+|\xi|^2\right)^{\sigma/2} \widehat{f_j}(\xi) \right)^\vee |L_p(\mathbb{R}^n) \right\| \\
&\le 2^{\sigma j - jn/p} \left\| \left(\varphi(\xi) \left(2^{-2j}+|\xi|^2\right)^{\sigma/2} \widehat{f^j}(\xi) \right)^\vee |L_p(\mathbb{R}^n) \right\| \\
&\le M\, 2^{\sigma j - jn/p} \, \|f^j \,|L_p(\mathbb{R}^n)\| \\
&\le M\, 2^{\sigma j} \, \|f_j \,|L_p(\mathbb{R}^n)\|,
\end{aligned} \tag{7.204}$$

where the last but one estimate is a Fourier multiplier assertion with respect to

$$m_j(\xi) = \varphi(\xi) \, \left(2^{-2j}+|\xi|^2\right)^{\sigma/2}, \qquad j \in \mathbb{N}. \tag{7.205}$$

According to [Triβ, Theorem 1.5.2, p. 26] the above multiplier constant M can be estimated from above by

$$M \le \|m_j \,|W_2^k(\mathbb{R}^n)\|, \quad \mathbb{N} \ni k = 1 + \left[n \left(\frac{1}{\min(p,1)} - \frac{1}{2} \right) \right], \tag{7.206}$$

where $W_2^k(\mathbb{R}^n)$ are the classical Sobolev spaces normed by (1.4). By the support properties of φ one gets for all $j \in \mathbb{N}$ and all σ with $|\sigma| \le N$ that $M \le c(N,t)$, hence (7.201) and finally (7.198).

Remark 7.51. The Remarks 7.49 and 7.50 justify (7.167) and complete the proof of Theorem 7.48 and also of Theorem 1.199(ii). One may consult Section 1.18 and Remark 1.200 for a few further comments and some references. The definition (7.154), now admitting $-\infty \le s_f(t) \le \infty$, makes sense for arbitrary $f \in S'(\mathbb{R}^n)$ which are not necessarily compactly supported. The characterisation of all resulting curves $s(t) = s_f(t)$ has been given recently in [Ved06]. The basic ideas for proving Theorem 7.48 are quite simple: One lifts (7.163), gets (7.165) and approximates the given curve $s(t)$ from above by concave polygonal lines. However the justification of this approximation procedure, based on (7.167), is somewhat cumbersome, fulfilling the two preceding remarks. On the other hand on this way one can extract additional information such as (7.161), (7.162) and the following one. Preparing for what follows we recall that a compactly supported non-negative function $f \in L_1(\mathbb{R}^n)$ can be interpreted as a (non-negative) regular measure $f\mu_L$, where μ_L is the Lebesgue measure.

Corollary 7.52. *Let $s(t)$ be a curve as in Theorem 7.48. Then there is a compactly supported (regular or singular) Radon measure μ in $\mathbb{R}^n$ with $s_\mu(t) = s(t)$ for all $0 \le t < \infty$ if, and only if, $s(1) \ge 0$.*

Proof. According to Proposition 1.127 we have $s_\mu(1) \ge 0$ for any compactly supported measure. We prove the converse and suppose $s(1) \ge 0$. Then we may assume that all line segments in the proof of Theorem 7.48 are generated by $f = I_\sigma \mu$ with (singular) measures μ and liftings I_σ with $\sigma < 0$. By (3.251) one can represent

I_σ as a convolution operator with positive kernel. Then $I_\sigma \mu \in L_1(\mathbb{R}^n)$ is positive. The approximation procedure in the proof of Theorem 7.48 and the multiplication with a non-negative cut-off function results in a compactly supported non-negative distribution. But this is a measure. As for details of the last assertion we refer to Step 2 of the proof of Theorem 3.48. □

Remark 7.53. In particular, $s(t)$ with $s(1) = 0$ can be represented as $s(t) = s_\mu(t)$ where μ is a compactly supported measure. But it is not clear whether such a representation can be achieved even with a singular measure according to (7.100) where Γ is the support of μ. But one has in any case (7.161) for $\Gamma = \operatorname{sing\,supp} \mu$.

Remark 7.54. For $f \in S'(\mathbb{R}^n)$ which materialises the given curve $s(t)$ according to Theorem 7.48 we got the extra information (7.161). If $s_f(1) > 0$ then f is an L_1-function; if $s_f(0) > 0$ then f is even a real continuous function. It is one of the favorite subjects of fractal geometry to study the graph of such functions in terms of fractal quantities such as the Hausdorff dimension of the graph of f in $\mathbb{R}^{n+1}$, or diverse types of box dimensions etc. There is a huge literature and something can be found in the books mentioned in Section 1.12.1. But it is not the subject of this book. Nearest to us and related to Theorem 7.48 are the two recent papers [Car05, Car06], where one finds also further references. In particular one may ask for fractal properties of continuous, real, compactly supported functions belonging for given $s \in \mathbb{R}$ and $0 \le t = 1/p < \infty$ to the residual sets

$$R^s_p(\mathbb{R}^n) = \big\{ f \in B^s_{p\infty}(\mathbb{R}^n),\ \operatorname{supp} f \text{ compact},\ s = s_f(t) \big\} \tag{7.207}$$

or

$$\bar{R}^s_p(\mathbb{R}^n) = \{ f \in S'(\mathbb{R}^n),\ \operatorname{supp} f \text{ compact},\ s = s_f(t) \}\,. \tag{7.208}$$

Of course it is always assumed that f is not C^∞, such that the curve $s_f(t)$ exists. Using the wavelet isomorphism according to Theorem 3.5 one gets

$$\emptyset \neq R^s_p(\mathbb{R}^n) \subsetneqq \bar{R}^s_p(\mathbb{R}^n), \tag{7.209}$$

where the first assertion is also a consequence of (7.161). One of the main aims of [Car05, Car06] is the detailed study of how bizarre the graphs in $\mathbb{R}^{n+1}$ of real continuous, compactly supported functions f can be and have to be if they belong to the above residual sets.

7.3 Operators

7.3.1 Potentials and the regularity of measures

In Section 7.1.2 we dealt with the interrelations between Radon measures of type (7.1), Bessel potentials and (truncated) Riesz potentials. This is one of the favorite subjects of fractal analysis. One may consult the literature in the references given at the beginning of Section 7.1.2. Now we return to this subject in a modified way. We always assume that μ is a singular measure in $\mathbb{R}^n$,

$$\Gamma = \operatorname{supp} \mu \subset \{x : \ |x| < 1\}, \quad 0 < \mu(\mathbb{R}^n) < \infty, \quad |\Gamma| = 0, \tag{7.210}$$

where $|\Gamma|$ is the Lebesgue measure of Γ. Again let Q_{jm}, where $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$, be the closed cubes in $\mathbb{R}^n$ with sides parallel to the axes of coordinates, centred at $2^{-j}m$ and having side-length 2^{-j+1}. As before we put

$$\mu_j = \sup_{m \in \mathbb{Z}^n} \mu(Q_{jm}), \qquad j \in \mathbb{N}_0. \tag{7.211}$$

We have by (7.9)

$$\|\mu \,|B^{-n}_{\infty,1}(\mathbb{R}^n)\| \sim \mu^0_{\infty,1} = \sum_{j=0}^{\infty} \mu_j \tag{7.212}$$

as a characterisation. In particular, $\mu \in B^{-n}_{\infty,1}(\mathbb{R}^n)$ if the right-hand side of (7.212) is finite. Since

$$\mu \in \mathcal{C}^{-n}(\mathbb{R}^n) = B^{-n}_{\infty,\infty}(\mathbb{R}^n)$$

for any measure μ with (7.210) the assumption $\sum_{j=0}^{\infty} \mu_j < \infty$ is a mild additional restriction. It ensures that

$$z_\mu(x) = J_n \mu \in B^0_{\infty,1}(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n), \tag{7.213}$$

where

$$J_n g = (\mathrm{id} - \Delta)^{-n/2} g, \qquad g \in S'(\mathbb{R}^n), \tag{7.214}$$

is a special Bessel-potential with the isomorphism (7.18). The last embedding in (7.213) is covered by (4.25). Although the characterisations in Theorem 7.5, the related applications in Section 7.1.2 and also the considerations about characteristics of measures in Section 7.2.1 are rather satisfactory it comes out that a closer look at the behavior of the partial sums of the (converging) right-hand side of (7.212) provides an even finer tuning.

Definition 7.55. *Let μ be a measure according to* (7.210) *with $\sum_{j=0}^{\infty} \mu_j < \infty$. Then*

$$Z_\mu(t) = \sum_{j \ge |\log t|} \mu_j, \qquad 0 < t \le 1, \tag{7.215}$$

and

$$z_\mu(x) = (J_n \mu)(x), \qquad x \in \mathbb{R}^n, \tag{7.216}$$

where J_n is given by (7.214).

Remark 7.56. It follows by (7.213) that $z_\mu(x)$ is a continuous function in $\mathbb{R}^n$. It is the main aim to judge the regularity of μ in terms of smoothness properties of $z_\mu(x)$. Since μ is a measure in $\mathbb{R}^n$ with (7.210) it follows that

$$\mu_j \le c\, 2^{n(J-j)}\, \mu_J \qquad \text{for} \quad 0 \le j \le J, \tag{7.217}$$

where c is independent of j, J, and μ. In particular,

$$\mu_j \sim \mu_{j+1}, \qquad j \in \mathbb{N}_0, \tag{7.218}$$

although the measure μ itself need not be doubling in the understanding of (1.493). This supports (7.215). Obviously, $Z_\mu(t)$ is a positive increasing (means non-decreasing) function on $(0,1]$ with

$$Z(1) = \sum_{j=0}^{\infty} \mu_j \quad \text{and} \quad Z(0) = \lim_{t\downarrow 0} Z(t) = 0. \tag{7.219}$$

As usual, $\Delta = \sum_{r=1}^n \partial^2/\partial x_r^2$ stands for the Laplacian in $\mathbb{R}^n$ and Δ_h^l are the differences in $\mathbb{R}^n$ according to (4.32). Recall that log is taken to base 2.

Theorem 7.57. *Let μ be a Radon measure in $\mathbb{R}^n$ according to* (7.210) *with $\sum_{j=0}^\infty \mu_j < \infty$. Let μ_j, $Z_\mu(t)$ and $z_\mu(x)$ as in* (7.211) *and in Definition* 7.55.

(i) *Then $z_\mu \in C(\mathbb{R}^n)$ is a positive continuous function in $\mathbb{R}^n$. Furthermore there is a positive number c such that for all measures μ, all $x \in \mathbb{R}^n$, and all $h \in \mathbb{R}^n$ with $0 < |h| \le 1$,*

$$|\Delta_h^{n+1} z_\mu(x)| \le c\, Z_\mu(|h|). \tag{7.220}$$

(ii) *Let $l \in \mathbb{N}$. Then there is a positive number c such that for all measures μ, all $x \in \mathbb{R}^n$, all $J \in \mathbb{N}_0$ and all $h \in \mathbb{R}^n$ with $0 < |h| \le 1$,*

$$|\Delta_h^l z_\mu(x)| \le c \sum_{j\ge J} \mu_j + c|h|^l \sum_{j<J} 2^{lj}\, \mu_j\, . \tag{7.221}$$

Proof. Step 1. By (7.213) and (7.19) with $\sigma = n$ and the positive kernels $G_n(x)$ according to [AdH96, p. 10] it follows that $z_\mu(x)$ is a positive continuous function. We prove part (ii) in Step 2 and take it temporarily for granted. By (7.217) the last term in (7.221) with $l = n+1$ and $|h| \sim 2^{-J}$ can be estimated from above by

$$c\, \mu_J\, 2^{-J(n+1)} \sum_{j<J} 2^{(n+1)j}\, 2^{n(J-j)} \le c\, \mu_J \tag{7.222}$$

and incorporated in the first term on the right-hand side in (7.221). With $J \sim |\log|h||$ one gets (7.220).

Step 2. We prove part (ii). For this purpose we represent μ according to Theorem 3.26 as

$$\mu = \sum_{\beta,j,m} \lambda_{jm}^\beta(\mu)\, \Phi_{jm}^\beta, \tag{7.223}$$

where $\Phi_{jm}^\beta \in S(\mathbb{R}^n)$ are the same functions as in (3.89), and by (3.88), (3.139),

$$\lambda_{jm}^\beta(\mu) = 2^{jn} \int_\Gamma k^\beta \left(2^j\gamma - m\right)\, \mu(\mathrm{d}\gamma). \tag{7.224}$$

In particular,

$$\lambda_{jm}^\beta(\mu) \le c\, 2^{jn}\, \mu(Q_{jm}), \tag{7.225}$$

where c is independent of j, m, β and μ. Here we may assume that k in (3.66) is chosen appropriately, otherwise Q_{jm} in (7.225) must be replaced by a cube $\tilde{Q}_{jm}$ concentric with Q_{jm} and side-length $b2^{-j}$ for some suitable $b > 0$. But this is immaterial and will be ignored. With (7.216) one gets

$$z_\mu = \sum_{\beta,j,m} \lambda^\beta_{jm}(\mu)\, J_n \Phi^\beta_{jm}, \tag{7.226}$$

unconditional convergence at least in $S'(\mathbb{R}^n)$. The liftings $J_n\Phi^\beta_{jm}$ and $D_L^{-1}\Phi^\beta_{jm}$ in (3.183), (3.187) with $2L = n$, and based on (3.154), are of the same type resulting in

$$J_n \Phi^\beta_{jm}(x) = 2^{-jn}\tilde{\Phi}^\beta_{jm}(x), \quad j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n,\ \beta \in \mathbb{N}^n_0, \tag{7.227}$$

where $\tilde{\Phi}^\beta_{jm}$ are of the same type as $\Phi^{\beta,L}_{jm}$ in Definition 3.29. For this purpose one has to replace $1 + |\xi|^{2L}$ with $2L = n$ by $(1 + |\xi|^2)^{n/2}$ in (3.181), (3.186), which does not change the arguments. In particular, for any $\alpha \in \mathbb{N}^n_0$, $\varkappa > 0$, and $a > 0$ there is a constant c such that

$$|D^\alpha \tilde{\Phi}^\beta_{jm}(x)| \le c\, 2^{j|\alpha|}\, 2^{-\varkappa|\beta|}\, (1 + |2^j x - m|)^{-a} \tag{7.228}$$

for all $\beta \in \mathbb{N}^n_0$, $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$ and $x \in \mathbb{R}^n$. By (7.225), (7.227) and (7.228) with $\alpha = 0$ one gets uniformly for all $x \in \mathbb{R}^n$ and $J \in \mathbb{N}$,

$$\Big| \sum_{\beta,j\ge J,m} \lambda^\beta_{jm}(\mu)\, J_n\Phi^\beta_{jm}(x) \Big| \le c \sum_{\beta,j\ge J,m} 2^{-\varkappa|\beta|}\, \frac{\mu(Q_{jm})}{(1 + |2^j x - m|)^a}\,. \tag{7.229}$$

With $a > n$ and $\varkappa > 0$ the right-hand side converges uniformly and we have

$$\Big| \sum_{\beta,j\ge J,m} \lambda^\beta_{jm}(\mu)\, J_n\Phi^\beta_{jm}(x) \Big| \le c \sum_{j\ge J} \mu_j. \tag{7.230}$$

With $J = 0$ it follows that (7.226) converges even absolutely (and hence unconditionally) in $C(\mathbb{R}^n)$. Furthermore one gets the first term on the right-hand side of (7.221). As for the second term we use that for some $c > 0$, all, say, C^∞ functions Φ in $\mathbb{R}^n$, all $x \in \mathbb{R}^n$ and all $h \in \mathbb{R}^n$ with $|h| \le 1$,

$$|\Delta^l_h \Phi(x)| \le c\, |h|^l \sum_{|\gamma|=l} \sup |D^\gamma \Phi(y)|, \tag{7.231}$$

where the supremum is taken over all $y \in \mathbb{R}^n$ with $|x - y| \le l|h|$. This can be proved by elementary reasoning, for example by induction with respect to $l \in \mathbb{N}$. It follows by (7.227), (7.228) and (7.231) that

$$\begin{aligned} \Big| \sum_{\beta,j<J,m} \lambda^\beta_{jm}(\mu)\, \Delta^l_h\, J_n\Phi^\beta_{jm}(x) \Big| &\le c\, |h|^l \sum_{\beta,j<J,m} 2^{-\varkappa|\beta|}\, \frac{2^{jl}\mu(Q_{jm})}{(1 + |2^j x - m|)^a} \\ &\le c'\, |h|^l \sum_{j<J} 2^{lj}\mu_j. \end{aligned} \tag{7.232}$$

This is the second term in (7.221). □

Remark 7.58. This is an improved version of [Tri04e, Theorem 1]. In particular we added now part (i) of the above theorem. On the other hand the basic idea to use wavelet expansions of type (7.223) and (7.226) is taken over from [Tri04e]. We add a discussion. If

$$\sum_{j=0}^{\infty} 2^{jd}\,\mu_j < \infty \quad \text{for some} \quad 0 \le d \le n, \tag{7.233}$$

then one gets by the above arguments for $d \ge l \in \mathbb{N}_0$,

$$|\Delta_h^l z_\mu(x)| \le c\,|h|^l \sum_{j=0}^{\infty} 2^{lj}\,\mu_j \le c'\,|h|^l, \quad 0 < |h| \le 1. \tag{7.234}$$

But this is not a surprise. In this case one can apply D^α with $|\alpha| \le l$ to (7.226). By (7.227), (7.228) one gets

$$z_\mu \in C^l(\mathbb{R}^n) = \{f \in C(\mathbb{R}^n) : \; D^\alpha f \in C(\mathbb{R}^n), \; |\alpha| \le l\}. \tag{7.235}$$

More interesting seems to be the case when

$$\sum_{j<J} 2^{jl}\,\mu_j \sim 2^{Jl}\mu_J \quad \text{for some} \quad n \ge l \in \mathbb{N}. \tag{7.236}$$

Then one gets by the same arguments as in Step 1 of the proof of the above theorem,

$$|\Delta_h^l z_\mu(x)| \le c Z_\mu(|h|), \qquad |h| \le 1. \tag{7.237}$$

If, in addition,

$$\sum_{j \ge J} \mu_j \sim \mu_J, \qquad J \in \mathbb{N}, \tag{7.238}$$

then one gets

$$|\Delta_h^l z_\mu(x)| \le c\,\mu_J \quad \text{with} \quad |h| \sim 2^{-J}. \tag{7.239}$$

Remark 7.59. If one has (7.236) then one can reduce the $n+1$ differences in (7.220) to l differences in (7.237). But this effect is well known. Of interest for us in connection with the above considerations are the equivalent norms in (1.12) for the Hölder-Zygmund spaces $\mathcal{C}^s(\mathbb{R}^n)$. There is the following generalisation.

Corollary 7.60. *Let μ be a measure as in Theorem 7.57 and let for some $k \in \mathbb{N}$, some $l \in \mathbb{N}$ with $l \le n$, some $0 < \varepsilon \le 2^{-k}$, and some $0 < \varkappa < 1$,*

$$Z_\mu(2^k t) \le \varkappa 2^{lk} Z_\mu(t) \quad \text{for all} \quad 0 < t \le \varepsilon. \tag{7.240}$$

Then

$$\begin{aligned} &\sup \frac{|\Delta_h^l z_\mu(x)|}{Z_\mu(|h|)} + \|z_\mu\,|L_\infty(\mathbb{R}^n)\| \\ &\sim \sup \frac{|\Delta_h^{n+1} z_\mu(x)|}{Z_\mu(|h|)} + \|z_\mu\,|L_\infty(\mathbb{R}^n)\| < \infty \end{aligned} \tag{7.241}$$

where the suprema are taken over $x \in \mathbb{R}^n$ and $h \in \mathbb{R}^n$ with $0 < |h| \le 1$.

Proof. By (7.220) the right-hand side of (7.241) is finite and can be estimated from above by the left-hand side. As for the converse estimate we recall that

$$\left(\Delta_h^l f\right)(x) = 2^{-l}\left(\Delta_{2h}^l f\right)(x) + \Delta_h^{l+1}\left(\sum_{r=0}^{2l} a_r f(x+rh)\right), \qquad (7.242)$$

where $a_r \in \mathbb{R}$ are suitable numbers, [Triβ, (46), p. 99]. Iteration gives

$$\left(\Delta_h^l f\right)(x) = 2^{-lk}\left(\Delta_{2^k h}^l f\right)(x) + \Delta_h^{l+1}\left(\sum a_r^k f(x+rh)\right). \qquad (7.243)$$

Denoting temporarily the norm on the left-hand side of (7.241) by $\|z_\mu\|_l$ it follows from (7.243) that

$$\|z_\mu\|_l \le 2^{-lk}\left(\sup_{0<t<\varepsilon}\frac{Z_\mu(2^k t)}{Z_\mu(t)}\right)\cdot\|z_\mu\|_l + c\,\|z_\mu\|_{l+1}. \qquad (7.244)$$

By (7.240) the factor of the first term on the right-hand side is smaller than 1. Then one gets

$$\|z_\mu\|_l \le c'\,\|z_\mu\|_{l+1}. \qquad (7.245)$$

Iteration by decreasing l, beginning with $l = n$, and restricting first $|h|$ to $|h| \ge \delta > 0$, with δ tending afterwards to zero, gives the desired result. □

Example 7.61. The first examples to be tested might be the isotropic measures according to Definition 7.18. Obviously, $\mu_j \sim h(2^{-j}) = h_j$. If μ is strongly isotropic then (7.238) coincides with (7.58). Of peculiar interest are the d-sets with the related measures $\mu = \mathcal{H}^d|\Gamma$ according to (7.97) where $h_j = 2^{-jd}$. We adopt here a slightly more general point of view and assume that

$$\mu_j \sim 2^{-jd} j^b, \quad j \in \mathbb{N}, \quad \text{where} \quad 0<d<n, \quad b \in \mathbb{R}. \qquad (7.246)$$

Then one has both (7.236) with $d < l \in \mathbb{N}$ and (7.238). Since always $Z_\mu(t) \sim Z_\mu(2t)$ one gets for $t \sim 2^{-J}$ that

$$Z_\mu(t) \sim \mu_J \sim 2^{-Jd} J^b \sim t^d\,|\log t|^b, \qquad 0 < t \le 1/2. \qquad (7.247)$$

Now it follows from (7.239), or (7.220) combined with Corollary 7.60, that

$$|\Delta_h^l z_\mu(x)| \le c\,|h|^d \cdot |\log|h||^b, \quad 0<|h|\le 1/2, \quad x \in \mathbb{R}^n. \qquad (7.248)$$

The same arguments apply to

$$\mu_j \sim 2^{-jn} j^b, \qquad j \in \mathbb{N}, \qquad b > 0. \qquad (7.249)$$

Then one gets for $n+1 \le l \in \mathbb{N}$,

$$|\Delta_h^l z_\mu(x)| \le c\,|h|^n \cdot |\log|h||^b, \quad 0<|h|\le 1/2, \quad x \in \mathbb{R}^n, \qquad (7.250)$$

and for $t \sim 2^{-J}$,

$$Z_\mu(t) \sim \mu_J \sim 2^{-Jn} J^b \sim t^n\,|\log t|^b, \quad 0<t\le 1/2. \qquad (7.251)$$

Finally in case of

$$\mu_j \sim j^{-a}, \qquad a > 1, \qquad j \in \mathbb{N}, \tag{7.252}$$

one has $\sum \mu_j < \infty$ and also (7.236) for any $l \in \mathbb{N}$. Now it follows that

$$\left|\Delta_h^l z_\mu(x)\right| \le c\,|\log|h|\,|^{1-a}, \quad 0 < |h| \le 1/2, \quad x \in \mathbb{R}^n, \tag{7.253}$$

and

$$Z_\mu(t) \sim |\log t|^{1-a}, \qquad 0 < t \le 1/2. \tag{7.254}$$

As mentioned above in case of isotropic measures according to Definition 7.18 one has $\mu_j \sim h(2^{-j}) = h_j$. Further information about isotropic measures may be found in Section 1.15. Especially we have the criterion in Theorem 1.155 and the examples in Remark 1.154. A list with further standard and non-standard isotropic measures and generating functions h may be found in [Bri03, Bri04]. In particular, according to these examples there are singular isotropic measures with (7.246), (7.249), (7.252). By (1.494) we may also assume that we have (7.210). Otherwise we again followed [Tri04e] with some modifications. There one finds also further information.

Remark 7.62. One may ask to which extent the assertions in Theorem 7.57, Corollary 7.60 and Example 7.61 are sharp. In case of d-sets, hence (7.246) with $0 < d < n$ and $b = 0$ one has

$$\mu = \mathcal{H}^d|\Gamma \in B^{d-n}_{\infty,q}(\mathbb{R}^n) \quad \text{if, and only if,} \quad q = \infty. \tag{7.255}$$

Then one obtains that

$$z_\mu = J_n \mu \in B^{d}_{\infty,q}(\mathbb{R}^n) \quad \text{if, and only if,} \quad q = \infty, \tag{7.256}$$

and (7.248) with $d < l \in \mathbb{N}$ and $b = 0$ is sharp. This assertion can be strengthened as follows. Recall that $\mathcal{C}^s(\mathbb{R}^n) = B^s_{\infty\infty}(\mathbb{R}^n)$. As in Remark 2.30 we denote by $\mathring{\mathcal{C}}^s(\mathbb{R}^n)$ with $s \in \mathbb{R}$ the completion of $S(\mathbb{R}^n)$ in $\mathcal{C}^s(\mathbb{R}^n)$ getting a genuine subspace of $\mathcal{C}^s(\mathbb{R}^n)$. Then one can strengthen (7.255) by

$$\mu = \mathcal{H}^d|\Gamma \in \mathcal{C}^{d-n}(\mathbb{R}^n)\backslash\mathring{\mathcal{C}}^{d-n}(\mathbb{R}^n) \tag{7.257}$$

and one gets for z_μ in (7.256),

$$\limsup_{|h|\to 0} \left[\sup_{\gamma\in\Gamma} |h|^{-d} \cdot \left|\left(\Delta_h^l z_\mu\right)(\gamma)\right|\right] \sim 1 \tag{7.258}$$

with $d < l \in \mathbb{N}$. Since J_n is an elliptic operator one can restrict the supremum over Γ in (7.258) for any fixed $\gamma_0 \in \Gamma$ by $|\gamma - \gamma_0| \le \varepsilon$ with $\gamma \in \Gamma$, $\varepsilon > 0$. One may ask whether this local assertion can even be strengthened by pointwise assertions and whether these considerations can be extended to arbitrary measures covered by Theorem 7.57. In other words, one may ask whether (7.220) and (7.241), (7.240) have pointwise counterparts of type

$$\left|\Delta_h^l\, z_\mu(\gamma)\right| \sim Z_\mu(|h|), \qquad 0 < |h| \le 1, \tag{7.259}$$

for some or for all $\gamma \in \Gamma = \operatorname{supp} \mu$ with equivalence constants which are independent of h. Of course, the first candidates to be tested are isotropic measures. We refer in this context also to [Tri04e, p. 281]. To have a notation one could call $z_\mu(\gamma)$ the *Bessel characteristics* of μ with $\gamma \in \Gamma = \operatorname{supp} \mu$. It might well be the case that both $Z_\mu(t)$ and $z_\mu(\gamma)$ play a more significant role in the study of fractal and regularity properties of Radon measures in $\mathbb{R}^n$ than reflected by the above assertions.

7.3.2 Elliptic operators: general measures

First we collect some notation and assertions obtained so far. Let

$$H^\sigma(\mathbb{R}^n) = H^\sigma_2(\mathbb{R}^n) = B^\sigma_{2,2}(\mathbb{R}^n) = F^\sigma_{2,2}(\mathbb{R}^n), \qquad \sigma \in \mathbb{R}, \tag{7.260}$$

be the distinguished Hilbert spaces now furnished with the scalar product

$$(f, g)_{H^\sigma(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (\mathrm{id} - \Delta)^{\sigma/2} f(x) \cdot (\mathrm{id} - \Delta)^{\sigma/2} \overline{g(x)} \, \mathrm{d}x. \tag{7.261}$$

Again let for $s \in \mathbb{R}$,

$$J_s = (\mathrm{id} - \Delta)^{-s/2}, \qquad J_s H^\sigma(\mathbb{R}^n) = H^{\sigma+s}(\mathbb{R}^n), \tag{7.262}$$

be the usual Bessel potentials with the indicated isomorphic mapping properties according to (7.18). This justifies in particular (7.261). Let tr_μ be the trace operator and id_μ be the identification operator for Radon measures μ according to (7.210) as introduced in Section 7.1.3. Let μ_j be as in (7.211).

Proposition 7.63. *Let μ be a Radon measure in $\mathbb{R}^n$ according to* (7.210) *with $\sum_{j=0}^\infty \mu_j < \infty$. Then*

$$\mathrm{id}^\mu = \mathrm{id}_\mu \circ \mathrm{tr}_\mu : \quad H^{n/2}(\mathbb{R}^n) \hookrightarrow H^{-n/2}(\mathbb{R}^n) \tag{7.263}$$

is linear and compact. There is a positive number c such that for all μ,

$$\|\mathrm{id}^\mu\| \le c \sum_{j=0}^\infty \mu_j. \tag{7.264}$$

If, in addition, $\sum_{j=0}^\infty \mu_j^{1/2} < \infty$ then

$$\mathrm{id}^\mu : \quad H^{n/2}(\mathbb{R}^n) \hookrightarrow B^{-n/2}_{2,1}(\mathbb{R}^n) \tag{7.265}$$

is linear and compact.

Proof. Both (7.263), (7.264) and the compactness of id^μ follow from Theorem 7.16 with $p = q = 2$ and $s = n/2$. If μ satisfies the stronger condition $\sum \mu_j^{1/2} < \infty$ then one can apply part (i) of Theorem 7.16 with $p = q = 2$, $s = n/2$ to tr_μ and part (ii) with $p = 2$, $q = 1$, $s = n/2$ to id_μ. □

Remark 7.64. For measures μ according to the above proposition and the special Bessel potentials J_n it follows from (7.262) that

$$A = J_n \circ \mathrm{id}^\mu = (\mathrm{id} - \Delta)^{-n/2} \circ \mathrm{id}^\mu : \quad H^{n/2}(\mathbb{R}^n) \hookrightarrow H^{n/2}(\mathbb{R}^n) \tag{7.266}$$

is a linear and compact operator. If, in addition, $\sum_{j=0}^{\infty} \mu_j^{1/2} < \infty$ then one gets by the above proposition and (7.18) that

$$A = J_n \circ \mathrm{id}^\mu : \quad H^{n/2}(\mathbb{R}^n) \hookrightarrow B^{n/2}_{2,1}(\mathbb{R}^n) \tag{7.267}$$

and also

$$A = J_n \circ \mathrm{id}^\mu : \quad H^{n/2}(\mathbb{R}^n) \hookrightarrow H^{n/2}(\mathbb{R}^n) \cap C(\mathbb{R}^n) \tag{7.268}$$

are linear and compact. We used (1.203) in the latter assertion. We are now in a similar situation as in Section 1.13.2. There we dealt with bounded C^∞ domains Ω in the plane $\mathbb{R}^2$ instead of $\mathbb{R}^n$ and with the inverse $(-\Delta)^{-1}$ of the Dirichlet Laplacian in Ω instead of J_n. Then Proposition 1.141 and Theorem 1.143 are the counterpart of (7.263), (7.266), (7.268). According to Remark 1.144, Theorem 1.143 is covered by the reasoning and the references given there with exception of the compactness of tr_μ and part (iii) of this theorem. But these gaps are now sealed by the above arguments with $n = 2$ and $(-\Delta)^{-1}$ in place of J_2. It is the next aim to prove the counterpart of Theorem 1.145 with $\mathbb{R}^n$ in place of bounded C^∞ domains Ω in $\mathbb{R}^2$ and J_n instead of $(-\Delta)^{-1}$. But $n = 2$ has some peculiarities which are covered by the arguments and references given in Section 1.13.2. This applies in particular to the observation that the largest eigenvalue ϱ in (1.470) is simple and that the related eigenfunctions in (1.471) have no zeros. There are no assertions of this type for general $n \in \mathbb{N}$. But furnishing $H^{n/2}(\mathbb{R}^n)$ with the specific scalar product (7.261) one has a counterpart of the crucial observation (1.468) which now reads as follows,

$$(Af, g)_{H^{n/2}(\mathbb{R}^n)} = \int_\Gamma f(\gamma)\, \overline{g(\gamma)}\, \mu(\mathrm{d}\gamma), \quad f \in H^{n/2}(\mathbb{R}^n), \quad g \in H^{n/2}(\mathbb{R}^n). \tag{7.269}$$

This can be justified by the arguments given in [Triϵ, Section 19.3, p. 257] and [Triδ, Sections 28.6, 30.2, pp. 226, 234], which will not be repeated here. Hence one can generate the linear and compact operator A in $H^{n/2}(\mathbb{R}^n)$ according to (7.266) by the scalar product of $L_2(\Gamma, \mu)$ interpreted as a bounded symmetric quadratic form in $H^{n/2}(\mathbb{R}^n)$. As a consequence, A is a non-negative self-adjoint compact operator in $H^{n/2}(\mathbb{R}^n)$ and

$$\|\sqrt{A} f \,|H^{n/2}(\mathbb{R}^n)\| = \|\mathrm{tr}_\mu f \,|L_2(\Gamma, \mu)\|. \tag{7.270}$$

Otherwise we use the previous notation. In particular, $Z_\mu(t)$ has the same meaning as in (7.215) and the differences Δ_h^l in $\mathbb{R}^n$ are given by (4.32). As before μ_j is defined by (7.211).

Theorem 7.65. *Let μ be a Radon measure in $\mathbb{R}^n$ according to* (7.210) *with* $\sum_{j=0}^{\infty} \mu_j < \infty$ *and let*

$$A = J_n \circ \mathrm{id}^\mu = (\mathrm{id} - \Delta)^{-n/2} \circ \mathrm{id}^\mu. \tag{7.271}$$

(i) *Then A is a non-negative compact self-adjoint operator in $H^{n/2}(\mathbb{R}^n)$, generated by the quadratic form* (7.269). *Let ϱ_k be the positive eigenvalues of A, repeated according to multiplicity and ordered by decreasing magnitude and let u_k be the related eigenfunctions,*

$$Au_k = \varrho_k\, u_k, \qquad k \in \mathbb{N}. \tag{7.272}$$

Then

$$\varrho_1 \ge \varrho_2 \ge \cdots > 0, \qquad \varrho_k \to 0 \quad \text{if} \quad k \to \infty, \tag{7.273}$$

and there is a positive constant c such that for all measures μ,

$$\varrho_1 \le c\, Z_\mu(1) = c \sum_{j=0}^{\infty} \mu_j. \tag{7.274}$$

Furthermore, the u_k are C^∞ functions in $\mathbb{R}^n \backslash \Gamma$.

(ii) *Let, in addition, $\sum_{j=0}^{\infty} \mu_j^{1/2} < \infty$. Then u_k are continuous functions in $\mathbb{R}^n$. There is a positive constant c such that for all μ according to* (7.210) *with $\sum_{j=0}^{\infty} \mu_j^{1/2} < \infty$, all $x \in \mathbb{R}^n$, all $h \in \mathbb{R}^n$ with $0 < |h| \le 1$ and all eigenfunctions u_k normalised by $\|u_k \,|L_\infty(\mathbb{R}^n)\| \le 1$,*

$$\varrho_k \left|\Delta_h^{n+1} u_k(x)\right| \le c\, Z_\mu(|h|). \tag{7.275}$$

Proof. Step 1. We prove part (i). By the above preparations it follows that A is a non-negative compact self-adjoint operator, generated by (7.269). Furthermore, (7.274) follows from (7.264) and (7.262) with $\sigma = -n/2$ and $s = n$. Rewriting (7.272) as

$$\varrho_k\, (\mathrm{id} - \Delta)^{n/2} u_k = \mathrm{id}_\mu \circ \mathrm{tr}_\mu u_k, \tag{7.276}$$

one gets

$$(\mathrm{id} - \Delta)^{n/2} u_k = 0 \qquad \text{in} \quad D'(\mathbb{R}^n \backslash \Gamma) \tag{7.277}$$

and hence $u_k \in C^\infty(\mathbb{R}^n \backslash \Gamma)$.

Step 2. We prove part (ii). By (7.268) we have $u_k \in C(\mathbb{R}^n)$. Normalised by $\sup |u_k(x)| \le 1$ it follows that the $u_k \mu$ in

$$\varrho_k u_k = (\mathrm{id} - \Delta)^{-n/2} u_k \mu, \qquad k \in \mathbb{N}, \tag{7.278}$$

can be interpreted as uniformly bounded complex measures. We expand $u_k\mu$ as in (7.223) and get in generalisation of (7.225) that

$$|\lambda^\beta_{jm}(u_k\mu)| \le c\, 2^{jn} \mu(Q_{jm}), \tag{7.279}$$

where c is independent of j, m, β, μ and u_k. But then one obtains (7.275) as in the proof of Theorem 7.57. □

Remark 7.66. This is a modified version of [Tri04e, Theorem 2]. One can replace the right-hand side of (7.275) by the right-hand side of (7.221) or the corresponding terms in Corollary 7.60 if (7.240) is satisfied. In Example 7.61 we discussed a few special cases. Then we have the corresponding specifications of $Z_\mu(|h|)$ on the right-hand side of (7.275).

Remark 7.67. In Section 1.13.2 we considered the corresponding problem for the operator B in Proposition 1.141 with the inverse $(-\Delta)^{-1}$ of the Dirichlet Laplacian in a bounded C^∞ domain in the plane $\mathbb{R}^2$ in place of J_n in (7.271). One can complement Theorem 1.145 by related counterparts of (7.274), (7.275). Some assertions of this type may be found in [Tri04e, Theorem 3]. Of specific interest might be the positive eigenfunction $u(x) = u^\mu(x)$ of the largest eigenvalue ϱ, which is simple, according to (1.470), (1.471). This is the fractal counterpart of Courant's classical observation as described in Theorem 1.137. One may call $u^\mu(\gamma)$ with $\gamma \in \Gamma$ the *Courant characteristics* of μ complementing the Bessel characteristics $z_\mu(\gamma)$ at the end of Section 7.3.1, now restricted to the plane $\mathbb{R}^2$. One may ask in modification of (7.259) under which circumstances one has

$$\left|\Delta_h^l u^\mu(\gamma)\right| \sim Z_\mu(|h|), \qquad \gamma \in \Gamma, \quad 0 < |h| \le 1. \tag{7.280}$$

7.3.3 Elliptic operators: isotropic measures

Theorem 7.65 deals with general measures according to equation (7.210) with $\sum_{j=0}^\infty \mu_j < \infty$ and special Bessel potentials J_n such that the operator A in (7.271) acts just in the critical Hilbert spaces $H^{n/2}(\mathbb{R}^n)$ in the notation of Theorem 1.73. Now we assume that μ is an isotropic measure according to Definition 7.18, which includes (7.210) = (7.32). On the other hand we extend now the considerations from J_n in (7.266), (7.271) to J_{2s} according to (7.262) asking for operators B_s,

$$B_s = (\mathrm{id} - \Delta)^{-s} \circ \mathrm{id}^\mu : \qquad H^s(\mathbb{R}^n) \hookrightarrow H^s(\mathbb{R}^n), \tag{7.281}$$

where $0 < s \le n/2$. First we explain and ensure (7.281). We again assume that $H^s(\mathbb{R}^n)$ are the Hilbert spaces (7.260) furnished with the scalar product (7.261). Furthermore if μ satisfies the hypotheses of Theorem 7.22 then we have

$$\mathrm{tr}_\mu : \quad H^s(\mathbb{R}^n) \hookrightarrow L_2(\Gamma, \mu) \tag{7.282}$$

for the trace operator and by Theorem 7.16,

$$\mathrm{id}_\mu : \quad L_2(\Gamma, \mu) \hookrightarrow H^{-s}(\mathbb{R}^n) \tag{7.283}$$

for the related identification operator. Hence

$$\mathrm{id}^\mu = \mathrm{id}_\mu \circ \mathrm{tr}_\mu : \quad H^s(\mathbb{R}^n) \hookrightarrow H^{-s}(\mathbb{R}^n), \tag{7.284}$$

in modification of (7.263). Together with (7.262) one gets (7.281). With the same references as there one has the counterparts of (7.269), (7.270),

$$(B_s f, g)_{H^s(\mathbb{R}^n)} = \int_\Gamma f(\gamma)\,\overline{g(\gamma)}\,\mu(\mathrm{d}\gamma), \quad f \in H^s(\mathbb{R}^n), \quad g \in H^s(\mathbb{R}^n), \tag{7.285}$$

and

$$\left\| \sqrt{B_s} f \,|H^s(\mathbb{R}^n)\right\| = \|\mathrm{tr}_\mu f \,|L_2(\Gamma,\mu)\|\,. \tag{7.286}$$

Theorem 7.68. *Let* μ *be a strongly isotropic Radon measure according to Definition* 7.18 *with the generating function* h *and the inverse function* H *given by* (7.70)*. Let* $0 < s \le n/2$ *and*

$$\sum_{j \ge J} 2^{j(n-2s)} h_j \sim 2^{J(n-2s)} h_J, \qquad J \in \mathbb{N}_0, \tag{7.287}$$

where the equivalence constants are independent of J*. Then* B_s *as introduced above is a compact, non-negative self-adjoint operator in* $H^s(\mathbb{R}^n)$*. Let* ϱ_k *be the positive eigenvalues of* B_s*, repeated according to multiplicity and ordered by decreasing magnitude,*

$$\varrho_1 \ge \varrho_2 \ge \cdots > 0, \qquad \varrho_k \to 0 \quad \textit{if} \quad k \to \infty. \tag{7.288}$$

Then

$$\varrho_k \sim k^{-1}\, H(k^{-1})^{2s-n}, \qquad k \in \mathbb{N}. \tag{7.289}$$

Proof. Step 1. By Theorem 7.22 the trace operator tr_μ in (7.282) is compact. Then also its dual operator id_μ in (7.283) is compact. It follows that both id^μ in (7.284) and B_s in (7.281) are compact. Together with the above explanations one gets all the assertions of the theorem with the exception of (7.289).

Step 2. We prove that there is a number c such that

$$\varrho_k \le c k^{-1} H(k^{-1})^{2s-n}, \qquad k \in \mathbb{N}. \tag{7.290}$$

So far we have (7.73) for the approximation numbers $a_k(\mathrm{tr}_\mu)$. By the duality (7.37) and the well-known properties of the approximation numbers for dual operators, [EdE87, Proposition 2.5, p. 55], one gets

$$a_k(\mathrm{id}_\mu) = a_k(\mathrm{tr}_\mu) \sim k^{-1/2} H(k^{-1})^{s-\frac{n}{2}}, \qquad k \in \mathbb{N}. \tag{7.291}$$

Using the multiplication property for approximation numbers according to (1.285) and (1.293) one gets

$$\varrho_{2k} = a_{2k}(B_s) \le c\,\|J_s\|\,a_k(\mathrm{tr}_\mu)^2 \le c' k^{-1} H(k^{-1})^{2s-n}. \tag{7.292}$$

By (7.96) we have $H(t) \sim H(2t)$ if $0 < 2t \le 1$. Then (7.290) follows from (7.292).

Step 3. By the above considerations the converse of (7.290) follows from

$$a_k(\sqrt{B_s}) \ge c\,k^{-1/2} H(k^{-1})^{s-\frac{n}{2}}, \qquad k \in \mathbb{N}, \tag{7.293}$$

for some $c > 0$. But here we are in the same situation as in Step 2 of the proof of Theorem 7.22 with $p = 2$. Let as there $M_j \sim h_j^{-1}$ and

$$f_j(x) = \sum_{l=1}^{M_j} c_{jl} 2^{-j(s-\frac{n}{2})} \varkappa\left(2^j(x - \gamma^{j,l})\right), \quad c_{jl} \in \mathbb{C}, \quad x \in \mathbb{R}^n. \tag{7.294}$$

Then one gets by the counterparts of (7.87)–(7.89) and (7.286) that

$$\left\| \sqrt{B_s} f_j \,|H^s(\mathbb{R}^n) \right\| \sim 2^{-j(s-\frac{n}{2})} h_j^{1/2} \quad \text{if} \quad \|f_j \,|H^s(\mathbb{R}^n)\| \sim 1. \tag{7.295}$$

Now (7.293) follows from (7.295) in the same way as in Step 2 of the proof of Theorem 7.22. □

Remark 7.69. We followed essentially [Tri04c]. We refer also to Section 1.15.2 where we dealt with the special case $n = 2$ and $s = n/2 = 1$ in the setting of the operator B in (1.503). But there is no essential difference compared with B_1 according to (7.281) in the plane $\mathbb{R}^2$. There one finds also further references. We took the outcome $\varrho_k \sim k^{-1}$ to introduce in Definition 1.159 what we called a Weyl measure. Now we extend this notation from $\mathbb{R}^2$ to $\mathbb{R}^n$.

7.3.4 Weyl measures

We extend the notation of Weyl measures from $\mathbb{R}^2$ according to Definition 1.159 to $\mathbb{R}^n$. References, results and explanations may be found in Section 1.16.

Definition 7.70. *Let μ with $\sum_{j=0}^\infty \mu_j < \infty$ be a Radon measure in $\mathbb{R}^n$ according to* (7.210), (7.211)*. Then μ is said to be a Weyl measure if*

$$\varrho_k \sim k^{-1}, \qquad k \in \mathbb{N}, \tag{7.296}$$

for the eigenvalues of the operator A in Theorem 7.65*.*

Remark 7.71. According to Theorem 7.65 the operator A is non-negative self-adjoint and compact in $H^{n/2}(\mathbb{R}^n)$ and its positive eigenvalues ϱ_k are ordered by (7.273). Hence the question (7.296) makes sense. If $n = 2$ then one has an immaterial modification of Definition 1.159. There are the remarkable Theorem 1.161 and the disillusion originating from Proposition 1.163 that there might be simple natural necessary and sufficient conditions for measures μ to be Weylian. We restrict ourselves here to a straightforward conclusion of Theorem 7.68.

Corollary 7.72. *Any strongly isotropic Radon measure μ according to Definition* 7.18 *is a Weyl measure.*

Proof. By Proposition 7.20 we have (7.58). Since $A = B_{n/2}$ it follows from Theorem 7.68 that μ is a Weyl measure. □

Remark 7.73. This is the n-dimensional generalisation of a corresponding assertion in Theorem 1.157(iii).

Chapter 8

Function Spaces on Quasi-metric Spaces

8.1 Spaces on d-sets

8.1.1 Introduction

So far we dealt mainly with function spaces on $\mathbb{R}^n$ and on (Lipschitz) domains in $\mathbb{R}^n$, and, to a lesser extent, with spaces

$$B^s_{pq}(\Gamma,\mu), \qquad s>0, \quad 1<p<\infty, \quad 0<q\le\infty, \tag{8.1}$$

on compact sets $\Gamma = \operatorname{supp}\mu$ in $\mathbb{R}^n$ according to Definition 1.178, introduced as trace spaces

$$B^s_{pq}(\Gamma,\mu) = \operatorname{tr}_\mu B^{s+|s_\mu(1-t)|}_{pq}(\mathbb{R}^n), \qquad 0<1/p=t<1, \tag{8.2}$$

where the Besov characteristcs s_μ of μ are coming in naturally. We characterised these spaces in Theorem 1.185 in terms of quarkonial representations. It is our first aim to return to this subject in greater detail specifying Γ to be d-sets in $\mathbb{R}^n$ with $0<d<n$, and the spaces from (8.1) to be

$$B^s_p(\Gamma) = B^s_{pp}(\Gamma,\mu), \qquad s>0, \quad 1<p<\infty, \tag{8.3}$$

notationally in good agreement with (3.196) and (7.69). Recall that a compact set Γ in $\mathbb{R}^n$ is called a d-set with $0<d<n$ if there is a Radon measure μ in $\mathbb{R}^n$ with

$$\operatorname{supp}\mu = \Gamma, \quad \mu\left(B(\gamma,r)\right) \sim r^d, \quad \gamma\in\Gamma, \quad 0<r<1, \tag{8.4}$$

where $B(x,r)$ is a ball in $\mathbb{R}^n$ centred at $x\in\mathbb{R}^n$ and of radius $r>0$. Then (8.2), (8.3) reduces to

$$B^s_p(\Gamma) = \operatorname{tr}_\mu B^{s+(n-d)t}_p(\mathbb{R}^n), \quad s>0, \quad 0<1/p=t<1, \tag{8.5}$$

using now the abbreviation (7.69). Details and references may be found in Remark 1.179 and also in Theorem 1.181, Remark 1.182. Furthermore, (8.3) is justified by the well-known observation that for given Γ any two Radon measures μ with (8.4) are equivalent to each other and equivalent to the restriction of the Hausdorff measure $\mathcal{H}^d$ in $\mathbb{R}^n$ to Γ. A short proof may be found in [Triδ, Theorem 3.4, p. 5]. This gives the possibility to fix in what follows,

$$\mu = \mathcal{H}^d_\Gamma = \mathcal{H}^d|\Gamma, \quad \Gamma \text{ compact } d\text{-set in } \mathbb{R}^n, \quad 0 < d < n. \tag{8.6}$$

First we adapt in this Section 8.1 the quarkonial representations for the spaces in (8.5), (8.6) to our later needs, complemented by intrinsic decompositions in term of non-smooth atoms. However it is the main aim of Chapter 8 to develop a new approach to function spaces on abstract quasi-metric spaces. This can be done on a large scale. But we are more interested in presenting basic ideas than in the most general formulations and spaces. For this purpose we rely on the snowflaked transform as described in Section 1.17.4, especially Theorem 1.192, Remark 1.193, transferring the spaces from (8.3) to some classes of abstract quasi-metric spaces. This will be done in Sections 8.2–8.4, including some applications, returning in detail to what we indicated roughly in Section 1.17.6. (One may also look at Section 1.19). In the following Chapter 9 we describe a different approach to spaces on arbitrary compact sets in $\mathbb{R}^n$. One can combine this approach with general snowflaked transforms which apply to huge classes of abstract quasi-metric spaces. But this will not be done. In this Chapter 8 we follow essentially [Tri05c].

8.1.2 Quarkonial characterisations

In Section 1.17.3 we described quarkonial characterisations of the spaces (8.1), (8.2) for rather general measures μ according to (1.515). Now we specify these considerations to compact d-sets in $\mathbb{R}^n$ according to (8.4), (8.6) and to the related spaces in (8.3), (8.5). We adapt this approach to our later needs. For $\delta > 0$ let

$$\Gamma_\delta = \{x \in \mathbb{R}^n : \ \text{dist}\ (x, \Gamma) < \delta\} \tag{8.7}$$

be a δ-neighborhood of the compact d-set Γ in $\mathbb{R}^n$. Let $\varepsilon > 0$ be fixed (later on ε will be the same number as in Theorem 1.192, in particular $0 < \varepsilon < 1$). We now modify the construction at the beginning of Section 1.17.3 as follows. Let for $k \in \mathbb{N}_0$,

$$\left\{\gamma^{k,m}\right\}_{m=1}^{M_k} \subset \Gamma \quad \text{and} \quad \left\{\psi^{k,m}\right\}_{m=1}^{M_k} \tag{8.8}$$

be (appropriate) lattices and subordinated resolutions of unity with the following properties:

(i) For some $c_1 > 0$,

$$|\gamma^{k,m_1} - \gamma^{k,m_2}| \geq c_1\, 2^{-\varepsilon k}, \qquad k \in \mathbb{N}_0, \quad m_1 \neq m_2. \tag{8.9}$$

(ii) For some $k_0 \in \mathbb{N}$, some $c_2 > 0$ and $\delta_k = c_2 2^{-\varepsilon k}$,

$$\Gamma_{\delta_k} \subset \bigcup_{m=1}^{M_k} B\left(\gamma^{k,m}, 2^{-\varepsilon(k+2k_0)}\right), \qquad k \in \mathbb{N}_0, \tag{8.10}$$

where again $B(x,r)$ are $\mathbb{R}^n$-balls centred at $x \in \mathbb{R}^n$ and of radius $r > 0$.

(iii) Furthermore, $\psi^{k,m}$ are non-negative C^∞ functions in $\mathbb{R}^n$ with

$$\operatorname{supp} \psi^{k,m} \subset B\left(\gamma^{k,m}, 2^{-\varepsilon(k+k_0)}\right), \qquad k \in \mathbb{N}_0; \quad m = 1, \ldots, M_k, \tag{8.11}$$

$$|D^\alpha \psi^{k,m}(x)| \le c_\alpha 2^{k\varepsilon|\alpha|}, \qquad k \in \mathbb{N}_0; \quad m = 1, \ldots, M_k, \tag{8.12}$$

for all multi-indices $\alpha \in \mathbb{N}_0^n$ and suitable positive constants c_α, and

$$\sum_{m=1}^{M_k} \psi^{k,m}(x) = 1, \qquad k \in \mathbb{N}_0, \quad x \in \Gamma_{\delta_k}. \tag{8.13}$$

Since Γ is a compact d-set we may assume that $M_k \sim 2^{\varepsilon k d}$. Furthermore, (8.13) can be extended consistently to a corresponding resolution of unity in $\mathbb{R}^n$. As usual we put

$$\gamma^\beta = \gamma_1^{\beta_1} \cdots \gamma_n^{\beta_n}, \qquad \gamma \in \Gamma \quad \text{and} \quad \beta \in \mathbb{N}_0^n, \tag{8.14}$$

and abbreviate

$$B_{k,m} = B\left(\gamma^{k,m}, 2^{-\varepsilon k}\right), \qquad k \in \mathbb{N}_0; \quad m = 1, \ldots, M_k. \tag{8.15}$$

Definition 8.1. *Let Γ be a d-set in $\mathbb{R}^n$ according to (8.4), (8.6) and let $\varepsilon > 0$.*

(i) *Let $\left\{\gamma^{k,m}\right\}_{m=1}^{M_k}$ and $\left\{\psi^{k,m}\right\}_{m=1}^{M_k}$ be as above. Then*

$$\varepsilon\text{-}\Psi_\Gamma = \left\{\psi_\beta^{k,m} : \ \beta \in \mathbb{N}_0^n, \ k \in \mathbb{N}_0; \ m = 1, \ldots, M_k\right\} \tag{8.16}$$

with

$$\psi_\beta^{k,m}(\gamma) = \mathcal{H}_\Gamma^d(B_{k,m})^{-|\beta|/d} \left(\gamma - \gamma^{k,m}\right)^\beta \psi^{k,m}(\gamma), \qquad \gamma \in \Gamma, \tag{8.17}$$

and for $s > 0$, $1 < p < \infty$,

$$\varepsilon\text{-}\Psi_\Gamma^{s,p} = \left\{(\beta\text{-qu})_{km}^\Gamma : \ \beta \in \mathbb{N}_0^n, \ k \in \mathbb{N}_0; \ m = 1, \ldots, M_k\right\} \tag{8.18}$$

with the (s,p)-β-quarks on Γ,

$$(\beta\text{-qu})_{km}^\Gamma = \mathcal{H}_\Gamma^d(B_{k,m})^{\frac{s}{d} - \frac{1}{p}} \psi_\beta^{k,m}(\gamma), \qquad \gamma \in \Gamma. \tag{8.19}$$

(ii) *Let*

$$\nu = \left\{\nu_{km}^\beta \in \mathbb{C} : \ \beta \in \mathbb{N}_0^n, \ k \in \mathbb{N}_0; \ m = 1, \ldots, M_k\right\} \tag{8.20}$$

and $1 < p < \infty$. *Then*

$$\ell_p^\Gamma = \left\{ \nu : \ \|\nu \,|\ell_p^\Gamma\| = \left(\sum_{\beta,k,m} |\nu_{km}^\beta|^p \right)^{1/p} < \infty \right\}. \tag{8.21}$$

Remark 8.2. As before, $\mathbb{C}$ is the complex plane. This is the adapted version of corresponding definitions in Section 1.17.3 in the specification of (1.574). We fixed now the number ϱ in Definition 1.183(ii) and replaced (1.570), (1.571) with $q = p$ (and $\varrho = 0$) by (8.21). This is possible if $k_0 \in \mathbb{N}$ in (8.10), (8.11) is chosen appropriately. In particular one may assume that we have for some $c > 0$ the exponential decay

$$\left| \psi_\beta^{k,m}(\gamma) \right| \le c\, 2^{-\varepsilon|\beta|}, \qquad \beta \in \mathbb{N}_0^n, \ k \in \mathbb{N}_0; \ m = 1, \ldots, M_k. \tag{8.22}$$

The (s,p)-β-quarks $(\beta\text{-qu})_{km}^\Gamma$ are the modifications of the corresponding building blocks in Definition 1.183, (1.574) and Theorem 1.185. Of course, $(\beta\text{-qu})_{km}^\Gamma$ depends on s and p, but this will not be indicated. By (8.4) one can replace

$$\psi_\beta^{k,m}(\gamma) \quad \text{by} \quad 2^{\varepsilon k|\beta|} (\gamma - \gamma^{k,m})^\beta \, \psi^{k,m}(\gamma) \tag{8.23}$$

and

$$(\beta\text{-qu})_{km}^\Gamma \quad \text{by} \quad 2^{-\varepsilon k(s-d/p) + \varepsilon k|\beta|} \, (\gamma - \gamma^{k,m})^\beta \, \psi^{k,m}(\gamma). \tag{8.24}$$

But the above version might be more transparent when snowflaked transforms are applied. On the other hand if one prefers the right-hand sides of (8.23), (8.24) then one can choose $k_0 = 1$ in (8.10), (8.11). But this is immaterial.

After these preparations we characterise now the trace spaces from (8.5) in terms of the above building blocks in specification and modification of Theorem 1.185. As far as traces are concerned we refer also to Section 7.1.3. As before $L_r(\Gamma, \mu)$ with $1 \le r < \infty$ are the usual complex Banach spaces normed by (1.523), (7.35). Specialised to the above d-sets furnished with the canonical measure (8.6) we write simply $L_r(\Gamma)$ (in good agreement with (8.5)), normed by

$$\|f \,|L_r(\Gamma)\| = \left(\int_\Gamma |f(\gamma)|^r \mathcal{H}_\Gamma^d(\mathrm{d}\gamma) \right)^{1/r}. \tag{8.25}$$

Theorem 8.3. *Let* Γ *be a compact* d*-set in* $\mathbb{R}^n$ *with* $0 < d < n$ *according to* (8.4), (8.6). *Let* $B_p^s(\Gamma)$ *be as in* (8.5). *Let* $\varepsilon > 0$ *and let* $(\beta\text{-qu})_{km}^\Gamma$ *be the* (s,p)*-*β*-quarks and* ℓ_p^Γ *be the sequence spaces as in Definition* 8.1.

(i) *Then* $B_p^s(\Gamma)$ *is the collection of all* $f \in L_1(\Gamma)$ *which can be represented as*

$$f(\gamma) = \sum_{\beta \in \mathbb{N}_0^n} \sum_{k=0}^\infty \sum_{m=1}^{M_k} \nu_{km}^\beta \cdot (\beta\text{-qu})_{km}^\Gamma(\gamma), \qquad \|\nu \,|\ell_p^\Gamma\| < \infty, \tag{8.26}$$

$\gamma \in \Gamma$ (*absolute convergence being in* $L_1(\Gamma)$). *Furthermore,*

$$\|f \,|B^s_p(\Gamma)\| \sim \inf \|\nu \,|\ell^\Gamma_p\|, \tag{8.27}$$

where the infimum is taken over all admissible representations (8.26).

(ii) *Let, in addition,* $0 < s < 1$. *Then there is a linear and bounded mapping*

$$f \mapsto \nu(f) = \left\{\nu^\beta_{km}(f)\right\}: \quad B^s_p(\Gamma) \mapsto \ell^\Gamma_p, \tag{8.28}$$

such that

$$f(\gamma) = \sum_{\beta,k,m} \nu^\beta_{km}(f) \cdot (\beta\text{-qu})^\Gamma_{km}(\gamma), \qquad \gamma \in \Gamma, \tag{8.29}$$

with

$$\|f \,|B^s_p(\Gamma)\| \sim \|\nu(f) \,|\ell^\Gamma_p\| \tag{8.30}$$

(*equivalent norms where the equivalence constants do not depend on* f).

Proof. Part (i) with $\varepsilon = 1$ and the modified (s,p)-β-quarks as indicated on the right-hand sides of (8.23), (8.24) is covered by Theorem 1.185 and the references and explanations in Remark 1.186, specialised to d-sets and (8.5). In particular, the series in (8.26) converges absolutely in $L_1(\Gamma)$ if $\|\nu \,|\ell^\Gamma_p\| < \infty$. Then it follows that (8.26) and (8.29) converge unconditionally in the spaces considered. The step from $\varepsilon = 1$ to $\varepsilon > 0$ is a technical matter adapting formulations and normalising factors. If $0 < s < 1$ then one can apply Theorem 1.181 and Remark 1.182. One has a common extension operator

$$\text{ext}_\Gamma : \quad B^s_p(\Gamma) \hookrightarrow B^{s+(n-d)t}_p(\mathbb{R}^n), \quad g = \text{ext}_\Gamma f, \tag{8.31}$$

where again $0 < 1/p = t < 1$. By Theorem 1.39 and Corollary 1.42 one gets the frame representation

$$g = \sum_{\beta \in \mathbb{N}^n_0} \sum_{k=0}^\infty \sum_{m \in \mathbb{Z}^n} \left(g, \Psi^\beta_{km}\right) \cdot (\beta\text{-qu})_{km}, \tag{8.32}$$

where $(\beta\text{-qu})_{km}$ are $(\sigma,p)_L$-β-quarks in $\mathbb{R}^n$ with $s + (n-d)t = \sigma < L \in \mathbb{N}$, and

$$\|g \,|B^\sigma_p(\mathbb{R}^n)\| \sim \left(\sum_{\beta,k,m} \left|\left(g, \Psi^\beta_{km}\right)\right|^p \right)^{1/p} \tag{8.33}$$

(equivalent norms). We always assumed that the covering (8.10) and the related resolution of unity in (8.11)–(8.13) can be naturally extended to corresponding coverings according to (1.98) and Definition 1.36 (modified obviously in case of general $\varepsilon > 0$). Then the restriction of (8.32), (8.33) to Γ results in (8.29), (8.30). □

Remark 8.4. One may doubt whether the above proof deserves to be called a proof. We simply reduced the above theorem to previous assertions in Chapter 1 where we referred in turn to [Triϵ]. However we tried to give in Chapter 1 careful formulations which may justify the introduction of the spaces $B^s_p(\Gamma)$ simply by (8.5) (a tiny little bit against our intention that part 2 of this book from Chapter 2 onwards should be independent of Chapter 1).

Remark 8.5. Decompositions of type (8.29), (8.30) are called *frame representations.* This may justify calling ε-$\Psi^{s,p}_\Gamma$ in (8.18) a *frame* consisting of normalised (s,p)-β-quarks.

8.1.3 Atomic characterisations

We consider atomic decompositions for the spaces $B^s_p(\Gamma)$ as introduced in (8.5) where again Γ is a d-set in $\mathbb{R}^n$ according to (8.4), (8.6). First we deal with smooth atoms obtained by restriction of corresponding atoms in $\mathbb{R}^n$ to Γ. Smooth atoms in $\mathbb{R}^n$ have been described in detail in Section 1.5.1 including some references. Let

$$s > 0, \quad 0 < 1/p = t < 1, \quad \sigma = s + (n-d)t < N \in \mathbb{N}, \tag{8.34}$$

as in (8.5). Let again $\varepsilon > 0$ and $B_{k,m}$ be the same balls as in (8.15). Then $a^{k,m}_\Gamma$, defined on $\mathbb{R}^n$, is called a smooth (s,p)-atom, more precisely a smooth (s,p)-ε-atom, on Γ if

$$\operatorname{supp} a^{k,m}_\Gamma \subset B_{k,m}\,, \qquad k \in \mathbb{N}_0; \quad m = 1, \ldots, M_k, \tag{8.35}$$

and

$$|D^\alpha a^{k,m}_\Gamma(x)| \le \mathcal{H}^d_\Gamma(B_{k,m})^{\frac{s}{d} - \frac{1}{p} - \frac{|\alpha|}{d}}, \qquad x \in \mathbb{R}^n, \quad |\alpha| \le N, \tag{8.36}$$

where the latter can also be written as

$$|D^\alpha a^{k,m}_\Gamma(x)| \le c\, 2^{-\varepsilon k(s - \frac{d}{p} - |\alpha|)} = c 2^{-\varepsilon k(\sigma - \frac{n}{p} - |\alpha|)}, \quad x \in \mathbb{R}^n, \quad |\alpha| \le N. \tag{8.37}$$

In particular it follows from Definition 1.15 that $a^{k,m}_\Gamma$ are also smooth (σ,p)-atoms in $\mathbb{R}^n$ with respect to the balls $B_{k,m}$ without moment conditions. Again the replacement of the mesh-length 2^{-k} by $2^{-\varepsilon k}$ is immaterial and the right-hand sides of (8.36), (8.37) are the correct normalising factors.

Proposition 8.6. *Let Γ be a compact d-set in $\mathbb{R}^n$ according to* (8.4), (8.6)*. Let $B^s_p(\Gamma)$ be the spaces introduced in* (8.5)*. Let $\varepsilon > 0$. Then $B^s_p(\Gamma)$ is the collection of all $f \in L_1(\Gamma)$ which can be represented as*

$$f(\gamma) = \sum_{k=0}^\infty \sum_{m=1}^{M_k} \lambda_{km}\, a^{k,m}_\Gamma(\gamma), \qquad \gamma \in \Gamma, \tag{8.38}$$

where $a^{k,m}_\Gamma$ are smooth (s,p)-ε-atoms on Γ according to (8.34)–(8.36) *with*

$$\lambda = \{\lambda_{km} \in \mathbb{C} : \ k \in \mathbb{N}_0; \ m = 1, \ldots, M_k\} \tag{8.39}$$

and

$$\|\lambda\,|\ell_p^{\Gamma,0}\| = \left(\sum_{k=0}^{\infty}\sum_{m=1}^{M_k}|\lambda_{km}|^p\right)^{1/p} < \infty, \tag{8.40}$$

(*absolute convergence being in* $L_1(\Gamma)$). *Furthermore,*

$$\|f\,|B_p^s(\Gamma)\| \sim \inf \|\lambda\,|\ell_p^{\Gamma,0}\|, \tag{8.41}$$

where the infimum is taken over all admissible representations.

Proof. Step 1. If f is given by (8.38), (8.40) then it follows by (8.35), (8.36) (or (8.37)) with $\alpha = 0$, that the right-hand side of (8.38) converges absolutely in $L_p(\Gamma)$ and hence in $L_1(\Gamma)$. Then it converges unconditionally in $B_p^s(\Gamma)$. Extending the right-hand side of (8.38) to $\mathbb{R}^n$ then $a_\Gamma^{k,m}(x)$ from (8.35) and (8.37) with $\sigma < N$ are correctly normalised (σ, p)-atoms in $B_{pp}^\sigma(\mathbb{R}^n)$ and it follows from Theorem 1.19 and (8.5) that

$$\|f\,|B_p^s(\Gamma)\| \le c\,\|\lambda\,|\ell_p^{\Gamma,0}\|. \tag{8.42}$$

Step 2. In case of quarkonial decompositions it is quite clear that restrictions of corresponding representations in $B_{pp}^\sigma(\mathbb{R}^n)$ to Γ result in optimal representations for $B_p^s(\Gamma)$ of type (8.26), (8.27). This optimality is not so obvious for atoms. However one can derive optimal atomic decompositions of type (8.38) from optimal quarkonial decompositions (8.26), (8.27), rewriting the latter by

$$\begin{aligned} f(\gamma) &= \sum_{k=0}^{\infty}\sum_{m=1}^{M_k}\left(\sum_{\eta\in\mathbb{N}_0^n} 2^{-\varepsilon'|\eta|}\,|\nu_{km}^\eta|\right)\cdot\left(\sum_{\beta\in\mathbb{N}_0^n}\frac{\nu_{km}^\beta\cdot(\beta\text{-qu})_{km}^\Gamma(\gamma)}{\sum_{\eta\in\mathbb{N}_0^n}2^{-\varepsilon'|\eta|}\,|\nu_{km}^\eta|}\right) \\ &= \sum_{k=0}^{\infty}\sum_{m=1}^{M_k}\lambda_{km}\,a_\Gamma^{k,m}(\gamma), \end{aligned} \tag{8.43}$$

where $0 < \varepsilon' < \varepsilon$. Here λ_{km} refer to the first brackets and $a_\Gamma^{k,m}$ to the second ones. Since $\varepsilon' > 0$ one obtains that

$$\|\lambda\,|\ell_p^{\Gamma,0}\| \le c\,\|\nu\,|\ell_p^\Gamma\|. \tag{8.44}$$

We have (8.22) and corresponding estimates for $D^\alpha\psi_\beta^{k,m}$ with $\tilde\varepsilon < \varepsilon$, where we may assume that $\varepsilon' < \tilde\varepsilon < \varepsilon$. Then one gets (8.35) from (8.11), and (8.36) (or (8.37)) from (8.19) (or (8.24)). Hence $a_\Gamma^{k,m}$ are (s,p)-ε-atoms. By (8.44) any optimal quarkonial decomposition gives an optimal atomic decomposition. □

Remark 8.7. Despite the technicalities at the end of the above proof the outcome is not a surprise: the reduction of smooth atomic decompositions for $B_{pp}^\sigma(\mathbb{R}^n)$ to Γ gives smooth atomic decompositions for $B_p^s(\Gamma)$. On the other hand comparing (s,p)-β-quarks $(\beta\text{-qu})_{km}^\Gamma$ according to (8.19) with the corresponding atoms in (8.35), (8.36) (or (8.37)) then there is the following difference. Although the functions $(\beta\text{-qu})^\Gamma_{km}$ originate from the surrounding $\mathbb{R}^n$ they can be regarded as intrinsic building blocks for the spaces $B^s_p(\Gamma)$ as described in Theorem 8.3. But one can hardly accept the restriction of $a^{k,m}_\Gamma$ with (8.35), (8.36) to Γ as intrinsic building blocks. Especially (8.36) has no intrinsic meaning. In general it is not clear what a better adapted substitute of (8.36) may look like. But if $0 < s < 1$ there is a satisfactory solution of this question which we are going to discuss now and which will play a decisive role in the theory of function spaces on d-sets and on some related abstract quasi-metric spaces developed later on.

Definition 8.8. *Let Γ be a compact d-set in $\mathbb{R}^n$ according to* (8.4), (8.6). *Let*

$$\varepsilon > 0, \quad 1 < p < \infty, \quad 0 < s < 1. \tag{8.45}$$

Let

$$B^\Gamma_{k,m} = \left\{\gamma \in \Gamma : \ |\gamma - \gamma^{k,m}| \le 2^{-\varepsilon k}\right\}, \quad k \in \mathbb{N}_0; \quad m = 1, \ldots, M_k\,, \tag{8.46}$$

be the intersection of the balls $B_{k,m}$ in (8.15) *with Γ, where the lattices $\{\gamma^{k,m}\}^{M_k}_{m=1}$ have the same meaning as in* (8.8)–(8.10). *Then a Lipschitz-continuous function $a^{k,m}_\Gamma$ on Γ is called an $(s,p)^*$-atom, more precisely an $(s,p)^*$-ε-atom, if for $k \in \mathbb{N}_0$ and $m = 1, \ldots, M_k$,*

$$\operatorname{supp} a^{k,m}_\Gamma \subset B^\Gamma_{k,m}, \tag{8.47}$$

$$|a^{k,m}_\Gamma(\gamma)| \le \mathcal{H}^d_\Gamma\left(B^\Gamma_{k,m}\right)^{\frac{s}{d}-\frac{1}{p}}, \qquad \gamma \in \Gamma, \tag{8.48}$$

and

$$|a^{k,m}_\Gamma(\gamma) - a^{k,m}_\Gamma(\delta)| \le \mathcal{H}^d_\Gamma\left(B^\Gamma_{k,m}\right)^{\frac{s-1}{d}-\frac{1}{p}} |\gamma - \delta| \tag{8.49}$$

with $\gamma \in \Gamma$, $\delta \in \Gamma$.

Remark 8.9. To avoid any misunderstanding we remark that $|\gamma - \delta| = \varrho_n(\gamma, \delta)$ is the usual Euclidean distance in $\mathbb{R}^n$. These $(s,p)^*$-atoms are the intrinsic counterparts of the smooth (s,p)-atoms according to (8.35), (8.36) with $|\alpha| = 1$. In particular, the restriction of any smooth (s,p)-atom to Γ is an $(s,p)^*$-atom on Γ.

It is the main aim of this Section 8.1.3 to prove the counterpart of Proposition 8.6 for spaces $B^s_p(\Gamma)$ where $1 < p < \infty$ and $0 < s < 1$ with $(s,p)^*$-atoms in place of (s,p)-atoms. For this purpose we need some preparation. First we recall that $B^s_p(\Gamma)$ with $1 < p < \infty$ and $0 < s < 1$ can be equivalently normed by $\|f\,|B^s_p(\Gamma)\|_*$ with

$$\|f\,|B^s_p(\Gamma)\|^p_* = \int_\Gamma |f(\gamma)|^p\,\mu(\mathrm{d}\gamma) + \int\limits_{\Gamma\times\Gamma} \frac{|f(\gamma) - f(\delta)|^p}{|\gamma - \delta|^{d+sp}}\,\mu(\mathrm{d}\gamma)\,\mu(\mathrm{d}\delta) \tag{8.50}$$

with $\mu = \mathcal{H}^d_\Gamma$ for short. We refer to [JoW84, Chapter V]. In particular, $a^{k,m}_\Gamma \in B^s_p(\Gamma)$ for the above $(s,p)^*$-atoms for all $1 < p < \infty$ and $0 < s < 1$. Let

$$0 < c_1 < c_2 \qquad \text{and} \qquad 0 < c_3 < c_4 \tag{8.51}$$

be four given numbers and let Γ be a d-set with

$$\operatorname{supp}\Gamma \subset \{x: \ |x|<1\}, \qquad c_1 \le \mu(\Gamma) \le c_2 \tag{8.52}$$

and

$$c_3 r^d \le \mu\left(B(\gamma,r)\right) \le c_4 r^d \quad \text{for } \gamma\in\Gamma,\ 0<r\le \operatorname{diam}\Gamma, \tag{8.53}$$

in specification of (8.4), (8.6). We denote the extension operator ext_μ from Theorem 1.181 in the sequel by ext_Γ (since μ is now fixed by (8.6)). Then it follows from the proof of [JoW84, Theorem 1, p. 103] that the norm of ext_Γ,

$$\mathrm{ext}_\Gamma: \quad B^s_p(\Gamma) \hookrightarrow B^\sigma_p(\mathbb{R}^n) = B^\sigma_{pp}(\mathbb{R}^n), \tag{8.54}$$

$$0<s<1, \quad 0<1/p=t<1, \quad \sigma = s+(n-d)t, \tag{8.55}$$

can be estimated from above uniformly for all d-sets Γ with (8.52), (8.53). The corresponding constants may depend on c_1, c_2, c_3, c_4, but not on other peculiarities of Γ. We use this observation in the proof of the following assertion.

Proposition 8.10. *Let Γ be a d-set in $\mathbb{R}^n$ according to* (8.4), (8.6). *Let for $0<r<1$,*

$$B^\Gamma(r) = \left\{\gamma\in\Gamma: \ |\gamma-\gamma^0|<r\right\} \quad \textit{for some} \quad \gamma^0\in\Gamma, \tag{8.56}$$

and

$$B(2r) = \left\{x\in\mathbb{R}^n: \ |x-\gamma^0|<2r\right\}. \tag{8.57}$$

Let s,p,σ be as in (8.55) *and let*

$$f\in B^s_p(\Gamma) \quad \textit{with} \quad \operatorname{supp} f\subset B^\Gamma(r). \tag{8.58}$$

Then

$$\begin{aligned} &\|f\,|B^s_p(\Gamma)\|^p \\ &\sim r^{-sp}\int_{B^\Gamma(r)} |f(\gamma)|^p\,\mu(\mathrm{d}\gamma) + \int\limits_{\gamma,\delta\in B^\Gamma(2r)} \frac{|f(\gamma)-f(\delta)|^p}{|\gamma-\delta|^{d+sp}}\,\mu(\mathrm{d}\gamma)\,\mu(\mathrm{d}\delta). \end{aligned} \tag{8.59}$$

Furthermore,

$$\|f\,|B^s_p(\Gamma)\| \sim \inf \|g\,|B^\sigma_p(\mathbb{R}^n)\|, \tag{8.60}$$

where the infimum is taken over all

$$g\in B^\sigma_p(\mathbb{R}^n), \quad g|\Gamma = f, \quad \operatorname{supp} g\subset B(2r). \tag{8.61}$$

The equivalence constants both in (8.59) *and* (8.60) *are independent of $B^\Gamma(r)$ and of f with* (8.58).

Proof. Step 1. We prove (8.59) where we may assume $r = 2^{-k}$ for some $k \in \mathbb{N}$. Then

$$\begin{aligned}\int\limits_{\Gamma\times\Gamma} \frac{|f(\gamma)-f(\delta)|^p}{|\gamma-\delta|^{d+sp}}\,\mu(\mathrm{d}\gamma)\mu(\mathrm{d}\delta) &\sim \int\limits_{\gamma,\delta\in B^\Gamma(2r)} \frac{|f(\gamma)-f(\delta)|^p}{|\gamma-\delta|^{d+sp}}\,\mu(\mathrm{d}\gamma)\,\mu(\mathrm{d}\delta)\\ &+ \int\limits_{\gamma\in B^\Gamma(r)} |f(\gamma)|^p \int\limits_{|\gamma-\delta|\geq 2^{-k}} \frac{\mu(\mathrm{d}\delta)}{|\gamma-\delta|^{d+sp}}\,\mu(\mathrm{d}\gamma).\end{aligned} \tag{8.62}$$

The inner integral over $|\delta-\gamma|\geq 2^{-k}$ is equivalent to

$$\sum_{l=0}^{L} 2^{(k-l)(d+sp)}\,2^{d(l-k)} \sim 2^{ksp} = r^{-sp}. \tag{8.63}$$

Then (8.59) follows from (8.50) and (8.62).

Step 2. We prove (8.60) with (8.61) where we may assume $\gamma^0 = 0$ in (8.56), (8.57) and $r = 2^{-k}$ for some $k \in \mathbb{N}$. Obviously, $B^\Gamma(2^{-k})$ is again a d-set with total mass $\mu\left(B^\Gamma(2^{-k})\right) \sim 2^{-kd}$. Let

$$D_k: \quad x \mapsto 2^k x, \quad x\in\mathbb{R}^n, \tag{8.64}$$

be a dyadic dilation mapping in particular the d-set $B^\Gamma(2^{-k})$ onto the d-set

$$\Gamma_k = D_k B^\Gamma(2^{-k}) \quad \text{with the measure} \quad \mu_k = 2^{kd}\,\mu\circ D_k^{-1}. \tag{8.65}$$

Here the image measure $\mu^k = \mu\circ D_k^{-1}$ is multiplied with the factor 2^{kd} compensating the total mass $\mu^k(\Gamma_k)\sim 2^{-kd}$. We may assume that all these d-sets Γ_k fit in the scheme of (8.52), (8.53) for some c_1, c_2, c_3, c_4 which are independent of k. Let f',

$$f'(\gamma') = f(\gamma), \qquad \gamma = 2^{-k}\gamma' \quad \text{with} \quad \gamma'\in\Gamma_k, \tag{8.66}$$

be the transferred functions with (8.58). Then it follows by (8.59) with $r = 2^{-k}$ that

$$\begin{aligned}&\|f\,|B^s_p(\Gamma)\|^p\\ &\sim 2^{ksp}\int_{\Gamma_k}|f'(\gamma')|^p\,\mu^k(\mathrm{d}\gamma') + 2^{k(d+sp)}\int\limits_{\Gamma_k\times\Gamma_k}\frac{|f'(\gamma')-f'(\delta')|^p}{|\gamma'-\delta'|^{d+sp}}\,\mu^k(\mathrm{d}\gamma')\,\mu^k(\mathrm{d}\delta')\\ &\sim 2^{k(sp-d)}\left[\int_{\Gamma_k}|f'(\gamma')|^p\,\mu_k(\mathrm{d}\gamma') + \int\limits_{\Gamma_k\times\Gamma_k}\frac{|f'(\gamma')-f'(\delta')|^p}{|\gamma'-\delta'|^{d+sp}}\,\mu_k(\mathrm{d}\gamma')\,\mu_k(\mathrm{d}\delta')\right]\\ &\sim 2^{k(sp-d)}\,\|f'\,|B^s_p(\Gamma_k)\|^p\end{aligned} \tag{8.67}$$

where all equivalence constants are independent of $k \in \mathbb{N}$ and f with (8.58). We apply (8.51)–(8.55) and get

$$\|f' \,|B^s_p(\Gamma_k)\| \sim \inf \|g' \,|B^\sigma_p(\mathbb{R}^n)\| \tag{8.68}$$

where the infimum is taken over all

$$g' \in B^\sigma_p(\mathbb{R}^n) \quad \text{with} \quad g'|\Gamma_k = f' \quad \text{and} \quad \operatorname{supp} g' \subset \{x: \; |x| < 2\}. \tag{8.69}$$

As in (8.66) we put

$$g'(x') = g(x) \quad \text{with} \quad x = 2^{-k}x', \qquad |x'| < 2. \tag{8.70}$$

By (2.28), (2.29) we have

$$\|g \,|B^\sigma_p(\mathbb{R}^n)\| \sim 2^{k(\sigma - n/p)} \|g' \,|B^\sigma_p(\mathbb{R}^n)\| = 2^{k(s-d/p)} \|g' \,|B^\sigma_p(\mathbb{R}^n)\|. \tag{8.71}$$

Now (8.60), (8.61) follows from (8.68) and (8.67), (8.71). □

It is the main aim of this Section 8.1.3 to find intrinsic atomic decompositions of some spaces $B^s_p(\Gamma)$ replacing in Proposition 8.6 the smooth atoms in (8.38) by the intrinsic non-smooth atoms according to Definition 8.8. We use the previous notation. In particular M_k has the same meaning as in (8.15), (8.46) and $\|\lambda \,|\ell^{\Gamma,0}_p\|$ is given by (8.39), (8.40).

Theorem 8.11. *Let Γ be a compact d-set in $\mathbb{R}^n$ according to* (8.4), (8.6). *Let $B^s_p(\Gamma)$ with $1 < p < \infty$, $0 < s < 1$, be the spaces introduced in* (8.5). *Let $\varepsilon > 0$. Then $B^s_p(\Gamma)$ is the collection of all $f \in L_1(\Gamma)$ which can be represented as*

$$f(\gamma) = \sum_{k=0}^{\infty} \sum_{m=1}^{M_k} \lambda_{km} \, a^{k,m}_\Gamma(\gamma), \qquad \gamma \in \Gamma, \quad \|\lambda \,|\ell^{\Gamma,0}_p\| < \infty, \tag{8.72}$$

where $a^{k,m}_\Gamma$ are $(s,p)^*$-ε-atoms according to Definition 8.8. *Furthermore,*

$$\|f \,|B^s_p(\Gamma)\| \sim \inf \|\lambda \,|\ell^{\Gamma,0}_p\| \tag{8.73}$$

where the infimum is taken over all admissible representations (8.72).

Proof. According to Remark 8.9 the representations (8.38) with the smooth (s,p)-ε-atoms are special representations of type (8.72) resulting in (8.41). Hence it remains to show that this representation can be extended to all $(s,p)^*$-ε-atoms. As remarked after (8.50) the above $(s,p)^*$-ε-atoms belong to $B^s_p(\Gamma)$. We claim that they are normalised building blocks. In particular there is a number $c > 0$ such that

$$\|a^{k,m}_\Gamma \,|B^s_p(\Gamma)\| \le c \tag{8.74}$$

for all $(s,p)^*$-ε-atoms. We give a direct proof of this assertion in Remark 8.12 below. But this is not a surprise when comparing (8.36) for $|\alpha| = 0$ and $|\alpha| = 1$

with (8.48), (8.49) resulting in the same normalising factors. Furthermore for different values of s the corresponding $(s,p)^*$-ε-atoms differ only by their normalising factors. As a consequence one gets for the above $(s,p)^*$-ε-atoms

$$\|a_\Gamma^{k,m}\,|B_p^{\bar{s}}(\Gamma)\| \le c\,\mathcal{H}_\Gamma^d\left(B_{k,m}^\Gamma\right)^{\frac{s-\bar{s}}{d}} \sim r^{s-\bar{s}}, \tag{8.75}$$

where $r = 2^{-\varepsilon k}$ and $0 < s \le \bar{s} < 1$. We apply Proposition 8.10 to $a_\Gamma^{k,m}$. Then it follows that there are functions

$$a_{km} \in B_p^{\bar{\sigma}}(\mathbb{R}^n) \quad \text{where} \quad \bar{\sigma} = \bar{s} + (n-d)t, \quad 0 < s \le \bar{s} < 1 \tag{8.76}$$

such that for $r = 2^{-\varepsilon k}$,

$$a_{km}|\Gamma = a_\Gamma^{km}, \qquad \operatorname{supp} a_{km} \subset \left\{x \in \mathbb{R}^n :\ |x - \gamma^{k,m}| \le c_1 r\right\} \tag{8.77}$$

and

$$\|a_{km}\,|B_p^{\bar{\sigma}}(\mathbb{R}^n)\| \le c_2\, r^{s-\bar{s}} = c_2\, 2^{k(\bar{\sigma}-\sigma)\varepsilon}, \tag{8.78}$$

where c_1 and c_2 are positive constants which are independent of $r, \bar{s}$ with $s \le \bar{s} < 1$ and the atoms a_Γ^{km}. Then a_{km} are non-smooth atoms for $B_p^\sigma(\mathbb{R}^n)$ according to Definition 2.7. Now it follows from Theorem 2.13 and (8.5) that one has for some $c > 0$,

$$\|f\,|B_p^s(\Gamma)\| \le c\,\|\lambda\,|\ell_p^{\Gamma,0}\| \tag{8.79}$$

for any representation (8.72). □

Remark 8.12. We give a direct proof of (8.74). We rely on (8.59) with $f = a_\Gamma^{k,m}$ and $r \sim 2^{-\varepsilon k}$. It follows from (8.48) that

$$r^{-sp} \int_{B^\Gamma(r)} |f(\gamma)|^p\,\mu(\mathrm{d}\gamma) \le c r^{-sp}\, r^{sp-d}\, r^d = c \tag{8.80}$$

and from (8.49) (assuming that $B^\Gamma(r)$ is centred at the origin) that

$$\begin{aligned} &\int\limits_{\gamma,\delta\in B^\Gamma(2r)} \frac{|f(\gamma)-f(\delta)|^p}{|\gamma-\delta|^{d+sp}}\,\mu(\mathrm{d}\gamma)\,\mu(\mathrm{d}\delta) \\ &\le c\,r^{(s-1)p-d} \int\limits_{\gamma,\delta\in B^\Gamma(2r)} |\gamma-\delta|^{(1-s)p-d}\,\mu(\mathrm{d}\gamma)\,\mu(\mathrm{d}\delta) \\ &\le c\,r^{(s-1)p-d} \int\limits_{|\gamma|\le c'r}\ \int\limits_{|\delta|\le c'r} |\delta|^{(1-s)p-d}\,\mu(\mathrm{d}\delta)\,\mu(\mathrm{d}\gamma) \\ &\le c''\, r^{(s-1)p-d}\, r^{(1-s)p}\, r^d = c''. \end{aligned} \tag{8.81}$$

This proves (8.74).

8.2 Quasi-metric spaces

8.2.1 d-spaces

Let X be a set. A non-negative function $\varrho(x, y)$ on $X \times X$ is called a *quasi-metric* if it has the following properties:

$$\varrho(x, y) = 0 \qquad \text{if, and only if,} \qquad x = y, \tag{8.82}$$

$$\varrho(x, y) = \varrho(y, x) \qquad \text{for all } x \in X \text{ and all } y \in X, \tag{8.83}$$

there is a real number $A \geq 1$ such that for all $x \in X$, $y \in X$, $z \in X$,

$$\varrho(x, y) \leq A\,[\varrho(x, z) + \varrho(z, y)]\,. \tag{8.84}$$

If $A = 1$ is admissible, then ϱ is called a *metric*. We discussed in Section 1.17.4 some basic properties of quasi-metric spaces and gave relevant references which will not be repeated. According to Theorem 1.187 for any quasi-metric ϱ there is an equivalent quasi-metric $\bar{\varrho}$ in the understanding of (1.582) and a number ε_0 with $0 < \varepsilon_0 \leq 1$ such that $\bar{\varrho}^{\varepsilon}$ with $0 < \varepsilon \leq \varepsilon_0$ is a metric. Then we have (1.583) and it makes sense to generate a topology on X taking the balls

$$B(x, r) = \{y \in X : \ \bar{\varrho}(x, y) < r\}\,, \qquad x \in X, \quad r > 0, \tag{8.85}$$

as a basis of neighborhoods. More details may be found at the beginning of Section 1.17.4, especially in connection with Theorem 1.187 and Remark 1.188.

One of the most remarkable properties of quasi-metric spaces is the existence of snowflaked mappings onto subsets in $\mathbb{R}^n$. In the second part of Section 1.17.4 we gave a description including relevant references. Now we return to this point in a modified way. Furthermore we shall illustrate snowflaked transforms in the next Section 8.2.2. But first we fix some notation. Let

$$\varrho_n : \qquad \varrho_n(x, y) = |x - y|, \quad x \in \mathbb{R}^n, \quad y \in \mathbb{R}^n, \tag{8.86}$$

be the natural metric in $\mathbb{R}^n$. We reserve now the notation *d-set* to (non-empty) sets Γ in $\mathbb{R}^n$ according to (8.4), (8.6) furnished with the Euclidean metric ϱ_n. Then

$$\left(\Gamma, \varrho_n, \mathcal{H}^d_\Gamma\right), \qquad \Gamma \subset \mathbb{R}^n, \quad 0 < d < n, \tag{8.87}$$

becomes a compact complete metric space. Similarly one can complement (X, ϱ) consisting of an (abstract) set X and a quasi-metric ϱ, furnished as indicated above with a topology, by a Borel measure μ, getting a *quasi-metric space* (X, ϱ, μ). However instead of the specifications according to Definition 1.189 resulting in Theorem 1.192 and the explanations given in the Remarks 1.190, 1.191, 1.193, we prefer now the opposite order of reasoning. But of course what has been said in Section 1.17.4 and the references given there remain the background of what follows.

Definition 8.13. *Let (X, ϱ, μ) be a quasi-metric space and let $d > 0$.*

(i) *Then (X, ϱ, μ) is called a d-space if there is a number ε with $0 < \varepsilon \le 1$ and a bi-Lipschitzian map H,*

$$H: \qquad (X, \varrho^\varepsilon, \mu) \quad \text{onto} \quad (\Gamma, \varrho_n, \mathcal{H}_\Gamma^{d_\varepsilon}), \quad d_\varepsilon = d/\varepsilon, \tag{8.88}$$

of the snowflaked version $(X, \varrho^\varepsilon, \mu)$ of (X, ϱ, μ) onto a d_ε-set Γ in some $\mathbb{R}^n$ according to the above explanations such that $\mu \sim \mathcal{H}_\Gamma^{d_\varepsilon} \circ H$ (image measure).

(ii) *The above d-space is called regular if*

$$\mu = \mathcal{H}_\Gamma^{d_\varepsilon} \circ H \quad \text{and} \quad |H(x) - H(y)| = \varrho^\varepsilon(x, y), \quad x \in X, \quad y \in X. \tag{8.89}$$

Remark 8.14. We give some explanations. First we recall that to be a d_ε-set must always be understood as in (8.87) with d_ε in place of d. We always exclude $d = 0$. One may include $d = n$ in (8.87). But this is immaterial because one can interpret Γ in such a case as a subset of $\mathbb{R}^{n+1}$. In particular Γ is a compact set in some $\mathbb{R}^n$. As for the above bi-Lipschitzian map H we have

$$H: \qquad X \mapsto \mathbb{R}^n \quad \text{with} \quad HX = \Gamma \tag{8.90}$$

and

$$\varrho_n\left(H(x), H(y)\right) = |H(x) - H(y)| \sim \varrho^\varepsilon(x, y), \quad x \in X, \quad y \in X, \tag{8.91}$$

for the ε-power ϱ^ε of ϱ. In particular, ϱ^ε is equivalent to a metric. Furthermore,

$$\mu\left(B(x, r)\right) \sim r^{\varepsilon d_\varepsilon} = r^d, \quad x \in X, \quad 0 < r \le 1. \tag{8.92}$$

This justifies calling (X, ϱ, μ) a *d-space*. It is complete and compact, and the measure is doubling. Usually one begins with these properties as definitions and proves afterwards that there exist bi-Lipschitzian maps of the indicated type. We refer for details and the relevant literature to Section 1.17.4. For a given d-space (X, ϱ, μ) there exist many bi-Lipschitzian maps H of the indicated type for different values of ε and into different spaces $\mathbb{R}^n$. We call H a *Euclidean chart* or *ε-chart* of (X, ϱ, μ) and later on it will be indicated as

$$(X, \varrho, \mu; H): \qquad H(X, \varrho, \mu) = (\Gamma, \varrho_n, \mathcal{H}_\Gamma^{d_\varepsilon}). \tag{8.93}$$

In case of *regular d-spaces* according to (8.89) we transfer the Hausdorff measure and the Euclidean distance from the d_ε-set to (X, ϱ, μ). This is reasonable if one deals with smoothness larger than 1. Up to Lipschitz-smoothness, or ε-Hölder-smoothness the ambiguity of equivalent measures or equivalent quasi-metrics does not matter very much. But the situation is different for higher smoothness. We return to this point later on in Section 8.4.2. Then it will also be clearer what is meant by these cryptic comments.

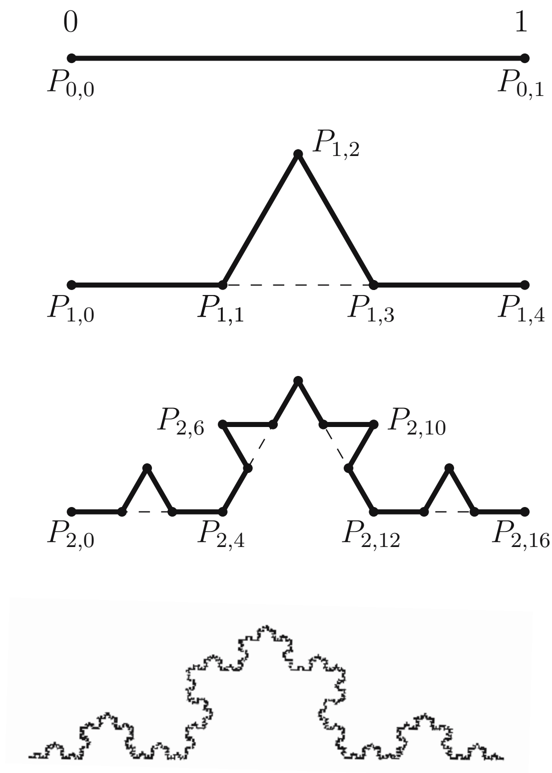

Figure 8.2.2

8.2.2 Snowflaked transforms

The remarkable Theorems 1.187 and 1.192 pave the way to export function spaces and wavelet frames from $\mathbb{R}^n$, domains in $\mathbb{R}^n$, or fractals in $\mathbb{R}^n$, to other abstract quasi-metric spaces. For this purpose the underlying building blocks must be robust enough to survive to be distorted and exported via snowflaked transforms to other quasi-metric worlds. One may consult Section 1.19 for a discussion of the underlying philosophy and Section 1.17.6 for a first description how to proceed. This will be done in Section 8.3 below. But first we wish to shed some extra light on snowflaked transforms and their impact on function spaces.

Example 8.15. We begin with the well-known construction of the classical *snowflake curve* or *Koch curve* (1906; Niels Fabian Helge von Koch, 1870–1924, Sweden, [Koch06]), adapted to our purpose. In the plane $\mathbb{R}^2$ the snowflake curve Γ can be constructed as indicated in Figure 8.2.2 as the closure of the sequence of points

$$P_{k,l}\,, \qquad \text{where} \quad k \in \mathbb{N}_0 \quad \text{and} \quad l = 0, \ldots, 4^k. \tag{8.94}$$

Hence $P_{0,0} = (0,0)$; $P_{0,1} = (1,0)$;

$$P_{1,0} = P_{0,0};\ P_{1,1} = (1/3,0);\ P_{1,2} = (1/2, 1/2\sqrt{3});\ P_{1,3} = (2/3,0);\ P_{1,4} = P_{0,1}; \tag{8.95}$$

and so on. On the other hand we subdivide the closed unit interval $X = [0,1]$ successively into 4^k intervals with endpoints $x_{k,l} = l \cdot 4^{-k}$ where $l = 0, \ldots, 4^k$. Then we map X continuously onto Γ,

$$H: \qquad X \mapsto \Gamma \quad \text{with} \quad H(x_{k,l}) = P_{k,l}\,, \tag{8.96}$$

where $k \in \mathbb{N}_0$ and $l = 0, \ldots, 4^k$. This can be done iteratively without contradiction since

$$P_{k+1,4l} = P_{k,l} \qquad \text{where} \quad l = 0, \ldots, 4^k, \tag{8.97}$$

in agreement with (8.96). Let

$$d = \frac{\log 4}{\log 3} \qquad \text{and} \qquad \eta = \frac{1}{d} = \frac{\log 3}{\log 4}\,. \tag{8.98}$$

Recall that d is the Hausdorff dimension of Γ. Furthermore $\mathcal{H}^d_\Gamma$ is the Hausdorff measure in $\mathbb{R}^2$ restricted to Γ and as in (8.86) the usual Euclidean metric in $\mathbb{R}^2$ is denoted by ϱ_2.

Proposition 8.16. *Let d and η be as in* (8.98). *The metric space (X, ϱ, μ) consisting of the interval $X = [0,1]$, the metric*

$$\varrho(x,y) = |x-y|^\eta, \qquad 0 \le x \le y \le 1, \tag{8.99}$$

and the Lebesgue measure μ, is a d-space according to Definition 8.13(i) *with respect to the bi-Lipschitzian map*

$$H: \qquad (X, \varrho, \mu) \quad \textit{onto} \quad \left(\Gamma, \varrho_2, \mathcal{H}^d_\Gamma\right), \tag{8.100}$$

where the Koch curve Γ and H have the same meaning as above.

Proof. We check (8.88) with $\varepsilon = 1$. First we remark that

$$\mu\left(B(x,r)\right) \sim r^d, \qquad x \in X, \quad 0 < r < 1, \tag{8.101}$$

for balls in the metric space (X, ϱ, μ). Furthermore by (8.96),

$$\begin{aligned} |H(x_{k,l}) - H(x_{k,l-1})| = |P_{k,l} - P_{k,l-1}| = 3^{-k} = 4^{-\eta k} \\ = \varrho(x_{k,l}\,,\, x_{k,l-1}) \end{aligned} \tag{8.102}$$

where $l = 1, \ldots, 4^k$, and one gets by geometrical reasoning

$$|H(x) - H(y)| \sim \varrho(x,y), \qquad 0 \le x \le y \le 1. \tag{8.103}$$

Now (8.101) and (8.103) prove (8.100). □

Remark 8.17. This example has been considered first in [Ass83]. As indicated briefly in Section 1.17.6 and considered in detail in Section 8.3 below we use mappings of type (8.100) to shift the function spaces covered by Theorem 8.3 from d-sets to (abstract) d-spaces. Even in the above simple case $X = [0,1]$ furnished with the non-standard metric $\varrho = \varrho_1^\eta$ where ϱ_1 is the Euclidean metric, and the Lebesgue measure μ, this way seems to be more effective than a direct approach whatever it may look like.

Remark 8.18. Let ϱ_n be as in (8.86) and let μ_n be the Lebesgue measure in $\mathbb{R}^n$. Let $0 < \varepsilon \le 1$ and let $X_1 = [0,1]$. Then it follows by Theorem 1.192 that there is a bi-Lipschitzian map H,

$$H: \quad (X_1, \varrho_1^\varepsilon, \mu_1) \quad \text{onto} \quad \Big(\Gamma, \varrho_N, \mathcal{H}_\Gamma^{d_\varepsilon}\Big), \quad d_\varepsilon = 1/\varepsilon, \tag{8.104}$$

of the snowflaked version $(X_1, \varrho_1^\varepsilon, \mu_1)$ of (X_1, ϱ_1, μ_1) onto a d_ε-set Γ in some $\mathbb{R}^N$. Of course, (X_1, ϱ_1, μ_1) is a 1-space (maybe formally interpreted as a subset of $\mathbb{R}^2$ to be consistent with (8.87)). But one may also begin with

$$(X, \varrho, \mu) = (X_1, \varrho_1^\varkappa, \mu_1), \qquad \varkappa > 0. \tag{8.105}$$

Then it follows by the above considerations that (X, ϱ, μ) is a d-space according to Definition 8.13(i) with $d = 1/\varkappa$. (If $\varkappa \ge 1$ then one may choose $\varepsilon = 1/\varkappa$ and $d_\varepsilon = 1$). One can use these comments to establish the following observation.

Example 8.19. Let

$$\alpha = (\alpha_1, \dots, \alpha_n) \quad \text{with} \quad 0 < \alpha_1 \le \cdots \le \alpha_n < \infty, \quad \sum_{j=1}^n \alpha_j = n, \tag{8.106}$$

be an anisotropy in $\mathbb{R}^n$ according to (5.10). We equip the closed unit cube $X_n = [0,1]^n$ in $\mathbb{R}^n$ with the anisotropic quasi-metric

$$\varrho_{\alpha,n}(x,y) = |x-y|_\alpha \quad \text{where} \quad |x|_\alpha = \sup_k |x_k|^{1/\alpha_k}. \tag{8.107}$$

This is in good agreement with the anisotropic rectangles $Q^\alpha_{\nu m}$ as described at the beginning of Section 5.1.5. Then

$$\big(X_n, \varrho^\varepsilon_{\alpha,n}, \mu_n\big) \quad \text{with} \quad \varepsilon = \alpha_1 \quad \text{and} \quad \varepsilon_k = \alpha_1/\alpha_k\,, \tag{8.108}$$

where $k = 1, \dots, n$, is a metric space (since $\varepsilon \le 1$ and $\varepsilon_k \le 1$). It might be considered as the snowflaked version of $(X_n, \varrho_{\alpha,n}, \mu_n)$. Let $X_1^k = [0,1]$ where k indicates that it is the unit interval with respect to the coordinate x_k. Then one has in obvious notation the product

$$\big(X_n, \varrho^\varepsilon_{\alpha,n}, \mu_n\big) = \prod_{k=1}^n \oplus \big(X_1^k, \varrho_1^{\varepsilon_k}, \mu_1\big) \tag{8.109}$$

of the (one-dimensional) spaces in (8.104) with the bi-Lipschitzian maps

$$H_k : \qquad \big(X_1^k, \varrho_1^{\varepsilon_k}, \mu_1\big) \quad \text{onto} \quad \Big(\Gamma^k, \varrho_{N_k}, \mathcal{H}^{d_k}_{\Gamma^k}\Big). \tag{8.110}$$

Since $\varepsilon_1 = 1$ one may assume that H_1 is the identity. Otherwise Γ^k are d_k-sets in some $\mathbb{R}^{N_k}$ with $d_k = 1/\varepsilon_k = \alpha_k/\alpha_1$. Let

$$H = (H_1, \ldots, H_n), \quad d = \sum_{k=1}^n d_k, \quad N = \sum_{k=1}^n N_k. \tag{8.111}$$

Proposition 8.20. *Let $X_n = [0,1]^n$ be the closed unit cube in $\mathbb{R}^n$. Let α be the anisotropy* (8.106) *and $\varrho_{\alpha,n}$ be the related quasi-metric according to* (8.107)*. Then H, given by* (8.111)*, is a bi-Lipschitzian map*

$$H : \qquad \big(X_n,\, \varrho^{\alpha_1}_{\alpha,n},\, \mu_n\big) \quad \text{onto} \quad \big(\Gamma,\, \varrho_N,\, \mathcal{H}^d_\Gamma\big) \tag{8.112}$$

of the indicated snowflaked version of $(X_n, \varrho_{\alpha,n}, \mu_n)$ onto a compact d-set in some $\mathbb{R}^N$ with $d = n/\alpha_1$.

Proof. We have (8.111) with (8.110). By (8.106) and $d_k = \alpha_k/\alpha_1$ it follows that $d = n/\alpha_1$. Then one gets (8.112) with

$$\Gamma = \prod_{k=1}^n \oplus \Gamma^k \quad \text{being a } d\text{-set in } \mathbb{R}^N \tag{8.113}$$

with N given by (8.111). □

Corollary 8.21. *The above quasi-metric space $(X_n, \varrho_{\alpha,n}, \mu_n)$ is an n-space according to Definition* 8.13(i)*.*

Proof. This follows immediately from Definition 8.13 and the above proposition with $\varepsilon = \alpha_1$ identifying H in (8.112) with H in (8.88). □

Example 8.22. We illustrate Proposition 8.20. Let $n = 2$ and $X_2 = [0,1]^2$. Let according to (8.106), (8.108),

$$\varepsilon_2 = \frac{\alpha_1}{\alpha_2} = \frac{\log 3}{\log 4} \quad \text{and} \quad \alpha_1 \left(1 + \frac{\log 4}{\log 3}\right) = 2. \tag{8.114}$$

Then one can identify H_2 in (8.110) with the bi-Lipschitzian map in (8.100) of $(X_1^2, \varrho_1^{\varepsilon_2}, \mu_1)$ onto the Koch curve in the plane. Together with the identity H_1 it follows in this case that $H = (H_1, H_2)$ in (8.112) maps X_2 onto a cylindrical d-set in $\mathbb{R}^3$ with $d = 1 + \frac{\log 4}{\log 3}$ (a number between 2 and 3) based on the Koch curve in the plane cylindrically extended in the third direction in $\mathbb{R}^3$. In the general case each of the maps H_k in (8.110) with $k \ge 2$ produces in some $\mathbb{R}^{N_k}$ a bizarre fractal curve which is a connected d_k-set. Then Γ in (8.113) is the cartesian product of these wired curves producing a connected fractal surface in $\mathbb{R}^N$.

8.3 Spaces on d-spaces

8.3.1 Frames

Recall that we reserved the notation *d-set* to concrete sets Γ in $\mathbb{R}^n$ according to (8.4), (8.6) whereas *d-space* refers to abstract quasi-metric spaces (X, ϱ, μ) as introduced in Definition 8.13. We discussed in Remark 8.14 in detail where the notation comes from and which decisive role is played by the snowflaked bi-Lipschitzian map H in (8.88). Now we wish to shift in this way the resolution of unity as described at the beginning of Section 8.1.2 and in particular in Definition 8.1 from the d_ε-set on the right-hand side of (8.88) to the d-space (X, ϱ, μ). Although, relying on the above preparations, it is more or less a technical matter that some (notational) care is necessary. For this purpose we repeat, adapt and complement previous notation. For a given d-space (X, ϱ, μ) we call

$$(X, \varrho, \mu; H) \qquad \text{or} \quad (X; H) \quad \text{for short,} \tag{8.115}$$

for an admitted bi-Lipschitzian map H according to (8.88) a corresponding *Euclidean chart* or *ε-chart*. In particular we have (8.90)–(8.92). As there (and in (8.85) with $\bar{\varrho} = \varrho$) we now reserve the notation

$$B(x, r) = \{y \in X,\ \varrho(x, y) < r\}, \qquad x \in X, \quad r > 0, \tag{8.116}$$

for balls in (X, ϱ, μ) whereas corresponding balls on d-sets Γ in $\mathbb{R}^n$ are indicated by an extra Γ,

$$B^\Gamma(\gamma, r) = \{\delta \in \Gamma :\ |\gamma - \delta| < r\}, \qquad \gamma \in \Gamma, \quad r > 0, \tag{8.117}$$

where $|\gamma - \delta| = \varrho_n(\gamma, \delta)$ is the Euclidean distance of that $\mathbb{R}^n$ in which Γ is located.

Proposition 8.23. *Let (X, ϱ, μ) be a d-space according to Definition 8.13. For any $k \in \mathbb{N}_0$ there are lattices and subordinated resolutions of unity*

$$\big\{x^{k,m}\big\}_{m=1}^{M_k} \subset X \qquad \text{and} \qquad \big\{\psi^{k,m}\big\}_{m=1}^{M_k} \tag{8.118}$$

such that for some $c_1 > 0$ and $c_2 > 0$ (which are independent of k),

$$\varrho(x^{k,m_1}, x^{k,m_2}) \ge c_1\, 2^{-k}, \quad k \in \mathbb{N}_0, \quad m_1 \ne m_2, \tag{8.119}$$

$$X = \bigcup_{m=1}^{M_k} B_{k,m} \quad \text{with} \quad B_{k,m} = B(x^{k,m}, c_2 2^{-k}) \quad \text{for} \quad k \in \mathbb{N}_0, \tag{8.120}$$

whereas $\psi^{k,m}$ are non-negative functions on X with

$$\operatorname{supp} \psi^{k,m} \subset B_{k,m}, \qquad k \in \mathbb{N}_0, \quad m = 1, \ldots, M_k, \tag{8.121}$$

and

$$\sum_{m=1}^{M_k} \psi^{k,m}(x) = 1 \qquad \text{where} \quad x \in X, \quad k \in \mathbb{N}_0. \tag{8.122}$$

Proof. This follows from the corresponding assertions for d-sets in some $\mathbb{R}^n$ as discussed at the beginning of Section 8.1.2 and the existence of ε-charts according to (8.88), (8.91). □

Remark 8.24. The above proposition is quite obvious if one relies on the existence of ε-charts. But the assertion itself is known on a larger scale and it is a cornerstone of the analysis on homogeneous spaces according to Definition 1.189(i). One may also consult [HaY02] and the references given there. By the above arguments it follows that one may assume that

$$M_k \sim 2^{dk} \qquad \text{and} \qquad \psi^{k,m} \in \mathrm{Lip}^\varepsilon(X), \quad k \in \mathbb{N}_0, \tag{8.123}$$

and $m = 1, \ldots, M_k$. Here $\mathrm{Lip}^\varepsilon(X)$ is the Banach space of all complex-valued continuous functions on X such that

$$\|f\,|\mathrm{Lip}^\varepsilon(X)\| = \sup_{x \in X} |f(x)| + \sup_{x \neq y} \frac{|f(x) - f(y)|}{\varrho^\varepsilon(x,y)} < \infty. \tag{8.124}$$

Let ε-Ψ_Γ and ε-$\Psi_\Gamma^{s,p}$ be the frames as introduced in Definition 8.1 and Remark 8.5 now with respect to d_ε-sets in some $\mathbb{R}^n$ as needed in connection with (8.88). We wish to transfer these frames from d_ε-sets Γ with $d_\varepsilon = d/\varepsilon$ to d-spaces. Let now

$$\psi_\beta^{k,m}(\gamma) = \mathcal{H}_\Gamma^{d_\varepsilon}(B_{k,m}^\Gamma)^{-|\beta|/d_\varepsilon} (\gamma - \gamma^{k,m})^\beta \psi^{k,m}(\gamma), \quad \gamma \in \Gamma, \tag{8.125}$$

with

$$B_{k,m}^\Gamma = B\left(\gamma^{k,m}, 2^{-\varepsilon k}\right), \quad k \in \mathbb{N}_0; \quad m = 1, \ldots, M_k, \tag{8.126}$$

be the adapted version of (8.17) or (8.23). Let $(X;H)$ be an ε-chart of the d-space (X, ϱ, μ) according to Definition 8.13, (8.115). We put

$$\psi_l^{k,m}(x) = \left(\psi_\beta^{k,m} \circ H\right)(x) = \psi_\beta^{k,m}(Hx), \qquad x \in X. \tag{8.127}$$

Here we assume that there is a one-to-one mapping of $\beta \in \mathbb{N}_0$ onto $l \in \mathbb{N}_0$ such that $l = 0$ if $\beta = 0$. In particular one may choose $\psi_0^{k,m} = \psi^{k,m}$ as the functions in (8.121), (8.122). Furthermore one may assume $\gamma^{k,m} = Hx^{k,m}$ in (8.118), (8.8) (with respect to the d_ε-set Γ).

Definition 8.25. *Let (X, ϱ, μ) be a d-space according to Definition* 8.13 *equipped with an ε-chart $\{X;H\}$ as in* (8.115), (8.88).

(i) *Let*

$$\varepsilon\text{-}\Psi_\Gamma = \left\{\psi_\beta^{k,m} : \ \beta \in \mathbb{N}_0^n, \ k \in \mathbb{N}_0; \ m = 1, \ldots, M_k\right\} \tag{8.128}$$

with (8.125), (8.126) *be a corresponding ε-frame* (8.16) *with respect to the above d_ε-set* Γ. *Then*

$$H\text{-}\Psi_X = (\varepsilon\text{-}\Psi_\Gamma) \circ H = \left\{\psi_l^{k,m} : \ l \in \mathbb{N}_0, \ k \in \mathbb{N}_0; \ m = 1, \ldots, M_k\right\} \tag{8.129}$$

with (8.127) *is called an H-frame. Let $s > 0$ and $1 < p < \infty$. Then*

$$H\text{-}\Psi_X^{s,p} = \big\{ (\text{l-qu})_{km}^X : \; l \in \mathbb{N}_0, \; k \in \mathbb{N}_0; \; m = 1, \ldots, M_k \big\} \tag{8.130}$$

with the (s,p)-l-quarks on X,

$$(\text{l-qu})_{km}^X(x) = \mu(B_{k,m})^{\frac{s}{d} - \frac{1}{p}} \, \psi_l^{k,m}(x), \qquad x \in X, \tag{8.131}$$

are the related H-(s,p)-frames.

(ii) *Let*

$$\nu = \{ \nu_{km}^l \in \mathbb{C} : \; l \in \mathbb{N}_0, \; k \in \mathbb{N}_0; \; m = 1, \ldots, M_k \} \tag{8.132}$$

and $1 < p < \infty$. Then

$$\ell_p^X = \left\{ \nu : \; \| \nu \, | \ell_p^X \| = \left(\sum_{l,k,m} |\nu_{km}^l|^p \right)^{1/p} < \infty \right\}. \tag{8.133}$$

Remark 8.26. This is the direct counterpart of Definition 8.1 with respect to the above d_ε-set Γ. Then ε in (8.16), (8.18) cancels out when H with (8.88), (8.91) is applied. One can organise the one-to-one map between $\beta \in \mathbb{N}_0^n$ and $l \in \mathbb{N}_0$ such that

$$|\psi_l^{k,m}(x)| \le c_1 \, 2^{-c_2 l^\eta}, \quad l \in \mathbb{N}_0, \; k \in \mathbb{N}_0; \; m = 1, \ldots, M_k, \tag{8.134}$$

for some $c_1 > 0$, $c_2 > 0$ and some $\eta > 0$ (for example $\eta = 1/n$) which are independent of l, k, m and $x \in X$. Furthermore, by (8.126), $\mu = \mathcal{H}_\Gamma^{d_\varepsilon} \circ H$ and the above mapping one has

$$\mathcal{H}_\Gamma^{d_\varepsilon}(B_{k,m}^\Gamma)^{\frac{s}{\varepsilon} \cdot \frac{1}{d_\varepsilon} - \frac{1}{p}} \sim \mu(B_{k,m})^{\frac{s}{d} - \frac{1}{p}} \sim 2^{-k(s - d/p)}. \tag{8.135}$$

Hence the (s,p)-l-quarks in (8.131) are the transferred $(s/\varepsilon, p)$-β-quarks on the d_ε-set Γ according to (8.19).

8.3.2 Spaces of positive smoothness

We summarise the procedure of the preceding Section 8.3.1. The starting point is an ε-chart $(X, \varrho, \mu; H)$ consisting of a d-space (X, ϱ, μ) and a bi-Lipschitzian map H (8.88) onto a d_ε-set Γ in some $\mathbb{R}^n$ mapping

$$H\{x^{k,m}\} = \{\gamma^{k,m}\} \subset \Gamma, \tag{8.136}$$

an arbitrary admitted lattice (8.118)–(8.120) onto an admitted (approximate) lattice on the d_ε-set Γ in some $\mathbb{R}^n$ where we may assume that (8.8)–(8.10) is satisfied. Afterwards one transforms via H^{-1} the ε-Ψ_Γ frame and the ε-$\Psi_\Gamma^{s,p}$ frame from the d_ε-set Γ to X including the indicated normalisations. By Theorem 8.3 we have for these d_ε-sets intrinsic descriptions of the spaces $B_p^s(\Gamma)$ with $s > 0$ and $1 < p < \infty$

in terms of ε-$\Psi_\Gamma^{s,p}$ frames according to Definition 8.1. This can be naturally transferred via ε-charts $(X, \varrho, \mu; H)$ to arbitrary d-spaces (X, ϱ, μ). If $1 \le r < \infty$ then $L_r(X, \varrho, \mu)$ has the usual meaning furnished with the norm

$$\|f \,|L_r(X, \varrho, \mu)\| = \Big(\int_X |f(x)|^r \, \mu(\mathrm{d}x) \Big)^{1/r} . \tag{8.137}$$

This is the counterpart of (8.25). If (X, ϱ, μ) is a regular d-set according to Definition 8.13(ii) then H in (8.88) gives the isometric map

$$\|f \,|L_r(X, \varrho, \mu)\|^r = \int_\Gamma |(f \circ H^{-1})(\gamma)|^r \, \mathcal{H}_\Gamma^{d_\varepsilon}(\mathrm{d}\gamma). \tag{8.138}$$

In analogy to Theorem 8.3 we use $L_1(X, \varrho, \mu)$ as the basic space.

Definition 8.27. *Let $(X, \varrho, \mu; H)$ be an ε-chart of the d-space (X, ϱ, μ) according to Definition* 8.13 *and* (8.115)*. Let $s > 0$ and $1 < p < \infty$. Let the H-(s,p)-frame H-$\Psi_X^{s,p}$ and the sequence space ℓ_p^X be as introduced in Definition* 8.25*. Then $B_p^s(X, \varrho, \mu; H)$, or $B_p^s(X; H)$ for short, is the collection of all $f \in L_1(X, \varrho, \mu)$ which can be represented as*

$$f(x) = \sum_{l=0}^\infty \sum_{k=0}^\infty \sum_{m=1}^{M_k} \nu_{km}^l \, (\text{l-qu})_{km}^X(x), \quad \|\nu \,|\ell_p^X\| < \infty, \tag{8.139}$$

$x \in X$ (absolute convergence being in $L_1(X, \varrho, \mu)$). Furthermore,

$$\|f \,|B_p^s(X; H)\| = \inf \|\nu \,|\ell_p^X\| \tag{8.140}$$

where the infimum is taken over all admissible representations (8.139).

Remark 8.28. This definition imitates Theorem 8.3. The (s,p)-l-quarks $(\text{l-qu})_{km}^X$ originate from the $(s/\varepsilon, p)$-β-quarks $(\beta\text{-qu})_{km}^\Gamma$ including the correct normalising factors in (8.19) as mentioned in (8.135). In particular, the absolute convergence of (8.139) in $L_1(X, \varrho, \mu)$ is not an additional assumption but a consequence of $\|\nu \,|\ell_p^X\| < \infty$.

Remark 8.29. The above definition is consistent with what we know so far. If (X, ϱ, μ) is a d-set $(\Gamma, \varrho_n, \mathcal{H}_\Gamma^d)$ in some $\mathbb{R}^n$ and if $H = \mathrm{id}$ is the identity then $B_p^s(X; H) = B_p^s(\Gamma)$ according to the above definition and Theorem 8.3. But otherwise one can introduce on $(\Gamma, \varrho_n, \mathcal{H}_\Gamma^d; H)$ many scales of spaces $B_p^s(X; H)$ which do not originate from trace spaces according to (8.5). For example, a Lipschitz diffeomorphism H of $\mathbb{R}^n$ onto itself preserves the property to be a d-set but destroys the differentiability assumptions (8.12) on which Definition 8.1 and Theorem 8.3 rest. Nevertheless as we shall see later on, in this particular case all spaces $B_p^s(X; H)$ coincide if $0 < s < 1$. But this is not necessarily the case if $s > 1$ (one may think of Hölder-Zygmund spaces $\mathcal{C}^s = B_{\infty,\infty}^s$ with $s > 1$ on pieces of hyperplanes).

Theorem 8.30. *Let $(X, \varrho, \mu; H)$ be an ε-chart of the d-space (X, ϱ, μ) according to Definition* 8.13 *and* (8.115)*. Let $B^s_p(X; H)$ be the spaces as introduced in Definition* 8.27 *where $s > 0$ and $1 < p < \infty$.*

(i) *Then $B^s_p(X; H)$ is a Banach space. It is independent of all admissible H-(s, p)-frames H-$\Psi^{s,p}_X$ according to Definition* 8.25 *and*

$$B^s_p(X, \varrho, \mu; H) = B^{s/\varepsilon}_p(\Gamma, \varrho_n, \mathcal{H}^{d_\varepsilon}_\Gamma) \circ H \tag{8.141}$$

where $d_\varepsilon = d/\varepsilon$.

(ii) *In addition let $0 < s < \varepsilon$. Then there is a linear and bounded map*

$$f \mapsto \nu(f) = \{\nu^l_{km}(f)\} : \quad B^s_p(X; H) \mapsto \ell^X_p\,, \tag{8.142}$$

such that

$$f(x) = \sum_{l,k,m} \nu^l_{km}(f) \cdot (\text{l-qu})^X_{km}, \qquad x \in X, \tag{8.143}$$

with

$$\|f \,|B^s_p(X; H)\| \sim \|\nu(f)\,|\ell^X_p\| \tag{8.144}$$

(*equivalent norms where the equivalence constants are independent of f*).

Proof. Here (8.141) means

$$\|f \,|B^s_p(X; H)\| \sim \|f \circ H^{-1}\,|B^{s/\varepsilon}_p(\Gamma)\|, \quad f \in B^s_p(X; H), \tag{8.145}$$

is an isomorphic map between the two spaces involved. Otherwise part (i) follows from Theorem 8.3, Definition 8.25, Remark 8.26 and (8.135), (8.136). Part (ii) is covered by Theorem 8.3(ii). □

8.3.3 Spaces of arbitrary smoothness

Chapter 8 is a modified and adapted version of [Tri05c]. In this paper we extended the above theory of spaces of positive smoothness on d-spaces to corresponding spaces of arbitrary smoothness. Basically this is a matter of duality of the spaces $B^s_p(X; H)$ according to Definition 8.27 and Theorem 8.30. But some care is necessary which will not be repeated here. We give a brief description referring for details and proofs to [Tri05c, Section 4.3].

Definition 8.31. *Let $(X, \varrho, \mu; H)$ be an ε-chart of the d-space (X, ϱ, μ) according to Definition* 8.13 *and* (8.115)*. Let $B^s_p(X; H)$ be the spaces introduced in Definition* 8.27*. Let $1 < p < \infty$. Then*

$$D(X; H) = \bigcap_{s>0} B^s_p(X; H), \tag{8.146}$$

naturally equipped with a locally convex topology.

Remark 8.32. By well-known embedding theorems $D(X;H)$ (including its locally convex topology) is independent of given p. In particular the topology of $D(X;H)$ can be generated by countably many norms of the Hilbert spaces $B^{s_k}_2(X;H)$,

$$\|f\,|B^{s_k}_2(X;H)\|, \quad k\in\mathbb{N}, \quad 0<s_1<s_2<\cdots<s_k\to\infty \tag{8.147}$$

if $k\to\infty$.

Proposition 8.33. *Under the above hypotheses, $D(X;H)$ is a nuclear locally convex space. It is dense in all spaces $B^s_p(X;H)$ with $s>0, 1<p<\infty$, and in all spaces $L_r(X,\varrho,\mu)$ with $1\le r<\infty$.*

Remark 8.34. This gives the possibility to introduce the dual space $D'(X;H)$ of $D(X;H)$ with the usual (weak or strong) topology and to interpret $f\in L_r(X)=L_r(X,\varrho,\mu)$ as a distribution $f\in D'(X;H)$ by the dual pairing

$$(f,g)=\int_X f(x)\cdot g(x)\,\mu(\mathrm{d}x), \qquad f\in L_r(X), \quad g\in L_{r'}(X), \tag{8.148}$$

$1\le r<\infty$, $\frac{1}{r}+\frac{1}{r'}=1$.

Definition 8.35. *Let $(X,\varrho,\mu;H)$ be an ε-chart of the d-space (X,ϱ,μ) according to Definition 8.13 and (8.115). Let $1<p<\infty$ and $s<0$. Then*

$$B^s_p(X;H)=\Big(B^{-s}_{p'}(X;H)\Big)', \qquad \frac{1}{p}+\frac{1}{p'}=1, \tag{8.149}$$

according to the dual pairing $(D(X;H),D'(X;H))$.

Remark 8.36. Again we refer to [Tri05c] for discussions and a justification of the definition and of the assertions. Furthermore we derived there some properties of the spaces $B^s_p(X;H)$ with $s<0$. The corresponding spaces with $s=0$ can be introduced by complex interpolation,

$$B^0_p(X;H)=\big[B^s_p(X;H),B^{-s}_p(X;H)\big]_{1/2}, \qquad 0<s<s_0, \tag{8.150}$$

where the outcome is independent of s and even of the ε-chart H if s_0 is sufficiently small, hence

$$B^0_p(X)=\big[B^s_p(X),B^{-s}_p(X)\big]_{1/2}, \qquad 1<p<\infty, \tag{8.151}$$

in terms of the spaces $B^s_p(X)=B^s_{pp}(X)$ according to Definition 1.194. This will be discussed in the following Section 8.3.4.

8.3.4 Spaces of restricted smoothness

For a d-space (X,ϱ,μ) according to Definition 8.13 there may be many incommensurable Euclidean charts or ε-charts $(X,\varrho,\mu;H)$ with (8.93), (8.115). The situation is quite similar to that of Riemannian geometry furnishing suitable topological spaces (Hausdorff spaces) with atlases of Riemannian charts creating incompatible smoothness structures (Riemannian geometries) on the same basic space. However if the underlying structure or space one is starting from has a natural (limited) smoothness up to some (restricted) order then this should be respected by any impressed higher, maybe C^∞, smoothness via local charts. In case of d-spaces and related ε-charts this suggests having a closer look at spaces $B^s_p(X;H)$ if $s>0$ is small or (including the sketched extension in Section 8.3.3 to $s \le 0$) if $|s|$ is small. So far we introduced d-spaces in Definition 8.13 by assuming that there is a number ε with $0<\varepsilon\le 1$ and a bi-Lipschitzian map H with (8.88). But as explained in Remark 8.14 the justification of this somewhat brutal procedure comes from Section 1.17.4, especially from Theorem 1.192 and Remark 1.193. In particular, if (X,ϱ,μ) is a d-space in the understanding of Definition 1.189 and if ε_0 with $0<\varepsilon_0\le 1$ has the same meaning as in Theorems 1.187, 1.192 then (X,ϱ,μ) is also a d-space according to Definition 8.13 and for any ε with $0<\varepsilon<\varepsilon_0$ there are bi-Lipschitzian maps H in (8.88). Recall that we have (8.92). Under these circumstances one has the intrinsically introduced spaces

$$B^s_{pq}(X), \quad 1\le p\le\infty, \quad 1\le q\le\infty, \quad |s|<\varepsilon, \tag{8.152}$$

according to Definition 1.194. By Theorem 1.196 they are compatible with corresponding spaces on d-sets in $\mathbb{R}^n$. Now we wish to establish the counterpart of this assertion for the above spaces $B^s_p(X;H)$. We use the abbreviation

$$B^s_p(X) = B^s_{pp}(X), \qquad 1<p<\infty, \quad |s|<\varepsilon, \tag{8.153}$$

for the spaces in (8.152) with $p=q$. We rely on intrinsic atomic characterisations of the spaces in question. For this purpose we transfer the intrinsic atomic decomposition for the spaces $B^s_p(\Gamma)$ on d-sets in $\mathbb{R}^n$ according to Theorem 8.11 to d-spaces. The number $\varepsilon>0$ in Definition 8.8 will now be identified with ε in (8.88). Otherwise we transfer Definition 8.8 now for d_ε-sets Γ, where $d_\varepsilon = d/\varepsilon$, to $X=H^{-1}\Gamma$ in the same way as this has been done in Definition 8.25 in case of frames.

Definition 8.37. *Let (X,ϱ,μ) be a d-space according to Definition 8.13 equipped with an ε-chart $\{X;H\}$ as in (8.115), (8.88). Let $B_{k,m}$ be the same balls as in (8.120) and let $\mathrm{Lip}^\varepsilon(X)$ as introduced in (8.124). Let $0<s<\varepsilon$ and $1<p<\infty$. Then $a^{k,m}_X \in \mathrm{Lip}^\varepsilon(X)$ is called an (s,p)-ε-atom if for $k\in\mathbb{N}_0$ and $m=1,\ldots,M_k$,*

$$\mathrm{supp}\, a^{k,m}_X \subset B_{k,m}, \tag{8.154}$$

$$|a^{k,m}_X(x)| \le \mu\,(B_{k,m})^{\frac{s}{d}-\frac{1}{p}}, \qquad x\in X, \tag{8.155}$$

and

$$|a^{k,m}_X(x) - a^{k,m}_X(y)| \le \mu\,(B_{k,m})^{\frac{s}{d}-\frac{1}{p}-\frac{\varepsilon}{d}}\,\varrho^\varepsilon(x,y) \tag{8.156}$$

with $x\in X$, $y\in X$.

Remark 8.38. This is the direct counterpart of Definition 8.8. By (8.88) it follows that

$$a_X^{k,m} = a_\Gamma^{k,m} \circ H, \qquad k \in \mathbb{N}_0; \quad m = 1, \ldots, M_k, \tag{8.157}$$

are the transferred $(s/\varepsilon, p)^*$-ε-atoms from the d_ε-set Γ on X including the correct normalising factors according to (8.135) (with $\frac{s}{\varepsilon} - 1$ in place of s/ε). Just as in (8.39), (8.40) we put

$$\lambda = \{\lambda_{km} \in \mathbb{C} : \ k \in \mathbb{N}_0; \ m = 1, \ldots, M_k\} \tag{8.158}$$

and

$$\|\lambda \,|\ell_p^{X,0}\| = \left(\sum_{k=0}^{\infty} \sum_{m=1}^{M_k} |\lambda_{km}|^p \right)^{1/p} \tag{8.159}$$

in modification of (8.133).

Theorem 8.39. *Let $(X, \varrho, \mu; H)$ be an ε-chart of the d-space (X, ϱ, μ) according to Definition* 8.13 *and* (8.115)*. Let $1 < p < \infty$ and $0 < s < \varepsilon$. Then $B_p^s(X; H)$ is the collection of all $f \in L_1(X, \varrho, \mu)$ which can be represented as*

$$f(x) = \sum_{k=0}^{\infty} \sum_{m=1}^{M_k} \nu_{km} \, a_X^{k,m}(x), \quad \|\nu \,|\ell_p^{X,0}\| < \infty, \tag{8.160}$$

$x \in X$ (absolute convergence being in $L_1(X, \varrho, \mu)$), where the $a_X^{k,m}$ are (s,p)-ε-atoms on X according to Definition 8.37*. Furthermore,*

$$\|f \,|B_p^s(X; H)\| \sim \inf \|\nu \,|\ell_p^{X,0}\| \tag{8.161}$$

where the infimum is taken over all admissible representations (8.160).

Proof. We have (8.88) with the d_ε-set Γ, (8.141) now with $s < \varepsilon$ and (8.157). Then the above assertion follows from Theorem 8.11 applied to $B_p^{s/\varepsilon}(\Gamma)$. □

Corollary 8.40. *Under the hypotheses of Theorem* 8.39*, especially $1 < p < \infty$ and $0 < s < \varepsilon$, the spaces $B_p^s(X; H)$ are independent of H. Furthermore,*

$$B_p^s(X; H) = B_p^s(X), \qquad 1 < p < \infty, \quad 0 < s < \varepsilon, \tag{8.162}$$

where $B_p^s(X)$ are the spaces according to (8.153), (8.152) *and Definition* 1.194.

Proof. The independence of $B_p^s(X; H)$ of H is an immediate consequence of the above theorem. The above intrinsic atomic characterisation coincides with a corresponding atomic representation of the spaces $B_p^s(X)$ in [HaY02, Theorem 1.1]. We refer also to [Yang03, Lemma 2.3]. □

Remark 8.41. This assertion extends Theorem 1.196 from d-sets to d-spaces. In Section 8.3.3 we outlined briefly how to extend the spaces $B^s_p(X;H)$ from $s>0$ to $s<0$ by duality. By [HaY02, Lemma 1.8, p. 18] there is a counterpart for the spaces $B^s_p(X)$ as introduced in Definition 1.194,

$$\big(B^s_p(X)\big)' = B^{-s}_{p'}(X), \quad -\varepsilon < s < \varepsilon, \quad 1<p<\infty, \quad \frac{1}{p}+\frac{1}{p'}=1. \tag{8.163}$$

Then one gets by (8.149) and (8.162) that also

$$B^s_p(X;H) = B^s_p(X), \qquad -\varepsilon < s < 0, \quad 1<p<\infty. \tag{8.164}$$

As for the interpolation mentioned in (8.151) we refer to [HaS94, Theorem 7.7]. (The formulation given there is in terms of homogeneous spaces. But the arguments apply also to the inhomogeneous spaces preferred here.)

8.4 Applications

8.4.1 Entropy numbers

Entropy numbers of compact operators have been introduced in Definition 4.43. Further information may be found in Section 1.10. We dealt several times with entropy numbers of compact embeddings between function spaces, Theorem 1.97 (bounded domains in $\mathbb{R}^n$), Section 4.4.1 (relations to other numbers), and Theorem 5.30 (anisotropic spaces). We now complement these assertions by looking briefly at compact embeddings between the above spaces $B^s_p(X;H)$. Again we express equivalences by $\sim$ as explained in (1.307), (1.308).

Theorem 8.42. *Let $(X,\varrho,\mu;H)$ be an ε-chart of the d-space (X,ϱ,μ) according to Definition 8.13, (8.115) and let $B^s_p(X;H)$ with $s>0$, $1<p<\infty$ be the spaces introduced in Definition 8.27. Let*

$$0<s_2<s_1<\infty, \qquad 1<p_1<\infty, \qquad 1<p_2<\infty, \tag{8.165}$$

and

$$s_1 - d/p_1 > s_2 - d/p_2\,. \tag{8.166}$$

Then the embedding

$$\mathrm{id}^X: \quad B^{s_1}_{p_1}(X;H) \hookrightarrow B^{s_2}_{p_2}(X;H) \tag{8.167}$$

is compact and

$$e_k(\mathrm{id}^X) \sim k^{-\frac{s_1-s_2}{d}} \qquad \textit{for} \quad k\in\mathbb{N}. \tag{8.168}$$

Proof. By (8.141),

$$H: \quad f \mapsto f \circ H: \quad B^{s/\varepsilon}_p(\Gamma, \varrho_n, \mathcal{H}^{d_\varepsilon}_\Gamma) \hookrightarrow B^s_p(X; H) \tag{8.169}$$

is an isomorphic map where Γ is a d_ε-set in some $\mathbb{R}^n$. Recall that $d_\varepsilon = d/\varepsilon$. Then

$$\mathrm{id}^X = H \circ \mathrm{id}^\Gamma \circ H^{-1} \tag{8.170}$$

with

$$\mathrm{id}^\Gamma: \qquad B^{s_1/\varepsilon}_{p_1}(\Gamma) \hookrightarrow B^{s_2/\varepsilon}_{p_2}(\Gamma). \tag{8.171}$$

Hence,

$$e_k(\mathrm{id}^X) \sim e_k(\mathrm{id}^\Gamma) \sim k^{-\frac{s_1-s_2}{d}}, \qquad k \in \mathbb{N}, \tag{8.172}$$

where the latter equivalence is covered by [Triδ, Theorem 20.6, p. 166] (here ε cancels out). □

Remark 8.43. The corresponding assertion for arbitrary bounded domains in $\mathbb{R}^n$ may be found in Theorem 1.97.

8.4.2 Riesz potentials

In Section 7.1.2 we discussed mapping properties of Bessel potentials and (truncated) Riesz potentials in $\mathbb{R}^n$ in relation to singular measures. There one finds also some references especially about the close connection between Riesz potentials and fractal analysis. Now we return to this subject in the context of d-spaces. But first we describe some results in connection with d-sets.

Let $\Gamma = \big(\Gamma, \varrho_n, \mathcal{H}^d_\Gamma\big)$ be a compact d-set in $\mathbb{R}^n$ according to (8.4), (8.6). Recall that $\varrho_n(x,y) = |x-y|$ is the usual metric in $\mathbb{R}^n$. Let $I^\Gamma_\varkappa$,

$$(I^\Gamma_\varkappa f)(\gamma) = \int_\Gamma \frac{f(\delta)}{|\gamma-\delta|^{d-\varkappa}}\, \mathcal{H}^d_\Gamma(\mathrm{d}\delta), \qquad 0 < \varkappa < d, \tag{8.173}$$

$\gamma \in \Gamma$, be Riesz potentials on Γ. These operators have been studied in detail in [Zah02, Zah04]. It comes out, in particular, that $I^\Gamma_\varkappa$ is a positive self-adjoint compact operator in $L_2(\Gamma)$. Let

$$\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_k \ge \cdots \to 0 \quad \text{if} \quad k \to \infty, \tag{8.174}$$

be the positive eigenvalues of $I^\Gamma_\varkappa$ repeated according to multiplicity and ordered by magnitude. Then

$$\lambda_k \sim k^{-\varkappa/d}, \qquad k \in \mathbb{N}. \tag{8.175}$$

We refer for details and proofs to [Zah02, Theorem 3.3, Corollary 3.4] and [Zah04].

We developed in [TrY02] a corresponding theory for d-spaces (X, ϱ, μ) in the original version of Definition 1.189 without using the existence of ε-charts as in Definition 8.13. This direct (long and complicated) approach resulted in the following assertion. For any $\varkappa$ with $0 < \varkappa < d$ there is an equivalent quasi-metric $\varrho_\varkappa$, $\varrho \sim \varrho_\varkappa$, such that the Riesz potential $I^X_\varkappa$,

$$(I^X_\varkappa f)(x) = \int_X \frac{f(y)}{\varrho^{d-\varkappa}_\varkappa(x,y)}\, \mu(\mathrm{d}y), \qquad x \in X, \tag{8.176}$$

is a non-negative self-adjoint compact operator in $L_2(X)$ with (8.174), (8.175) for its positive eigenvalues (not excluding that $I^X_\varkappa$ may have a non-trivial kernel). But it was not clear to which extent one can or has to replace a given quasi-metric ϱ by an equivalent one to get the above result. However even in the Euclidean case (8.173) it seems to be doubtful whether one can replace the Euclidean metric $\varrho_n(\gamma,\delta) = |\gamma-\delta|$ by an arbitrary equivalent quasi-metric $\varrho(\gamma,\delta)$ (which might be non-continuous) if one wishes to get the above assertion. We discussed this situation by some examples in [TrY02, Remark 3.8]. This is the point where the *regular d-spaces* according to Definition 8.13(ii) enter the scene transferring d-sets (or better d_ε-sets) according to (8.88) not only isomorphically onto related d-spaces but isometrically in the understanding of (8.89). It seems to be that

> *a substantial analysis on d-sets in* $\mathbb{R}^n$ *should be based on the Euclidean metric and related Hausdorff measures and a substantial analysis on (abstract) d-spaces should be based on their regular images according to Definition* 8.13(ii).

The following assertion may also serve to support this opinion.

Theorem 8.44. *Let* $(X,\varrho,\mu;H)$ *be an* ε*-chart of the regular d-space* (X,ϱ,μ) *according to Definition* 8.13(ii) *and* (8.115). *Let* $0<\varkappa<d$. *Then the Riesz potential* $I^X_\varkappa$,

$$(I^X_\varkappa f)(x) = \int_X \frac{f(y)}{\varrho^{d-\varkappa}(x,y)}\, \mu(\mathrm{d}y), \qquad x \in X, \tag{8.177}$$

is a positive self-adjoint operator in $L_2(X)$ *with* (8.174), (8.175) *for its positive eigenvalues.*

Proof. We adapt (8.173) to (8.88), (8.89), hence

$$(I^\Gamma_{\varkappa/\varepsilon} g)(\gamma) = \int_\Gamma \frac{g(\delta)}{|\gamma-\delta|^{\frac{d-\varkappa}{\varepsilon}}}\, \mathcal{H}^{d_\varepsilon}_\Gamma(\mathrm{d}\delta), \qquad \gamma \in \Gamma, \tag{8.178}$$

where Γ is the d_ε-set in (8.88). By (8.89) and (8.138) it follows that H and H^{-1} generate unitary maps between $L_2(\Gamma)$ and $L_2(X) = L_2(X,\varrho,\mu)$ (denoted again by H and H^{-1}). We claim that $I^X_\varkappa$ in (8.177) and $I^\Gamma_{\varkappa/\varepsilon}$ in (8.178) are unitarily equivalent,

$$I^X_\varkappa = H \circ I^\Gamma_{\varkappa/\varepsilon} \circ H^{-1}. \tag{8.179}$$

For this purpose we apply $f \in L_2(X)$ to the right-hand side,

$$(H \circ I^\Gamma_{\varkappa/\varepsilon} \circ H^{-1} f)(x) = \int_\Gamma \frac{f(H^{-1}\delta)}{|Hx-\delta|^{\frac{d-\varkappa}{\varepsilon}}}\, \mathcal{H}^{d_\varepsilon}_\Gamma(\mathrm{d}\delta). \tag{8.180}$$

We transform the integral over Γ into an integral over $\delta = Hy$. Using (8.89) it follows that the right-hand side of (8.180) equals

$$\int_X \frac{f(y)}{|Hx - Hy|^{\frac{d-\varkappa}{\varepsilon}}}\, \mu(\mathrm{d}y) = \int_X \frac{f(y)}{\varrho(x,y)^{d-\varkappa}}\, \mu(\mathrm{d}y). \tag{8.181}$$

This proves (8.179). Hence $I^X_\varkappa$ is unitarily equivalent to $I^\Gamma_{\varkappa/\varepsilon}$ on the d_ε-set Γ. Then the theorem follows from (8.173)–(8.175) where $\varepsilon > 0$ cancels out. □

8.4.3 Anisotropic spaces

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be an anisotropy in $\mathbb{R}^n$ as introduced in (5.10) and used in Example 8.19. Let

$$B^{s,\alpha}_p(\mathbb{R}^n) = B^{s,\alpha}_{pp}(\mathbb{R}^n), \qquad 1 < p < \infty, \quad s > 0, \tag{8.182}$$

be special anisotropic spaces as introduced in Definition 5.1, complemented by

$$\mathcal{C}^{s,\alpha}(\mathbb{R}^n) = B^{s,\alpha}_{\infty\infty}(\mathbb{R}^n), \qquad s > 0. \tag{8.183}$$

These spaces can be normed by (5.6) and as in Theorem 5.8, Remark 5.10. On the other hand we snowflaked $X_n = [0,1]^n$ in Example 8.19, Proposition 8.20 and Corollary 8.21 anisotropically onto a compact d-set in some $\mathbb{R}^N$ with $d = n/\alpha_1$, illuminated by Example 8.22. By Theorem 1.192 and Definition 1.189 one can extend this procedure from X_n to $\mathbb{R}^n$. In modification of (8.112) one gets a bi-Lipschitzian map H,

$$H: \quad \left(\mathbb{R}^n, \varrho^{\alpha_1}_{\alpha,n}, \mu_n\right) \quad \text{onto} \quad \left(\Gamma, \varrho_N, \mathcal{H}^d_\Gamma\right), \tag{8.184}$$

where Γ is a closed (but not compact) d-set in some $\mathbb{R}^N$ with $d = n/\alpha_1$ in the understanding of (8.4), (8.6) where *compact* is replaced by *closed*. There is also an obvious counterpart of Corollary 8.21 saying that $(\mathbb{R}^n, \varrho_{\alpha,n}, \mu_n)$ is an (closed, complete, but no longer compact) n-space. We take all this for granted. We also assume that there are immediate counterparts of the spaces $B^s_p(\Gamma)$ as trace spaces according to (8.5) and their quarkonial and atomic characterisations according to the Theorems 8.3, 8.11 (the latter restricted to $0 < s < 1$). The only point where some additional care might be needed is the question where the corresponding series converge. There is no immediate substitute of $L_1(X)$ serving in the compact case as the largest space in which everything happens. This point can be settled, but we do not discuss the details. This applies also to the natural extensions of Definition 8.27 and Theorem 8.30. Then one gets spaces

$$B^s_p(\mathbb{R}^n; H) = B^s_p(\mathbb{R}^n, \varrho_{\alpha,n}, \mu_n; H) = B^{s/\alpha_1}_p(\Gamma, \varrho_N, \mathcal{H}^{n/\alpha_1}_\Gamma) \circ H, \tag{8.185}$$

with $s > 0$, $1 < p < \infty$, which also deserve to be called *anisotropic*. The question arises as to how these spaces are related to the spaces in (8.182).

Proposition 8.45. *Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be an anisotropy in $\mathbb{R}^n$ according to (8.106). Let $B^{s,\alpha}_p(\mathbb{R}^n)$ and $B^s_p(\mathbb{R}^n; H)$ be the above spaces. Let $1 < p < \infty$ and $0 < s < \alpha_1$. Then*

$$B^{s,\alpha}_p(\mathbb{R}^n) = B^s_p(\mathbb{R}^n; H). \tag{8.186}$$

Proof. (Outline). We apply (the extended) Theorem 8.39 to the n-space $(\mathbb{R}^n, \varrho_{\alpha,n}, \mu; H)$ with $1 < p < \infty$ and $\varepsilon = \alpha_1$. Then we have the atomic decomposition (8.160) based on the atoms $a^{k,m}_X$ in Definition 8.37. By $\varrho = \varrho_{\alpha,n}$ with (8.107) it follows that the balls $B_{k,m}$ in (8.120) can be identified with the rectangles Q^α_{km} at the beginning of Section 5.1.5. Furthermore $\mu = \mu_n$ is Lebesgue measure and $|Q^\alpha_{km}| \sim 2^{-kn}$. Since $d = n$ and $\varepsilon = \alpha_1$, one gets for the atoms in Definition 8.37,

$$\operatorname{supp} a^{k,m}_X \subset cQ^\alpha_{km}, \qquad X = \mathbb{R}^n, \tag{8.187}$$

$$|a^{k,m}_X(x)| \le 2^{-k(s-n/p)}, \qquad x \in \mathbb{R}^n, \tag{8.188}$$

$$|a^{k,m}_X(x) - a^{k,m}_X(y)| \le 2^{-k(s-n/p)+\alpha_1 k} \sup_r |x_r - y_r|^{\alpha_1/\alpha_r}. \tag{8.189}$$

Let $0 < s < \sigma < \alpha_1$. We claim that for some $c > 0$ and all $a^{k,m}_X$,

$$\|a^{k,m}_X \,|B^{\sigma,\alpha}_p(\mathbb{R}^n)\| \le c\, 2^{k(\sigma-s)}. \tag{8.190}$$

We apply (5.6) with $\bar{\sigma} = (\sigma_1, \ldots, \sigma_n)$ in place of $\bar{s}$, where $\sigma_r \alpha_r = \sigma$ according to (5.24). Since $\sigma_r < 1$ we may choose $M_r = 1$ in (5.5), (5.6). We split the integral over $t \in (0,1)$ into an integral over $(2^{-k\alpha_r}, 1)$ and an integral over $(0, 2^{-k\alpha_r})$. Let e_r be the unit vector in x_r-direction. Then one gets for $a = a^{k,m}_X$,

$$\begin{aligned} &\int_{2^{-k\alpha_r}}^{1} t^{-\sigma_r p}\, \|a(\cdot + te_r) - a\,|L_p(\mathbb{R}^n)\|^p \frac{\mathrm{d}t}{t} \\ &\le c 2^{k\alpha_r \sigma_r p}\, 2^{-kp(s-n/p)}\, 2^{-kn} \\ &= c 2^{k(\sigma-s)p} \end{aligned} \tag{8.191}$$

and

$$\begin{aligned} &\int_0^{2^{-k\alpha_r}} t^{-\sigma_r p}\, \|a(\cdot + te_r) - a\,|L_p(\mathbb{R}^n)\|^p \,\frac{\mathrm{d}t}{t} \\ &\le c 2^{-k(s-n/p)p+\alpha_1 kp}\, 2^{-nk} \int_0^{2^{-k\alpha_r}} t^{-\sigma_r p}\, t^{p\alpha_1/\alpha_r} \,\frac{\mathrm{d}t}{t} \\ &= c 2^{-ksp+\alpha_1 kp} \int_0^{2^{-k\alpha_r}} t^{(\alpha_1-\sigma)p/\alpha_r} \,\frac{\mathrm{d}t}{t} \\ &= c' 2^{k(\sigma-s)p}. \end{aligned} \tag{8.192}$$

This proves (8.190). We take it for granted that the theory of non-smooth atoms for the isotropic spaces $B^s_p(\mathbb{R}^n)$ according to Definition 2.7 and Theorem 2.13 can be extended to the anisotropic spaces $B^{s,\alpha}_p(\mathbb{R}^n)$. Then (8.190) is the direct counterpart of (2.26). Now (8.186) follows from Theorem 8.39 and the anisotropic counterpart of Theorem 2.13. □

Remark 8.46. The anisotropic spaces $B^{s,\alpha}_{pq}(\mathbb{R}^n)$ and $F^{s,\alpha}_{pq}(\mathbb{R}^n)$ according to Definition 5.1 have a long and rich history. There are decompositions in terms of smooth atoms as described in Theorem 5.15. Also the theory of wavelet frames for isotropic spaces as developed in Section 3.2 has a counterpart for some of these anisotropic spaces. We refer to [HaTa05]. There is hardly any doubt that decompositions of the isotropic spaces $B^s_p(\mathbb{R}^n)$ in terms of non-smooth atoms according to Theorem 2.13 have anisotropic counterparts, which we took for granted in connection with (8.190). On the other hand based on the anisotropic distance $\varrho_{\alpha,n}$ in (8.107) one gets by the snowflaked bi-Lipschitzian map (8.184) the spaces in (8.185) which also deserve to be denoted as (non-standard) anisotropic spaces. According to Proposition 8.45 they coincide with the spaces $B^{s,\alpha}_p(\mathbb{R}^n)$ if $0 < s < \alpha_1$. But this is presumably not the case if $s \geq \alpha_1$ and one gets non-standard anisotropic spaces based on counterparts of Definition 8.27.

Chapter 9

Function Spaces on Sets

9.1 Introduction and reproducing formula

9.1.1 Introduction

Chapter 8 dealt with spaces $B^s_p(\Gamma)$, $s > 0$, $1 < p < \infty$, on compact d-sets in $\mathbb{R}^n$ and their snowflaked counterparts $B^s_p(X; H)$ on d-spaces according to Definition 8.27 and Theorem 8.30. We avoided any additional complication caused by more general compact sets Γ in $\mathbb{R}^n$ or by introducing more general spaces, say, $B^s_{pq}(\Gamma)$ and their snowflaked counterparts. In some sense we now return to this subject, at least as far as spaces on arbitrary closed sets Γ in $\mathbb{R}^n$ are concerned, but based on a new approach employing the technique of wavelet frames according to Section 3.2. As in Chapter 8 we restricted ourselves also in Section 3.2 to the simplest case, say, the spaces $B^s_p(\mathbb{R}^n)$ in (3.69). Now we are interested in more general spaces and in a new method with a reproducing formula as the crucial ingredient. We follow partly [Tri06b].

9.1.2 Reproducing formula

We rely on the same notation as in Section 3.2.1 adapted to our purposes. Let $\mathbb{R}^n_{++}$ be as in (3.65) and let k be a non-negative C^∞ function in $\mathbb{R}^n$ with

$$\operatorname{supp} k \subset \left\{ y \in \mathbb{R}^n : \ |y| < 2^{J-\varepsilon} \right\} \cap \mathbb{R}^n_{++} \tag{9.1}$$

for some fixed $\varepsilon > 0$ and some fixed $J \in \mathbb{N}$ (one may choose $J = n$ once and for all), and

$$\sum_{m \in \mathbb{Z}^n} k(x - m) = 1 \qquad \text{where} \quad x \in \mathbb{R}^n. \tag{9.2}$$

Let

$$k^\beta(x) = (2^{-J}x)^\beta k(x) \qquad \text{where} \quad x \in \mathbb{R}^n \quad \text{and} \quad \beta \in \mathbb{N}^n_0, \tag{9.3}$$

and

$$k^\beta_{jm}(x) = k^\beta(2^j x - m) \qquad \text{where} \quad j \in \mathbb{N}_0 \quad \text{and} \quad m \in \mathbb{Z}^n. \tag{9.4}$$

Obviously $k^\beta_{jm}(x) \geq 0$. Let ω be a C^∞ function in $\mathbb{R}^n$ with

$$\operatorname{supp} \omega \subset (-\pi, \pi)^n \qquad \text{and} \qquad \omega(x) = 1 \quad \text{if} \quad |x| \leq 2, \tag{9.5}$$

and for the same $J \in \mathbb{N}$ as before let

$$\omega^\beta(x) = \frac{i^{|\beta|}\, 2^{J|\beta|}}{(2\pi)^n\, \beta!}\, x^\beta \omega(x), \qquad x \in \mathbb{R}^n, \quad \beta \in \mathbb{N}_0^n, \tag{9.6}$$

and

$$\Omega^\beta(x) = \sum_{m \in \mathbb{Z}^n} (\omega^\beta)^\vee(m)\, e^{-imx}, \qquad x \in \mathbb{R}^n. \tag{9.7}$$

With the same function φ_0 as in (3.78) and $\varphi(x) = \varphi_0(x) - \varphi_0(2x)$ we introduce for $\beta \in \mathbb{N}_0^n$,

$$(\Phi^\beta_F)^\vee(\xi) = \varphi_0(\xi)\, \Omega^\beta(\xi), \qquad \xi \in \mathbb{R}^n, \tag{9.8}$$

$$(\Phi^\beta_M)^\vee(\xi) = \varphi(\xi)\, \Omega^\beta(\xi), \qquad \xi \in \mathbb{R}^n, \tag{9.9}$$

and for $r \in \mathbb{N}_0$, $\beta \in \mathbb{N}_0^n$, $m \in \mathbb{Z}^n$ and $\xi \in \mathbb{R}^n$,

$$\Phi^{\beta,r}_{jm}(x) = \begin{cases} \Phi^\beta_F(2^r x - m) & \text{if } j = r, \\ \Phi^\beta_M(2^j x - m) & \text{if } r < j \in \mathbb{N}. \end{cases} \tag{9.10}$$

Comments and properties of these functions may be found in Section 3.2.1. In particular we have $\Phi^{\beta,r}_{jm} \in S(\mathbb{R}^n)$. If $r = 0$ then we put $\Phi^{\beta,0}_{jm} = \Phi^\beta_{jm}$ in good agreement with (3.89). The usual dual pairing

$$\lambda^{\beta,r}_{jm}(f) = 2^{jn}\, \left(f, \Phi^{\beta,r}_{jm}\right), \qquad f \in S'(\mathbb{R}^n), \tag{9.11}$$

makes sense, where $r \in \mathbb{N}_0$, $r \leq j \in \mathbb{N}_0$, $\beta \in \mathbb{N}_0^n$ and $m \in \mathbb{Z}^n$. If $r = 0$ we put $\lambda^{\beta,0}_{jm}(f) = \lambda^\beta_{jm}(f)$ which coincides with (3.96).

Theorem 9.1. *Let K be a C^∞ function in $\mathbb{R}^n$ with*

$$\operatorname{supp} K \subset \{y: \ |y| < 2^{J-\varepsilon}\} \tag{9.12}$$

where $J \in \mathbb{N}$ and $\varepsilon > 0$ have the above meaning. Let K^γ and K^γ_{rl} with $\gamma \in \mathbb{N}_0^n$, $r \in \mathbb{N}_0$ and $l \in \mathbb{Z}^n$ be the obvious counterparts of (9.3), (9.4). Then

$$K^\gamma_{rl}(x) = \sum_{\beta, j \geq r, m} \lambda^{\beta,r}_{jm}\,(K^\gamma_{rl})\; k^\beta_{jm}(x), \qquad x \in \mathbb{R}^n, \tag{9.13}$$

unconditional convergence being in $S(\mathbb{R}^n)$. For any $\varkappa > 0$, $M > 0$, $b > 0$, there is a positive constant $c = c_{\varkappa,M,b}$ (depending on k and K) such that

$$\left|\lambda^{\beta,r}_{jm}\,(K^\gamma_{rl})\right| \leq c\, 2^{-\varepsilon|\gamma|}\, 2^{-\varkappa|\beta|}\, 2^{-(j-r)M}\, (1 + |m - 2^{j-r} l|)^{-b} \tag{9.14}$$

for all admitted β, γ; j, r; m, l.

Proof. Step 1. First we prove (9.13), (9.14) with $r = 0$ and $l = 0$, hence, for $K^\gamma(x) = (2^{-J}x)^\gamma K(x)$ where $\gamma \in \mathbb{N}_0^n$. By Theorem 3.21, (3.96) = (9.11) with $r = 0$ and (3.89) = (9.10) with $r = 0$ we have

$$K^\gamma(x) = \sum_{\beta,j,m} \lambda^\beta_{jm}(K^\gamma)\, k^\beta_{jm}(x), \qquad x \in \mathbb{R}^n, \tag{9.15}$$

$$\lambda^\beta_{jm}(K^\gamma) = 2^{jn} \int_{\mathbb{R}^n} K^\gamma(y)\, \Phi^\beta_M\left(2^j y - m\right) \mathrm{d}y, \quad j \in \mathbb{N}, \quad m \in \mathbb{Z}^n. \tag{9.16}$$

In case of $j = 0$ one has to modify appropriately here and in what follows. Let $A \in \mathbb{N}$ and $\varkappa > 0$. We claim that there is a constant $c_{A,\varkappa}$ such that

$$\left| D^\alpha \left(\Phi^\beta_M\right)^\vee(\xi)\right| \le c_{A,\varkappa}\, 2^{-\varkappa|\beta|} \quad \text{for} \quad |\alpha| \le 2A, \quad \beta \in \mathbb{N}_0^n, \quad \xi \in \mathbb{R}^n. \tag{9.17}$$

This follows from (9.9), based on (9.5)–(9.7), Stirling's formula according to (3.117), (3.118) applied to $\beta!$ and (3.124). One gets that for any $b > 0$ and any $\varkappa > 0$ there is a constant $c_{b,\varkappa}$ such that

$$\left|\Phi^\beta_M(y)\right| \le c_{b,\varkappa}\, 2^{-\varkappa|\beta|}\,(1+|y|)^{-b} \quad \text{for all } \beta \in \mathbb{N}_0^n \text{ and } y \in \mathbb{R}^n. \tag{9.18}$$

Recall that $\varphi(\xi) = 0$ if $|\xi| \le 1/2$. Then one has for any $L \in \mathbb{N}$,

$$(\Phi^\beta_M)^\vee(\xi) = |\xi|^{2L}\, \frac{\varphi(\xi)}{|\xi|^{2L}}\, \Omega^\beta(\xi) = |\xi|^{2L}\, \widetilde{\Omega}^\beta(\xi), \qquad \xi \in \mathbb{R}^n, \tag{9.19}$$

and hence

$$\Phi^\beta_M(y) = \Delta^L\, \widetilde{\Phi}^\beta_M(y), \qquad y \in \mathbb{R}^n, \tag{9.20}$$

where $\Delta^L = \left(\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}\right)^L$ is the Lth power of the Laplacian. Here $\widetilde{\Phi}^\beta_M$ has the same properties as Φ^β_M. In particular we have (9.18) with $\widetilde{\Phi}^\beta_M$ in place of Φ^β_M. Then it follows by partial integration that

$$\int_{\mathbb{R}^n} (\Delta^L K^\gamma)(y)\, \widetilde{\Phi}^\beta_M(2^j y - m)\, \mathrm{d}y = 2^{2jL} \int_{\mathbb{R}^n} K^\gamma(y)\, \Phi^\beta_M(2^j y - m)\, \mathrm{d}y. \tag{9.21}$$

By (9.16), (9.18) with $\widetilde{\Phi}^\beta_M$ in place of Φ^β_M, and (9.12) one obtains that

$$\begin{aligned} |\lambda^\beta_{jm}(K^\gamma)| &\le c\, 2^{jn-2jL}\, 2^{-\varepsilon|\gamma|}\, 2^{-\varkappa|\beta|} \int\limits_{|y|\le 2^J} (1 + |2^j y - m|)^{-b}\, \mathrm{d}y \\ &\le c'\, 2^{j(n+b-2L)}\, 2^{-\varepsilon|\gamma|}\, 2^{-\varkappa|\beta|}\, (1+|m|)^{-b}. \end{aligned} \tag{9.22}$$

First we choose $b > 0$, $\varkappa > 0$, then $L \in \mathbb{N}$, which are at our disposal. We get (9.15) with

$$|\lambda^\beta_{jm}(K^\gamma)| \le c\, 2^{-\varepsilon|\gamma|}\, 2^{-\varkappa|\beta|}\, 2^{-jM}\, (1+|m|)^{-b}, \tag{9.23}$$

hence (9.14) with $r = 0$ and $l = 0$.

Step 2. Let $r = 0$ and $l \in \mathbb{Z}^n$. Then it follows from (9.15), (9.16) that

$$\begin{aligned} K^\gamma_{0,l}(x) = K^\gamma(x-l) &= \sum_{\beta,j,m} \lambda^\beta_{jm}(K^\gamma)\, k^\beta_{j,m+2^j l}(x) \\ &= \sum_{\beta,j,m} \lambda^\beta_{j,m-2^j l}(K^\gamma)\, k^\beta_{jm}(x) \end{aligned} \tag{9.24}$$

and

$$\begin{aligned} \lambda^\beta_{j,m-2^j l}(K^\gamma) &= 2^{jn} \int_{\mathbb{R}^n} K^\gamma(y-l)\, \Phi^\beta_M(2^j y - m)\, \mathrm{d}y \\ &= \lambda^\beta_{jm}(K^\gamma_{0,l}). \end{aligned} \tag{9.25}$$

This proves (9.13) and (9.14) with $r = 0$. Let $r \in \mathbb{N}$ and $l \in \mathbb{Z}^n$. Then

$$\begin{aligned} K^\gamma_{r,l}(x) = K^\gamma_{0,l}(2^r x) &= \sum_{\beta,j,m} \lambda^\beta_{j,m}(K^\gamma_{0,l})\, k^\beta_{j+r,m}(x) \\ &= \sum_{\beta,j\geq r,m} \lambda^\beta_{j-r,m}(K^\gamma_{0,l})\, k^\beta_{jm}(x) \end{aligned} \tag{9.26}$$

and for $j > r$ one gets by (9.25), (9.11),

$$\begin{aligned} \lambda^\beta_{j-r,m}(K^\gamma_{0,l}) &= 2^{(j-r)n} \int_{\mathbb{R}^n} K^\gamma(y-l)\, \Phi^\beta_M(2^{j-r} y - m)\, \mathrm{d}y \\ &= \lambda^{\beta,r}_{jm}(K^\gamma_{rl}). \end{aligned} \tag{9.27}$$

If $j = r$ then one has to replace $\Phi^\beta_M(2^j y - m)$ by $\Phi^\beta_F(2^r y - m)$. This proves (9.13) and (9.14) with $r \in \mathbb{N}_0$ and $l \in \mathbb{Z}^n$.

Step 3. By (9.14) and the properties of k^β_{jm} it follows that (9.13) converges in any norm

$$\sup_{x\in\mathbb{R}^n, |\alpha|\leq L} (1+|x|^2)^{\sigma/2}\, |D^\alpha g(x)| \tag{9.28}$$

with $\sigma > 0$ and $L \in \mathbb{N}$. Hence (9.13) converges unconditionally in $S(\mathbb{R}^n)$. $\square$

Remark 9.2. One can replace the summation over $j \in \mathbb{N}_0$ in (3.97) by a corresponding summation over $r \leq j \in \mathbb{N}$ with $r \in \mathbb{N}$. This follows from the proof of Theorem 3.21. Then one gets (3.99) with $\lambda^{\beta,r}_{jm}(f)$ according to (9.11) in place of $\lambda^\beta_{jm}(f)$. Hence the representation (9.13) is not a surprise. The estimate (9.14) is the main point of the above theorem. The assumption (9.12) is convenient for us. It produces the factor $2^{-\varepsilon|\gamma|}$ in (9.14). But otherwise the above arguments work also for more general (compactly supported sufficiently smooth) functions K. In particular one can represent the above functions $k(x)$ or $k(x-l)$ with $l \in \mathbb{Z}^n$ by finer grids which is similar to the typical assertion for the scaling function

$$\Phi(x) = \sum_{m=-\infty}^{\infty} a_m\, \Phi(2x - m), \qquad x \in \mathbb{R}, \tag{9.29}$$

in the multiresolution analysis briefly described in Section 1.7.1. This results in the following observation.

Corollary 9.3. *Let k be a compactly supported C^∞ function in $\mathbb{R}^n$ generating the resolution of unity* (9.2). *Let $k_\beta(x) = x^\beta\, k(x)$ where $\beta \in \mathbb{N}_0^n$. Let $N \in \mathbb{N}$. Then there are complex numbers λ^β_{jm} such that*

$$k(x) = \sum_{\beta\in\mathbb{N}_0^n} \sum_{j=N}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda^\beta_{jm}\, k_\beta(2^j x - m), \qquad x \in \mathbb{R}^n, \tag{9.30}$$

unconditional convergence being in $S(\mathbb{R}^n)$. For any $\varkappa > 0$, $M > 0$, $b > 0$, there is a positive constant $c = c_{\varkappa,M,b}$ (depending on k and N) such that

$$|\lambda^\beta_{jm}| \le c\, 2^{-\varkappa|\beta|}\, 2^{-jM}\, (1 + |m|)^{-b}. \tag{9.31}$$

Proof. For this purpose one has to modify the arguments from Step 1 of the proof of the above theorem. One can start expanding the first $k(2^{-N}x)$ in place of $K^\gamma(x)$ in (9.15). □

9.2 Spaces on Euclidean n-space

9.2.1 Definitions and basic assertions

As said in the above Introduction 9.1.1 it is one aim of Chapter 9 to extend the considerations in Section 3.2 from the special spaces $B^s_p(\mathbb{R}^n)$ in (3.69) to more general B-spaces and F-spaces. For this purpose one has first to generalise Definition 3.13. As before, Q_{jm} denotes a cube in $\mathbb{R}^n$ with sides parallel to the axes of coordinates, centred at $2^{-j}m$ with side-length 2^{-j+1} where $m \in \mathbb{Z}^n$ and $j \in \mathbb{N}_0$. Let χ_{jm} be the characteristic function of Q_{jm}.

Definition 9.4. *Let $\varrho \ge 0$, $s \in \mathbb{R}$, $0 < p \le \infty$, $0 < q \le \infty$ and*

$$\lambda = \left\{ \lambda^\beta_{jm} \in \mathbb{C} :\ \beta \in \mathbb{N}_0^n,\ j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n \right\}. \tag{9.32}$$

Then

$$b^{s,\varrho}_{pq} = \{\lambda :\ \|\lambda\,|b^{s,\varrho}_{pq}\| < \infty\} \tag{9.33}$$

with

$$\|\lambda\,|b^{s,\varrho}_{pq}\| = \sup_{\beta\in\mathbb{N}_0^n} 2^{\varrho|\beta|} \left(\sum_{j=0}^{\infty} 2^{j(s-n/p)q} \left(\sum_{m\in\mathbb{Z}^n} |\lambda^\beta_{jm}|^p \right)^{q/p} \right)^{1/q} \tag{9.34}$$

and

$$f^{s,\varrho}_{pq} = \{\lambda :\ \|\lambda\,|f^{s,\varrho}_{pq}\| < \infty\} \tag{9.35}$$

with

$$\|\lambda\,|f^{s,\varrho}_{pq}\| = \sup_{\beta\in\mathbb{N}^n_0} 2^{\varrho|\beta|} \left\| \left(\sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} 2^{jsq}\, |\lambda^\beta_{jm}|^q\, \chi_{jm}(\cdot) \right)^{1/q} |L_p(\mathbb{R}^n) \right\| \tag{9.36}$$

with the usual modification if $p=\infty$ and/or $q=\infty$.

Remark 9.5. This is the generalisation of Definition 3.13. It is immaterial whether $2^{\varrho|\beta|}$ is taken with respect to an ℓ_p-quasi-norm or an ℓ_∞-norm. In contrast to Definition 1.17 and Theorem 1.39 we insert now the normalising factors in the sequence spaces. We have

$$b^{s,\varrho}_{pp} = f^{s,\varrho}_{pp}, \qquad s\in\mathbb{R}, \quad 0<p\le\infty. \tag{9.37}$$

Furthermore, $b^{s,\varrho}_{pq}$ and $f^{s,\varrho}_{pq}$ are quasi-Banach spaces.

Again we use standard notation. In particular, $L_p(\mathbb{R}^n)$ with $0<p\le\infty$ are the usual Lebesgue spaces in $\mathbb{R}^n$ quasi-normed by (2.1). Let $w_\sigma(x) = (1+|x|^2)^{\sigma/2}$ with $\sigma\in\mathbb{R}$. Then $L_\infty(\mathbb{R}^n, w_\sigma)$ is the weighted L_∞-space in $\mathbb{R}^n$ normed by $\|w_\sigma f\,|L_\infty(\mathbb{R}^n)\|$. As for the use of $L_\infty(\mathbb{R}^n, w_\sigma)$ one may consult the end of Remark 2.12.

Definition 9.6. *Let k be a non-negative C^∞ function in $\mathbb{R}^n$ with* (9.1) *for some $J\in\mathbb{N}$* (*one may fix $J=n$*) *and* (9.2). *Let k^β_{jm} as in* (9.4). *Let $\varrho\ge 0$, $0<q\le\infty$, and $s>0$.*

(i) *Let $0<p\le\infty$. Then $\mathfrak{B}^s_{pq}(\mathbb{R}^n)$ is the collection of all $f\in L_p(\mathbb{R}^n)$ which can be represented as*

$$f = \sum_{\beta,j,m} \lambda^\beta_{jm}\, k^\beta_{jm}, \qquad \lambda\in b^{s,\varrho}_{pq}, \tag{9.38}$$

absolute convergence being in $L_p(\mathbb{R}^n)$ if $p<\infty$ and in $L_\infty(\mathbb{R}^n, w_\sigma)$ with $\sigma<0$ if $p=\infty$. Let

$$\|f\,|\mathfrak{B}^s_{pq}(\mathbb{R}^n)\| = \inf \|\lambda\,|b^{s,\varrho}_{pq}\| \tag{9.39}$$

where the infimum is taken over all admissible representations (9.38).

(ii) *Let $0<p<\infty$. Then $\mathfrak{F}^s_{pq}(\mathbb{R}^n)$ is the collection of all $f\in L_p(\mathbb{R}^n)$ which can be represented as*

$$f = \sum_{\beta,j,m} \lambda^\beta_{jm}\, k^\beta_{jm}, \qquad \lambda\in f^{s,\varrho}_{pq}, \tag{9.40}$$

absolute convergence being in $L_p(\mathbb{R}^n)$. Let

$$\|f\,|\mathfrak{F}^s_{pq}(\mathbb{R}^n)\| = \inf \|\lambda\,|f^{s,\varrho}_{pq}\|, \tag{9.41}$$

where the infimum is taken over all admissible representations (9.40).

Remark 9.7. To avoid any misunderstanding we remark that the absolute convergence of (9.38) is a consequence of $\lambda \in b^{s,\varrho}_{pq}$ and not an extra requirement. Similarly for (9.40).

Theorem 9.8. *The above spaces* $\mathfrak{B}^s_{pq}(\mathbb{R}^n)$ *and* $\mathfrak{F}^s_{pq}(\mathbb{R}^n)$ *are quasi-Banach spaces. They are independent of* ϱ *and* k *(equivalent quasi-norms). Furthermore, for all admitted* s, p, q,

$$\mathfrak{B}^s_{pq}(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n), \qquad \mathfrak{F}^s_{pq}(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n), \tag{9.42}$$

and

$$\mathfrak{B}^s_{p,\min(p,q)}(\mathbb{R}^n) \hookrightarrow \mathfrak{F}^s_{pq}(\mathbb{R}^n) \hookrightarrow \mathfrak{B}^s_{p,\max(p,q)}(\mathbb{R}^n). \tag{9.43}$$

Proof. Step 1. Let $\varrho \geq 0$ be fixed. The continuous embedding (9.43) follows from a corresponding inclusion for the underlying sequence spaces which may be found in [Triδ, Proposition 13.6, p. 75]. We prove (9.42). By (9.43) it is sufficient to deal with the first inclusion. Let $0 < p \leq 1$. Then it follows from (9.1), (9.3) and (9.34) that

$$\begin{aligned} \Big\| \sum_{\beta,j,m} \lambda^\beta_{jm}\, k^\beta_{jm} \,|L_p(\mathbb{R}^n)\Big\|^p &\leq \sum_{\beta,j} \int_{\mathbb{R}^n} \Big| \sum_m \lambda^\beta_{jm}\, k^\beta(2^j x - m)\Big|^p \, \mathrm{d}x \\ &\leq c \sum_{\beta,j} 2^{-jn}\, 2^{-\varepsilon|\beta|p} \sum_m |\lambda^\beta_{jm}|^p \\ &\leq c'\, \|\lambda\,|b^{s,\varrho}_{pq}\|^p \sum_{\beta,j} 2^{-\varepsilon|\beta|p - jsp}. \end{aligned} \tag{9.44}$$

Since $\varepsilon > 0$ and $s > 0$ one gets (9.42). Similarly if $1 < p \leq \infty$. Then it follows by standard arguments that $\mathfrak{B}^s_{pq}(\mathbb{R}^n)$ and $\mathfrak{F}^s_{pq}(\mathbb{R}^n)$ are quasi-Banach spaces.

Step 2. We prove that the spaces $\mathfrak{B}^s_{pq}(\mathbb{R}^n)$ are independent of ϱ and k. Let K be a second non-negative C^∞ function in $\mathbb{R}^n$ satisfying (9.1), (9.2) with K in place of k and let K^β_{jm} be the related counterpart of (9.4). Let

$$f(x) = \sum_{\gamma,r,l} \nu^\gamma_{rl}\, K^\gamma_{rl}(x), \qquad \nu \in b^{s,\varrho}_{pq}, \tag{9.45}$$

be the corresponding representation which makes sense by Step 1. We insert (9.13) in (9.45) and get

$$f(x) = \sum_{\beta,j,m} \lambda^\beta_{jm}\, k^\beta_{jm}(x) \tag{9.46}$$

with

$$\lambda^\beta_{jm} = \sum_{\gamma\in\mathbb{N}^n_0} \sum_{r=0}^{j} \sum_{l\in\mathbb{Z}^n} \nu^\gamma_{rl} \cdot \lambda^{\beta,r}_{jm}(K^\gamma_{rl}). \tag{9.47}$$

Let $\nu^\gamma_{rl} = 0$ if $r < 0$ and let $t = j - r$. Then it follows from (9.14) that

$$\begin{aligned} &2^{j(s-n/p)}\, 2^{\varkappa|\beta|}\, |\lambda^\beta_{jm}| \\ &\le c \sum_\gamma 2^{-\varepsilon|\gamma|} \sum_{t=0}^\infty \sum_{l\in\mathbb{Z}^n} 2^{(j-t)(s-n/p)}\, |\nu^\gamma_{j-t,l}|\, 2^{-tM'}\, (1+|m-2^t l|)^{-b} \\ &\le c' \sum_\gamma 2^{-\varepsilon|\gamma|} \left(\sum_{t,l} 2^{(j-t)(s-n/p)p}\, |\nu^\gamma_{j-t,l}|^p\, 2^{-tM''p}\, \big(1+|m-2^t l|\big)^{-b'p} \right)^{1/p} \end{aligned} \tag{9.48}$$

where $\varkappa > 0$, $M'' > 0$, $b' > 0$ are at our disposal. We take the ℓ_p-quasi-norm with respect to $m \in \mathbb{Z}^n$. The related factor at the right-hand side of (9.48) can be estimated independently of $2^t l$ (choosing $b' > 0$ sufficiently large). Then the summation over $l \in \mathbb{Z}^n$ gives the desired ℓ_p-blocks for the coefficients $\nu^\gamma_{j-t,l}$. Afterwards one can do the same with respect to ℓ_q and the summation over j at the expense of $M'' > 0$. Since $\varepsilon > 0$ one gets

$$\|\lambda\, |b^{s,\varkappa}_{pq}\| \le c\, \|\nu\, |b^{s,0}_{pq}\| \le c\, \|\nu\, |b^{s,\varrho}_{pq}\| \tag{9.49}$$

for any $\varrho \ge 0$ and any $\varkappa \ge 0$. This proves the independence of the spaces $\mathfrak{B}^s_{pq}(\mathbb{R}^n)$ both of ϱ and of k.

Step 3. We prove that the spaces $\mathfrak{F}^s_{pq}(\mathbb{R}^n)$ are independent of ϱ and k. We need some preparations. Recall that for locally integrable functions in $\mathbb{R}^n$,

$$(Mg)(x) = \sup |Q|^{-1} \int_Q |g(y)|\, \mathrm{d}y, \qquad x \in \mathbb{R}^n, \tag{9.50}$$

is the Hardy-Littlewood maximal function, where the supremum is taken over all cubes Q centred at x. Let $0 < p < \infty$, $0 < q \le \infty$, and $0 < w < \min(p,q)$. Then there is a constant c such that for all such functions g_{jm} with $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$,

$$\left\| \left(\sum_{j,m} (M|g_{jm}|^w)\,(\cdot)^{q/w} \right)^{1/q} |L_p(\mathbb{R}^n) \right\| \le c \left\| \left(\sum_{j,m} |g_{jm}(\cdot)|^q \right)^{1/q} |L_p(\mathbb{R}^n) \right\|. \tag{9.51}$$

This vector-valued maximal inequality goes back to [FeS71]. A short proof may be found in [Tor86, pp. 303–305]. As for the use of (9.51) in the context of F-spaces and further references one may consult [Triγ, p. 89] and [Triδ, p. 79]. Let χ_r be the characteristic function of the cube of side-length 2^{-r} with $r \in \mathbb{N}_0$ centred at the origin. Then it follows from (9.50) that for some $c > 0$ and all $r \in \mathbb{N}_0$,

$$(M\chi_r)(x) \ge c\, \min\left[1, 2^{-rn}\, |x|^{-n}\right], \qquad x \in \mathbb{R}^n. \tag{9.52}$$

We compare χ_{jm} with $M\chi_{rl}$ as needed in (9.47) in connection with $f^{s,\varrho}_{pq}$ according to (9.36). Only the case $r \le j$ is of interest. With $x = 2^{-j}m$ one gets by (9.52) that

$$
\begin{aligned}
(M\chi_{rl})(x)^{1/w} &\geq c \min\left[1,\ \left(2^{-rn}|2^{-j}m - 2^{-r}l|^{-n}\right)^{1/w}\right] \\
&\geq c \min\left[1,\ |m - 2^{j-r}l|^{-n/w}\right].
\end{aligned}
\tag{9.53}
$$

This estimate remains valid for all $x \in \mathbb{R}^n$ with $\chi_{jm}(x) = 1$ on the left-hand side of (9.53) and the same right-hand side. With $t = j - r$ one gets

$$
\chi_{jm}(x) \leq c' \left(1 + |m - 2^t l|\right)^{n/w} (M\chi_{j-t,l})(x)^{1/w}, \quad x \in \mathbb{R}^n. \tag{9.54}
$$

With $\nu^\gamma_{rl} = 0$ if $r < 0$ one obtains by (9.47) and (9.14) in analogy to (9.48),

$$
\begin{aligned}
&2^{js}\, 2^{\varkappa|\beta|} |\lambda^\beta_{jm}|\, \chi_{jm}(x) \\
&\leq c \sum_\gamma 2^{-\varepsilon|\gamma|} \sum_{t=0}^\infty \sum_{l \in \mathbb{Z}^n} 2^{(j-t)s} |\nu^\gamma_{j-t,l}| (M\chi_{j-t,l})(x)^{1/w} \cdot 2^{-tM'} (1 + |m - 2^t l|)^{-b'} \\
&\leq c' \sum_\gamma 2^{-\varepsilon|\gamma|} \times \\
&\quad \left(\sum_{t,l} 2^{(j-t)sq} |\nu^\gamma_{j-t,l}|^q\, (M\chi_{j-t,l})(x)^{q/w}\, 2^{-tM''q} \left(1 + |m - 2^t l|\right)^{-b''q} \right)^{1/q}
\end{aligned}
\tag{9.55}
$$

where $\varkappa > 0$, $M'' > 0$, $b'' > 0$ are at our disposal. It follows for $0 < \varepsilon' < \varepsilon$ that

$$
\begin{aligned}
&2^{\varkappa|\beta|} \left(\sum_{j,m} 2^{jsq} |\lambda^\beta_{jm}|^q \chi_{jm}(x) \right)^{1/q} \\
&\leq c \sum_\gamma 2^{-\varepsilon'|\gamma|} \left(\sum_{j,m} 2^{jsq} |\nu^\gamma_{j,m}|^q (M\chi_{jm})(x)^{q/w} \right)^{1/q}.
\end{aligned}
\tag{9.56}
$$

Application of (9.51) with $g_{jm} = 2^{js} |\nu^\gamma_{j,m}| \chi_{jm}$ gives for $0 < \varepsilon'' < \varepsilon'$ that

$$
\begin{aligned}
&2^{\varkappa|\beta|} \left\| \left(\sum_{j,m} 2^{jsq} |\lambda^\beta_{jm}|^q \chi_{jm}(\cdot) \right)^{1/q} |L_p(\mathbb{R}^n) \right\| \\
&\leq c' \sum_\gamma 2^{-\varepsilon''|\gamma|} \left\| \left(\sum_{j,m} 2^{jsq} |\nu^\gamma_{jm}|^q \chi_{jm}(\cdot) \right)^{1/q} |L_p(\mathbb{R}^n) \right\|.
\end{aligned}
\tag{9.57}
$$

Since $\varepsilon > 0$ one gets by (9.36),

$$
\|\lambda \,| f^{s,\varkappa}_{pq}\| \leq c\, \|\nu \,| f^{s,0}_{pq}\| \leq c\, \|\nu \,| f^{s,\varrho}_{pq}\| \tag{9.58}
$$

for any $\varrho \geq 0$ and any $\varkappa \geq 0$. This proves that the spaces $\mathfrak{F}^s_{pq}(\mathbb{R}^n)$ are independent of ϱ and k. $\square$

Remark 9.9. The independence of $\mathfrak{B}^s_{pq}(\mathbb{R}^n)$ and $\mathfrak{F}^s_{pq}(\mathbb{R}^n)$ of ϱ looks a little bit curious at first glance. It comes from the estimates (9.49), (9.58) which in turn are based on (9.14), where c depends on $\varkappa$. In a slightly different but nearby context we estimated in[Triϵ, (2.81), p. 23] corresponding constants. This suggests that c in (9.14) can be replaced by $c_1 2^{c_2 \varkappa}$ with $c_1 > 0$, $c_2 > 0$, independent of $\varkappa$. Hence equivalence constants of $\mathfrak{B}^s_{pq}(\mathbb{R}^n)$ and $\mathfrak{F}^s_{pq}(\mathbb{R}^n)$ based on Definition 9.6 with large ϱ are of type $c_1\, 2^{c_2 \varrho}$ compared with, say, 0 in place of ϱ.

9.2.2 Properties

Again let

$$\sigma_p = n \left(\frac{1}{p} - 1 \right)_+ \qquad \text{and} \qquad \sigma_{pq} = n \left(\frac{1}{\min(p,q)} - 1 \right)_+ \tag{9.59}$$

where $0 < p \leq \infty$, $0 < q \leq \infty$. Let $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ be the spaces as introduced in Section 2.1.3.

Theorem 9.10. *Let $\mathfrak{B}^s_{pq}(\mathbb{R}^n)$ and $\mathfrak{F}^s_{pq}(\mathbb{R}^n)$ be the spaces according to Definition* 9.6. *Then*

$$\mathfrak{B}^s_{pq}(\mathbb{R}^n) = B^s_{pq}(\mathbb{R}^n) \qquad \text{if} \quad 0 < p \leq \infty, \quad 0 < q \leq \infty, \quad s > \sigma_p, \tag{9.60}$$

and

$$\mathfrak{F}^s_{pq}(\mathbb{R}^n) = F^s_{pq}(\mathbb{R}^n) \qquad \text{if} \quad 0 < p < \infty, \quad 0 < q \leq \infty, \quad s > \sigma_{pq}, \tag{9.61}$$

interpreted as subspaces of $S'(\mathbb{R}^n)$.

Proof. We described in Theorem 1.39 quarkonial representations of the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ with a reference to [Triϵ]. The building blocks k^β_{jm} in (9.38), (9.40) differ from the β-quarks $(\beta\text{-qu})_{jm}$ in (1.107) only by unimportant technicalities. In particular, the normalising factors $2^{-j(s-n/p)}$ are incorporated now in the sequence spaces according to Definition 9.4. Hence, the above theorem follows from Theorem 1.39. $\square$

Remark 9.11. The above building blocks k^β_{jm} in (9.4) coincide with corresponding functions in Definition 3.17, complemented by the dual system Φ^β_{jm} in (3.89). Furthermore, the spaces in (9.37) are essentially the same as the spaces $b^{s,\varrho}_p$ in Definition 3.13. In addition to the above theorem we have for the spaces $B^s_{pp}(\mathbb{R}^n)$ the frame representation in Theorem 3.21(ii) with the coefficients $\lambda^\beta_{jm}(f)$ in (3.96). One can expect that this frame representation can be extended to all spaces

$B^s_{pq}(\mathbb{R}^n)$ with $s > \sigma_p$ and all spaces $F^s_{pq}(\mathbb{R}^n)$ with $s > \sigma_{pq}$. But it is doubtful whether there are corresponding frame representations for the spaces

$$\mathfrak{B}^s_{pq}(\mathbb{R}^n) \quad \text{with } 0 < s < \sigma_p \qquad \text{and} \qquad \mathfrak{F}^s_{pq}(\mathbb{R}^n) \quad \text{with } 0 < s < \sigma_{pq} \tag{9.62}$$

with optimal coefficients depending linearly on f. If $0 < s < \sigma_p$ (in particular $0 < p < 1$) then the δ-distribution belongs to all spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$. Hence these spaces are incomparable with the spaces $\mathfrak{B}^s_{pq}(\mathbb{R}^n)$ and $\mathfrak{F}^s_{pq}(\mathbb{R}^n)$ which are subspaces of $L_p(\mathbb{R}^n)$ according to (9.42). But they are related to other spaces playing a role especially in approximation theory. We give a brief description and outline some assertions.

Let as before,

$$(\Delta^1_h f)(x) = f(x+h) - f(x), \qquad \Delta^{l+1}_h = \Delta^1_h \Delta^l_h\,, \tag{9.63}$$

where $x \in \mathbb{R}^n$, $h \in \mathbb{R}^n$, $l \in \mathbb{N}$ be the usual iterated differences and

$$\omega_l(f,t)_p = \sup_{|h|\le t} \|\Delta^l_h f \,|L_p(\mathbb{R}^n)\|, \qquad 0 < t < \infty, \quad 0 < p \le \infty, \tag{9.64}$$

be the related modulus of continuity. Let

$$d^l_{t,p} f(x) = \left(t^{-n} \int\limits_{|h|\le t} |(\Delta^l_h f)(x)|^p \, \mathrm{d}h \right)^{1/p}, \quad l \in \mathbb{N}, \quad 0 < p < \infty, \tag{9.65}$$

$x \in \mathbb{R}^n$, be the same ball means as in (1.377). By Theorem 1.116 the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ with p, q, s restricted as in (9.60), (9.61), can be characterised in terms of $\omega_l(f,t)_p$ and some means $d^l_{t,u} f$ of the above type. By the above comments and also by Remark 1.117 these characterisations cannot be extended to $s < \sigma_p$ or $s < \sigma_{pq}$. But on the other hand, the quasi-norms (1.379), (1.381), (1.383) make sense (with $u = p$) if $f \in L_p(\mathbb{R}^n)$. At least B-spaces of this type have some history. .

Definition 9.12.

(i) *Let $0 < p \le \infty$, $0 < q \le \infty$, $0 < s < l \in \mathbb{N}$. Then $\mathbf{B}^s_{pq}(\mathbb{R}^n)$ is the collection of all $f \in L_p(\mathbb{R}^n)$ such that*

$$\|f\,|\mathbf{B}^s_{pq}(\mathbb{R}^n)\|_l = \|f\,|L_p(\mathbb{R}^n)\| + \left(\int\limits_0^1 t^{-sq} \omega_l(f,t)^q_p \, \frac{\mathrm{d}t}{t} \right)^{1/q} < \infty \tag{9.66}$$

(*with the usual modification if $q = \infty$*).

(ii) *Let* $0 < p < \infty$, $0 < q \le \infty$, $0 < s < l \in \mathbb{N}$. *Then* $\mathbf{F}^s_{pq}(\mathbb{R}^n)$ *is the collection of all* $f \in L_p(\mathbb{R}^n)$ *such that*

$$\|f\,|\mathbf{F}^s_{pq}(\mathbb{R}^n)\|_l = \|f\,|L_p(\mathbb{R}^n)\| + \left\| \left(\int_0^1 t^{-sq}\, d^l_{t,p} f(\cdot)^q \, \frac{\mathrm{d}t}{t} \right)^{1/q} |L_p(\mathbb{R}^n) \right\| < \infty \tag{9.67}$$

(*with the usual modification if* $q = \infty$).

Remark 9.13. For $0 < p \le \infty$, $0 < q \le \infty$ one can complement (9.60) by

$$\mathbf{B}^s_{pq}(\mathbb{R}^n) = \mathfrak{B}^s_{pq}(\mathbb{R}^n) = B^s_{pq}(\mathbb{R}^n) \qquad \text{if} \quad s > \sigma_p, \tag{9.68}$$

(equivalent quasi-norms, also with respect to l in (9.66)). This follows from Theorem 1.116(i). In case of the F-spaces one has Theorem 1.116(iii). For $0 < p \le 1$, $0 < q \le \infty$ and with $r = 1$, $u = p$ in (1.382), (1.383) one can complement (9.61) by

$$\mathbf{F}^s_{pq}(\mathbb{R}^n) = \mathfrak{F}^s_{pq}(\mathbb{R}^n) = F^s_{pq}(\mathbb{R}^n) \qquad \text{if} \quad s > \sigma_{pq}. \tag{9.69}$$

The study of the spaces $\mathbf{B}^s_{pq}(\mathbb{R}^n)$ for all admitted p, q, s goes back to [StO78]. In particular they are independent of $l \in \mathbb{N}$ with $s < l$ in (9.66). One can prove this assertion using (7.242). This applies also to the independence of $\mathbf{F}^s_{pq}(\mathbb{R}^n)$ of l in (9.67). We refer also to [DeL93, Ch. 2, §10] and [DeS93] as far as the $\mathbf{B}^s_{pq}$-spaces are concerned. A new approach to these spaces were given in [Net89] and [HeN04] including atomic characterisations. We wish to apply in particular [HeN04, Theorem 1.1.14]. Restricted to the above situation one gets for the parameters ε_+, ε_- and r as used in this theorem that $\varepsilon_+ = \varepsilon_- = s$, and r with $r < p$ for the B-spaces and r with $r < \min(p, q)$ for the F-spaces. This results in $s > 0$ for the B-spaces and in

$$s > n \left(\frac{1}{\min(p,q)} - \frac{1}{p} \right) \qquad \text{for the } F\text{-spaces.}$$

Under these restrictions the quasi-norm in (9.67) coincides essentially with corresponding quasi-norms in [HeN04, Theorem 1.1.14] (discrete version). This gives the possibility to apply the atomic characterisations according to [Net89] and [HeN04] with the following outcome.

Proposition 9.14.

(i) *Let* $0 < p \le \infty$, $0 < q \le \infty$. *Then*

$$\mathbf{B}^s_{pq}(\mathbb{R}^n) = \mathfrak{B}^s_{pq}(\mathbb{R}^n) \qquad \textit{for } s > 0. \tag{9.70}$$

(ii) *Let* $0 < p < \infty$, $0 < q \le \infty$. *Then*

$$\mathbf{F}^s_{pq}(\mathbb{R}^n) = \mathfrak{F}^s_{pq}(\mathbb{R}^n) \qquad \textit{for } s > n \left(\frac{1}{\min(p,q)} - \frac{1}{p} \right). \tag{9.71}$$

Proof. By [Net89], [HeN04, Theorem 1.1.14] and the above comments one has for all these spaces $\mathbf{B}^s_{pq}(\mathbb{R}^n)$ and $\mathbf{F}^s_{pq}(\mathbb{R}^n)$ atomic characterisations as in Theorem 1.19 but based on atoms without moment conditions according to Definition 2.5. On the other hand shifting the normalising factors $2^{-j(s-n/p)}$ from the sequence spaces to the building blocks k^β_{jm}, both (9.38) and (9.40) are corresponding atomic decompositions in $\mathfrak{B}^s_{pq}(\mathbb{R}^n)$ and $\mathfrak{F}^s_{pq}(\mathbb{R}^n)$. The additional summation over β does not matter since one has the extra factors $2^{-\varepsilon|\beta|}$. In particular one gets

$$\|f\,|\mathbf{B}^s_{pq}(\mathbb{R}^n)\| \le c\,\|f\,|\mathfrak{B}^s_{pq}(\mathbb{R}^n)\| \tag{9.72}$$

and an F-counterpart. As for the converse one can argue as in the proof of Theorem 2.13. One expands arbitrary atoms a_{rl} with $r \in \mathbb{N}_0$, $l \in \mathbb{Z}^n$ by the smooth atoms $2^{-j(s-n/p)}k^\beta_{jm}$. Afterwards one can control the coefficients in the same way as in Step 2 of the proof of Theorem 2.13. We do not go into detail. But one gets in the same way as there the converse of (9.72) and its F-counterpart. □

Remark 9.15. We admit that the above proof is somewhat sketchy and that some details must be considered more carefully. Nevertheless we have characterisations in terms of the quasi-norms (9.66) for all spaces $\mathfrak{B}^s_{pq}(\mathbb{R}^n)$ with $s > 0$ in contrast to $B^s_{pq}(\mathbb{R}^n)$ where s is restricted by $s > \sigma_p$. By the above considerations $s = \sigma_p$ is a natural breaking line for B-spaces. For the F-spaces the situation is more complicated. On the one hand one has according to Theorem 1.116(iii) characterisations in terms of ball means for all spaces $F^s_{pq}(\mathbb{R}^n)$ with $s > \sigma_{pq}$. On the other hand (9.67) applies to all spaces $\mathfrak{F}^s_{pq}(\mathbb{R}^n)$ covered by (9.71). In case of $1 < p < \infty$ one may choose $r = u = p$ in (1.382) and (1.383). Then the spaces in (9.71) coincide with $F^s_{pq}(\mathbb{R}^n)$. Together with (9.69) one gets for $0 < p < \infty$, $0 < q \le \infty$ that

$$\mathbf{F}^s_{pq}(\mathbb{R}^n) = \mathfrak{F}^s_{pq}(\mathbb{R}^n) = F^s_{pq}(\mathbb{R}^n) \quad \text{if} \quad s > n\left(\frac{1}{\min(p,q)} - \frac{1}{\max(1,p)}\right).$$

Obviously no equivalent quasi-norms of type (9.67) can be expected for the spaces $F^s_{pq}(\mathbb{R}^n)$ if $s < \sigma_p$. The recent paper [ChS05] indicates that such a characterisation in terms of ball means is also impossible if $\sigma_p < s < \sigma_{pq}$ (in particular $0 < q < p$). We refer also to Remark 1.117. These observations suggest that

$$\mathbf{F}^s_{pq}(\mathbb{R}^n) \neq F^s_{pq}(\mathbb{R}^n) \qquad \text{if } 0 < p < \infty,\ 0 < q \le \infty,\ 0 < s < \sigma_{pq}, \tag{9.73}$$

as subspaces of $L_p(\mathbb{R}^n)$ or, if in addition $s > \sigma_p$, as subspaces of $S'(\mathbb{R}^n)$. Of course,

$$0 < q < p < \infty, \qquad \sigma_p < s < \sigma_{pq},$$

is the most surprising case in (9.73). Comparing the atomic decompositions for $F^s_{pq}(\mathbb{R}^n)$ in Theorem 1.19 with the indicated atomic decompositions for the spaces in (9.71) (without cancellation conditions) then it follows from (9.73) that the cancellation conditions for the spaces $F^s_{pq}(\mathbb{R}^n)$ with $s < \sigma_{pq}$ are indispensable. We refer also to Remark 1.20.

This settles a long standing question.

> *In particular,* $s = \sigma_{pq}$ *is a natural breaking line for equivalent quasi-norms in* $F^s_{pq}(\mathbb{R}^n)$ *of type* (1.383), (9.67) *and also in connection with cancellation conditions of type* (1.60), (1.72) *for atoms.*

By (9.1) all building blocks k^β_{jm} in (9.4) are non-negative. This paves the way to an immediate counterpart of Theorem 3.48 dealing with the so-called *positivity property* of function spaces according to Definition 3.46. Now we identify $A^s_{pq}(\mathbb{R}^n)$ in Definition 3.46 with one of the spaces in Definition 9.6.

Theorem 9.16. *All spaces* $\mathfrak{B}^s_{pq}(\mathbb{R}^n)$ *and* $\mathfrak{F}^s_{pq}(\mathbb{R}^n)$ *in Definition* (9.6) *have the positivity property.*

Proof. One can follow Step 1 of the proof of Theorem 3.48 based on $k^\beta_{jm} \geq 0$ mentioned above. □

Remark 9.17. By Theorem 3.48(ii) the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ with $s < \sigma_p$ do not have the positivity property. Then it follows from the above considerations that the representation in Theorem 1.39 cannot be extended to these spaces.

9.3 Spaces on sets

9.3.1 Preliminaries and sequence spaces

Let μ be a Radon measure in $\mathbb{R}^n$ according to (1.515) with compact support $\Gamma = \operatorname{supp} \mu$. We dealt in Sections 1.17.2, 1.17.3 with the spaces

$$B^s_{pq}(\Gamma, \mu) = \operatorname{tr}_\mu B^{s+|s_\mu(1-t)|}_{pq}(\mathbb{R}^n), \qquad 0 < 1/p = t < 1, \tag{9.74}$$

on Γ, defined in the indicated way as trace spaces, where s_μ is the Besov characteristics of μ. Here s, p, q are naturally restricted by

$$s > 0, \quad 1 < p < \infty, \quad 0 < q \leq \infty. \tag{9.75}$$

By Theorem 1.185 one has a satisfactory quarkonial representation for these spaces. In Section 8.1 we specified and detailed this approach to d-sets Γ in $\mathbb{R}^n$ with $\mu = \mathcal{H}^d|\Gamma$ and $q = p$ in (9.74). In particular there are several good reasons to restrict p by $1 < p < \infty$, where one defines spaces $B^s_{pq}(\Gamma, \mu)$ with $s > 0$ as traces of corresponding spaces $B^\sigma_{pq}(\mathbb{R}^n)$ with $\sigma > s > 0$. In this case the underlying spaces $B^\sigma_{pq}(\mathbb{R}^n)$ admit atomic or quarkonial representations where the corresponding building blocks need not satisfy cancellation (or moment) conditions. Such representations can be shifted from $\mathbb{R}^n$ to Γ resulting in corresponding representations for the trace spaces. We refer to Theorems 1.185 and 8.3. This procedure does not work in a satisfactory way if the building blocks, say, atoms, in the underlying spaces $B^\sigma_{pq}(\mathbb{R}^n)$ have to satisfy some cancellation conditions. This is the case if $p < 1$ and $0 < \sigma < n(\frac{1}{p} - 1)$, Theorem 1.19. Nevertheless something can be said. We refer to [Triδ, Section 20]. But the outcome is not really satisfactory and by this type of reasoning there is apparently no way to get counterparts of quarkonial representations as in Theorems 1.185, 8.3. Now the situation is much better if one replaces the above spaces $B^\sigma_{pq}(\mathbb{R}^n)$ where one is starting from by the spaces $\mathfrak{B}^\sigma_{pq}(\mathbb{R}^n)$ or $\mathfrak{F}^\sigma_{pq}(\mathbb{R}^n)$ according to Definition 9.6. All building blocks k^β_{jm} are non-negative and there is no trouble with required cancellations. Even better, there is no need to define $\mathfrak{B}^s_{pq}(\Gamma, \mu)$ and $\mathfrak{F}^s_{pq}(\Gamma, \mu)$ as traces of suitable spaces on $\mathbb{R}^n$. All this can be done directly on Γ following the scheme of Definition 9.6 and Theorem 9.8. It is the main aim of this Section 9.3 to discuss these new possibilities and to compare the outcome with previous results.

Definitions 9.4 for sequence spaces and 9.6 for function spaces can be extended to sets M in $\mathbb{R}^n$. Two cases are of interest for us. First $M = \Omega$ is a domain (= open set) in $\mathbb{R}^n$, and secondly,

$$M = \Gamma = \operatorname{supp} \mu \quad \text{compact}, \qquad |\Gamma| = 0, \tag{9.76}$$

is the support of a Radon measure μ in $\mathbb{R}^n$, where $|\Gamma|$ is the Lebesgue measure of Γ. We assume that

$$0 < \mu(\mathbb{R}^n) = \mu(\Gamma) < \infty \quad \text{and} \quad \mu \in \mathcal{C}^{-\sigma}(\mathbb{R}^n), \qquad 0 < \sigma \le n. \tag{9.77}$$

Here $\mathcal{C}^s(\mathbb{R}^n) = B^s_{\infty\infty}(\mathbb{R}^n)$ are the Hölder-Zygmund spaces (extended to $s \le 0$). Let Q_{jm} be the same cubes as recalled at the beginning of Section 9.2.1 with sides parallel to the axes of coordinates, centred at $2^{-j}m$ and with side-length 2^{-j+1}. Let

$$\mu_j = \sup_{m \in \mathbb{Z}^n} \mu(Q_{jm}), \qquad j \in \mathbb{N}_0. \tag{9.78}$$

Then

$$\mu \in \mathcal{C}^{-\sigma}(\mathbb{R}^n) \quad \text{if, and only if,} \quad \sup_{j \in \mathbb{N}_0} 2^{j(n-\sigma)} \mu_j < \infty, \tag{9.79}$$

where $0 < \sigma \le n$ (equivalent norms). This follows from the μ-property of the spaces $\mathcal{C}^{-\sigma}(\mathbb{R}^n)$ according to Theorems 1.131 and 7.5. In terms of the Besov characteristics introduced in Definitions 1.165, 7.26 one has

$$-n \le s_\mu(0) = \sup \big\{ -\sigma : \; \mu \in \mathcal{C}^{-\sigma}(\mathbb{R}^n) \big\} \le 0. \tag{9.80}$$

One may also consult Figure 1.17.1. One could adopt a different point of view furnishing a given (non-empty) compact set Γ in $\mathbb{R}^n$ with a Radon measure μ satisfying the doubling condition such that

$$\operatorname{supp} \mu = \Gamma \quad \text{compact}, \qquad 0 < \mu(\Gamma) < \infty. \tag{9.81}$$

This is possible. We refer to [VoK87]. But as always in this book we give preference to measures over sets. For $M \subset \mathbb{R}^n$ (either a domain Ω or a compact set Γ) we abbreviate

$$\sum_m^{M,j} = \sum_{m\in\mathbb{Z}^n, Q_{jm}\cap M\neq\emptyset} \qquad \text{where} \quad j\in\mathbb{N}_0. \tag{9.82}$$

Now we adapt Definition 9.4 to sets M. Let again χ_{jm} be the characteristic function of Q_{jm}.

Definition 9.18. *Let $\varrho\geq 0$, $s\in\mathbb{R}$, $0<p\leq\infty$, $0<q\leq\infty$. Let $M\subset\mathbb{R}^n$ and*

$$\lambda = \Big\{\lambda^\beta_{jm}\in\mathbb{C}:\ \beta\in\mathbb{N}^n_0,\ j\in\mathbb{N}_0,\ m\in\mathbb{Z}^n \text{ with } Q_{jm}\cap M\neq\emptyset\Big\}. \tag{9.83}$$

Then

$$b^{s,\varrho}_{pq}(M) = \big\{\lambda:\ \|\lambda\,|b^{s,\varrho}_{pq}(M)\|<\infty\big\} \tag{9.84}$$

with

$$\|\lambda\,|b^{s,\varrho}_{pq}(M)\| = \sup_{\beta\in\mathbb{N}^n_0} 2^{\varrho|\beta|}\left(\sum_{j=0}^\infty 2^{j(s-n/p)q}\left(\sum_m^{M,j}|\lambda^\beta_{jm}|^p\right)^{q/p}\right)^{1/q} \tag{9.85}$$

and

$$f^{s,\varrho}_{pq}(M) = \big\{\lambda:\ \|\lambda\,|f^{s,\varrho}_{pq}(M)\|<\infty\big\} \tag{9.86}$$

with

$$\|\lambda\,|f^{s,\varrho}_{pq}(M)\| = \sup_{\beta\in\mathbb{N}^n_0} 2^{\varrho|\beta|}\left\|\left(\sum_{j=0}^\infty\sum_m^{M,j} 2^{jsq}\,|\lambda^\beta_{jm}|^q\,\chi_{jm}(\cdot)\right)^{1/q}|L_p(\mathbb{R}^n)\right\| \tag{9.87}$$

with the usual modification if $p=\infty$ and/or $q=\infty$.

Remark 9.19. This is the modification of Definition 9.4 we are working with. There is an immediate counterpart of (9.37). But the above definition is by no means obvious. We add a discussion. One may ask what happens if one replaces $L_p(\mathbb{R}^n)$ in (9.87) by $L_p(\Gamma,\mu)$ quasi-normed by (7.35) in case of $M=\Gamma$ with (9.76), (9.77). In case of a domain $M=\Omega$ one may ask what happens if one replaces $L_p(\mathbb{R}^n)$ by $L_p(\Omega)$ naturally quasi-normed by (4.1). Let temporarily $f^{s,\varrho,\mu}_{pq}$ be the spaces originating from (9.87) with $L_p(\Gamma,\mu)$ in place of $L_p(\mathbb{R}^n)$ and let Γ be a d-set according to (7.97) with $0<d<n$. Then

$$f^{s,\varrho,\mu}_{pp} = f^{\sigma,\varrho}_{pp}(\Gamma) = b^{\sigma,\varrho}_{pp}(\Gamma), \quad 0<p<\infty, \quad s=\sigma-\frac{n}{p}+\frac{d}{p}. \tag{9.88}$$

We indicate a proof of this assertion. It follows by Proposition 1.33 that one can replace the characteristic functions χ_{jm} of the cubes Q_{jm} with $Q_{jm}\cap\Gamma\neq\emptyset$ in (9.87) by the characteristic functions of balls centred at Γ of radius $\sim 2^{-j}$, covering Γ, and having for each $j\in\mathbb{N}_0$ pairwise distance $\geq c\,2^{-j}$ for some $c>0$ and all $j\in$

$\mathbb{N}_0$. But then (9.88) follows from (7.97) and the counterpart of (9.37). But if $p \neq q$ then it is not so clear whether the spaces $f^{s,\varrho,\mu}_{pq}$ and $f^{\sigma,\varrho}_{pq}(\Gamma)$ are related. Maybe at least in case of isotropic measures according to Definition 7.18 the sequence spaces $f^{s,\varrho,\mu}_{pq}$ might be the better choice when it comes to $\mathfrak{F}$-spaces. Following the above arguments one would need a μ-counterpart of the vector-valued maximal inequality (9.51). This is available in the case of d-sets and covered by [GaM01, Theorem 3(ii), p. 473]. Furthermore, the recently discovered close connection between d-sets in $\mathbb{R}^n$ and Muckenhoupt weights in [HaPi05], [Pio06], suggests to have a closer look at spaces of type (9.87) where Lebesgue measure μ_L is replaced by $w\mu_L$ with a Muckenhoupt weight w. For corresponding vector-valued maximal inequalities one may consult [Kok78], [AnJ80] and [GaR85]. As for B-spaces and F-spaces based on Muckenhoupt weights we refer also to the literature in Remarks 1.48, 6.4, 6.17, especially to [BPT96, BPT97, Ry01, Rou02, Bow05, BoH05]. We stick here to Definition 9.18 adding first a few assertions originating from the geometry of M.

Definition 9.20.

(i) *A domain in $\mathbb{R}^n$ is said to be interior regular if there is a positive number c such that $|\Omega \cap B| \geq c\,|B|$ for any ball B centred at $\partial\Omega$ with radius less than 1.*

(ii) *A compact set Γ in $\mathbb{R}^n$ is said to be porous if there is a number η with $0 < \eta < 1$ such that one finds, for any ball $B(x,r)$, centred at $x \in \mathbb{R}^n$ and of radius r with $0 < r < 1$, a ball $B(y,\eta r)$ with*

$$B(y,\eta r) \subset B(x,r) \qquad \text{and} \qquad B(y,\eta r) \cap \Gamma = \emptyset. \tag{9.89}$$

Remark 9.21. Interior regularity of domains plays a role in connection with atomic decompositions of function spaces in non-smooth domains. We refer to [ET96, p. 59] and [TrW96]. So far we used the porosity condition in Remark 2.32 and (2.119) in the context of pointwise multipliers in function spaces. There one finds also some references. In particular, any d-set Γ with $d < n$ is porous. Furthermore if Γ is porous then $|\Gamma| = 0$.

Proposition 9.22.

(i) *Let $M = \Omega$ be an interior regular domain in $\mathbb{R}^n$ and let $\widetilde{\chi}^{\Omega}_{jm}$ be the characteristic function of $\widetilde{Q}_{jm} \cap \Omega$ where $\widetilde{Q}_{jm}$ is a cube with sides parallel to the axes of coordinates, centred at $2^{-j}m$ and of side-length 2^{-j+2}. Let $p < \infty$. Then the right-hand side of (9.87) with $\widetilde{\chi}^{\Omega}_{jm}$ in place of χ_{jm} is equivalent to the quasi-norm (9.87) with $M = \Omega$ (intrinsic characterisation).*

(ii) *Let $p < \infty$ and let $M = \Gamma$ be a compact porous set in $\mathbb{R}^n$. Then $f^{s,\varrho}_{pq}(\Gamma)$ is independent of q, in particular,*

$$f^{s,\varrho}_{pq}(\Gamma) = b^{s,\varrho}_{pp}(\Gamma). \tag{9.90}$$

Proof. Part (i) follows from Proposition 1.33(i). As for part (ii) we first remark that again by Proposition 1.33 the cubes Q_{jm} can be replaced by balls of radius 2^{-j} centred at Γ having pairwise distance $\geq c\,2^{-j}$ for some $c > 0$ and all $j \in \mathbb{N}_0$. For each such ball one can choose a sub-ball according to (9.89) of radius $\eta\,2^{-j}$ where one may assume in addition that each sub-ball has a distance to Γ of at least $c'\,2^{-j}$ for some $c' > 0$. All these sub-balls have a controlled overlapping, one may even assume that all these sub-balls have pairwise disjoint supports. Then one gets (9.90). □

Remark 9.23. In particular, if the domain $M = \Omega$ is interior regular then one can replace χ_{jm} in (9.87) by the characteristic function $\widetilde{\chi}_{jm}$ of $\widetilde{Q}_{jm}$ and $L_p(\mathbb{R}^n)$ by $L_p(\Omega)$ (equivalent quasi-norms).

9.3.2 Function spaces

Again M denotes either a domain Ω in $\mathbb{R}^n$ or a compact set Γ in $\mathbb{R}^n$ originating from a measure μ with (9.76)–(9.80). To unify our notation we agree on $L_p(M,\mu) = L_p(\Omega)$ if $M = \Omega$ is a domain. If the domain Ω is unbounded then we need again $L_\infty(\mathbb{R}^n, w_\sigma)$ being the collection of all f with $w_\sigma f \in L_\infty(\Omega)$ where $w_\sigma(x) = (1+|x|^2)^{\sigma/2}$. Let k^β_{jm} be the same functions as in (9.4) and let $b^{s,\varrho}_{pq}(M)$ and $f^{s,\varrho}_{pq}(M)$ be the sequence spaces according to Definition 9.18.

Definition 9.24. *Let $0 < p \leq \infty$ ($p < \infty$ for the $\mathfrak{F}$-spaces), $0 < q \leq \infty$ and $\varrho \geq 0$. Let either*

$$M = \Omega \quad \textit{domain in } \mathbb{R}^n \qquad \textit{and} \quad s > 0, \tag{9.91}$$

or

$$M = \Gamma \quad \textit{with} \quad \mu \in \mathcal{C}^{-\sigma}(\mathbb{R}^n), \quad 0 < \sigma \leq n, \quad s > \sigma/p, \tag{9.92}$$

according to (9.76)–(9.79). *Then $\mathfrak{B}^s_{pq}(M,\mu)$ is the collection of all $f \in L_p(M,\mu)$ which can be represented as*

$$f = \sum_{\beta,j} \sum_m{}^{M,j} \lambda^\beta_{jm}\, k^\beta_{jm}, \qquad \lambda \in b^{s,\varrho}_{pq}(M), \tag{9.93}$$

absolute convergence being in $L_p(M,\mu)$ (with the modification $L_\infty(\Omega, w_\varkappa)$, $\varkappa < 0$, if $p = \infty$ and Ω is an unbounded domain). Let

$$\|f\,|\mathfrak{B}^s_{pq}(M,\mu)\| = \inf \|\lambda\,|b^{s,\varrho}_{pq}(M)\| \tag{9.94}$$

where the infimum is taken over all admissible representations (9.93). *Similarly $\mathfrak{F}^s_{pq}(M,\mu)$ is the collection of all $f \in L_p(M,\mu)$ which can be represented as*

$$f = \sum_{\beta,j} \sum_m{}^{M,j} \lambda^\beta_{jm}\, k^\beta_{jm}, \qquad \lambda \in f^{s,\varrho}_{pq}(M), \tag{9.95}$$

absolute convergence being in $L_p(M,\mu)$. *Let*

$$\|f\,|\mathfrak{F}^s_{pq}(M,\mu)\| = \inf \|\lambda\,|f^{s,\varrho}_{pq}(M)\| \tag{9.96}$$

where the infimum is taken over all admissible representations (9.95).

Remark 9.25. This is the direct counterpart of Definition 9.6, including the comments in Remark 9.7. Obviously, in case of domains $M=\Omega$ one would prefer to write $\mathfrak{B}^s_{pq}(\Omega)$ and $\mathfrak{F}^s_{pq}(\Omega)$.

Theorem 9.26.

(i) *The above spaces* $\mathfrak{B}^s_{pq}(M,\mu)$ *and* $\mathfrak{F}^s_{pq}(M,\mu)$ *are quasi-Banach spaces. They are independent of* ϱ *and* k (*equivalent quasi-norms*). *Furthermore for all admitted* s,p,q,

$$\mathfrak{B}^s_{pq}(M,\mu) \hookrightarrow L_p(M,\mu), \qquad \mathfrak{F}^s_{pq}(M,\mu) \hookrightarrow L_p(M,\mu) \tag{9.97}$$

and

$$\mathfrak{B}^s_{p,\min(p,q)}(M,\mu) \hookrightarrow \mathfrak{F}^s_{pq}(M,\mu) \hookrightarrow \mathfrak{B}^s_{p,\max(p,q)}(M,\mu). \tag{9.98}$$

(ii) *Let, in addition,* Γ *in* (9.76) *be porous according to Definition* 9.20(ii). *Then* $\mathfrak{F}^s_{pq}(\Gamma,\mu)$ *is independent of* q, *in particular,*

$$\mathfrak{F}^s_{pq}(\Gamma,\mu) = \mathfrak{B}^s_{pp}(\Gamma,\mu), \qquad 0<p<\infty. \tag{9.99}$$

Proof. We prove the first inclusion in (9.97) in case of $M=\Gamma$ with (9.76)–(9.79). Let $p<\infty$. Then one gets in analogy to (9.44),

$$\begin{aligned}
\Big\| \sum_m{}^{\Gamma,j} \lambda^\beta_{jm} k^\beta_{jm}\,|L_p(\Gamma,\mu)\Big\|^p &\le c\,2^{-\varepsilon|\beta|p} \sum_m{}^{\Gamma,j} |\lambda^\beta_{jm}|^p\,\mu_j \\
&\le c' 2^{-\varepsilon|\beta|p}\,2^{-j(n-\sigma)}\,2^{-jsp+jn}\,\|\lambda\,|b^{s,0}_{pq}(\Gamma)\|^p \\
&\le c'\,2^{-\varepsilon|\beta|}\,2^{-j(sp-\sigma)}\,\|\lambda\,|b^{s,0}_{pq}(\Gamma)\|^p,
\end{aligned} \tag{9.100}$$

$j\in\mathbb{N}_0$. By (9.92) one gets the first inclusion in (9.97). If $p=\infty$ then one has to modify in an appropriate way. All other assertions in part (i) can be obtained in the same way as in the proof of Theorem 9.8. Part (ii) follows from Proposition 9.22(ii). □

Remark 9.27. Let $M=\Omega$ be a domain in $\mathbb{R}^n$. Let $0<p\le\infty$ ($p<\infty$ for the $\mathfrak{F}$-spaces), $0<q\le\infty$, $s>0$. Then it follows from Definitions 9.6, 9.24 that the spaces on Ω are restrictions of the corresponding spaces on $\mathbb{R}^n$ in the usual interpretation,

$$\|f\,|\mathfrak{B}^s_{pq}(\Omega)\| = \inf \|g\,|\mathfrak{B}^s_{pq}(\mathbb{R}^n)\|, \qquad g|\Omega = f, \tag{9.101}$$

as subspaces of $L_p(\Omega)$. Similarly for the $\mathfrak{F}$-spaces. Then one gets from Theorem 9.10 and Definition 4.1 of the spaces $B^s_{pq}(\Omega)$, $F^s_{pq}(\Omega)$ that

$$\mathfrak{B}^s_{pq}(\Omega) = B^s_{pq}(\Omega) \text{ if } s>\sigma_p \quad\text{and}\quad \mathfrak{F}^s_{pq}(\Omega) = F^s_{pq}(\Omega) \text{ if } s>\sigma_{pq}. \tag{9.102}$$

Remark 9.28. In case of compact sets Γ with (9.76), (9.77) we have also the trace spaces $B^s_{pq}(\Gamma,\mu)$ according to (9.74), (9.75). If Γ is a compact d-set with $0 < d < n$ then one gets by Proposition 7.32,

$$B^s_{pq}(\Gamma,\mu) = \mathrm{tr}_\mu\, B^{s+(n-d)/p}_{pq}(\mathbb{R}^n), \qquad \mu = \mathcal{H}^d|\Gamma \in \mathcal{C}^{-\sigma}(\mathbb{R}^n) \tag{9.103}$$

with $\sigma = n - d$. It follows by Theorem 9.10 and Definitions 9.6, 9.24 that

$$B^s_{pq}(\Gamma,\mu) = \mathfrak{B}^{s+(n-d)/p}_{pq}(\Gamma,\mu), \quad s > 0, \quad 1 < p < \infty, \quad 0 < q \le \infty. \tag{9.104}$$

In other words, the spaces introduced in Definition 9.24 are more general, extending the considerations in particular to $0 < p \le 1$. On the other hand, any d-set with $0 < d < n$ is porous, [Triϵ, Proposition 9.18, Remark 9.19, pp. 139/140], and one gets (9.99). This makes clear that Definition 9.24 for sets Γ with (9.76) is very reasonable in case of $\mathfrak{B}$-spaces, but somewhat doubtful when it comes to $\mathfrak{F}$-spaces. Finally we refer to the two recent papers [DHY05, HaY04] where one finds a theory of B-spaces and F-spaces based on measures μ in $\mathbb{R}^n$ satisfying (9.79). It relies on reproducing formulas and similar techniques as described in Section 1.17.5 and Definition 1.194. It is not clear how these spaces are related to the spaces introduced in Definition 9.24.

Remark 9.29. In Section 1.17.5 we mentioned several methods and proposals for introducing function spaces on general structures, say, beyond $\mathbb{R}^n$ and domains in $\mathbb{R}^n$. The references given there will not be repeated here. In the preceding chapters we dealt several times with questions of this type. In particular the trace spaces in (9.74), (9.75) have been discussed in some detail in Sections 1.17.2, 1.17.3. Specified as d-sets in $\mathbb{R}^n$ as in (9.103) and restricted to $q = p$ these spaces served as basic spaces to be transferred via snowflaked transforms to abstract d-spaces. This was the main subject of Chapter 8. However the snowflaked transform works on arbitrary spaces (X, ϱ, μ) of homogeneous type. One may consult Theorem 1.192, Definition 1.189 and the references given in Remark 1.193. On this basis one can use, say, the spaces $\mathfrak{B}^s_{pq}(\Gamma,\mu)$ as basic spaces subject to snowflaked transforms to introduce corresponding spaces on some abstract spaces (X, ϱ, μ) and to develop a respective analysis generalising Chapter 8. But nothing has been done so far.

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Notational Agreements

1. A formula in terms of A^s_{pq}-spaces refers both to the corresponding assertion with $A = B$ at all occurrences and with $A = F$ at all occurrences (if not specified differently). Usually formulas where B-spaces and F-spaces are mixed will be written down explicitly.
2. If there is no danger of confusion (which is mostly the case) we write A^s_{pq}, B^s_{pq}, F^s_{pq}, a^s_{pq} ... (spaces) instead of $A^s_{p,q}$, $B^s_{p,q}$, $F^s_{p,q}$, $a^s_{p,q}$ .... Similarly for $a_{\nu m}$, $\lambda_{\nu m}$, $Q_{\nu m}$ (functions, numbers, cubes) instead of $a_{\nu,m}$, $\lambda_{\nu,m}$, $Q_{\nu,m}$ etc.
3. Inconsequential positive constants, denoted by c (with subscripts and superscripts), may have different values in different formulas (but not in the same formula).
4. $a_j \lesssim b_j$ and $a_j \preceq b_j$ with $j \in J$ (where J is an index set) means that there is an inconsequential positive constant c such that $a_j \le c b_j$ for all $j \in J$. Furthermore, $a_j \sim b_j$ indicates equivalence, $a_j \preceq b_j \preceq a_j$.
5. *Domain* = open set (in $\mathbb{R}^n$ etc.)
6. *Optimal* in the context of inequalities, equalities or equivalences must always be understood up to inconsequential positive constants.
7. When quoted in the text we abbreviate Subsection k.l by Section k.l and Subsubsection k.l.m by Section k.l.m.
8. In connection with real functions in $\mathbb{R}$, *increasing* (decreasing) means *non-decreasing* (non-increasing).
9. References are ordered by names, not by labels, which roughly coincides, but may occasionally cause minor deviations.
10. The number(s) behind # in the References mark the pages where the corresponding entry is quoted (with exception of [Triα]–[Triϵ]).

Symbols

Operators

Functions, functionals

Numbers, relations

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